UDC 517.9

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EXISTENCE OF POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

ПРО ІСНУВАННЯ ДОДАТНИХ РОЗВ'ЯЗКІВ ЗВ'ЯЗАНИХ СИСТЕМ НЕЛІНІЙНИХ ДРОБОВО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We study the following nonlinear boundary-value problems for fractional differential equations

$$D^{\alpha}u(t) = f(t, v(t), D^{\beta-1}v(t)), \quad t > 0,$$

$$D^{\beta}v(t) = g(t, u(t), D^{\alpha-1}u(t)), \quad t > 0,$$

$$u > 0 \quad \text{and} \quad v > 0 \quad \text{in} \quad (0, \infty),$$

$$\lim_{t\to 0^+} u(t) = \lim_{t\to 0^+} v(t) = 0,$$

where $1 < \alpha \leq 2$ and $1 < \beta \leq 2$. Under certain conditions on f and g, the existence of positive solutions is obtained by applying the Schauder fixed-point theorem.

Вивчаються нелінійні граничні задачі для дробово-диференціальних рівнянь

$$D^{\alpha}u(t) = f(t, v(t), D^{\beta-1}v(t)), \quad t > 0,$$

$$D^{\beta}v(t) = g(t, u(t), D^{\alpha-1}u(t)), \quad t > 0,$$

$$u > 0 \quad \text{і} \quad v > 0 \quad \text{в} \quad (0, \infty),$$

$$\lim_{t\to 0^+} u(t) = \lim_{t\to 0^+} v(t) = 0,$$

де $1 < \alpha \leq 2$ та $1 < \beta \leq 2$. За деяких умов, накладених на f і g, існування додатних розв'язків встановлюється за допомогою теореми Шаудера про нерухому точку.

1. Introduction. Fractional differential equations are gaining much importance and attention since they can be applied in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc., can be modeled by fractional differential equations. They also serve as an excellent tool for the description of hereditary properties of various materials and processes. We refer the reader to [8, 11, 12, 20] and references therein for details.

Recently, many authors have investigated sufficient conditions for the existence of solutions for the following coupled systems of nonlinear fractional differential equations with different boundary conditions on finite domain

$$D^{\alpha}u(t) = f(t, v(t)),$$

$$D^{\beta}v(t) = g(t, u(t)),$$

and more generally,

$$D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)),$$

$$D^{\beta}v(t) = g(t, u(t), D^{\nu}u(t)),$$

where D^{α} is the standard Riemann – Liouville deravitive of order α, see, for example, [2, 4, 6, 9, 10, 16 – 19]. However, to the best of our knowledge few papers consider the existence of solutions of fractional differential equations on the half-line. Maagli in [13], studied the existence of solutions for differential equations involving the Riemann – Liouville fractional derivative on the half-line $\mathbb{R}^+ := (0, \infty)$

$$D^{\alpha}u(t) = f(t, u(t)) \quad \text{in} \quad (0, \infty),$$

$$u > 0 \quad \text{in} \quad (0, \infty),$$

$$\lim_{t \to 0} u(t) = 0,$$

where $1 < \alpha \leq 2$ and f is a Borel measurable function in $(0, \infty) \times (0, \infty)$.

Maagli and Dhifli [14] considered the following boundary-value problem for fractional differential equations:

$$D^{\alpha}u(t) = f\big(t, u(t), D^{\alpha-1}u(t)\big) \quad \text{in} \quad (0, \infty),$$

$$u > 0 \quad \text{in} \quad (0, \infty),$$

$$\lim_{t \to 0} u(t) = 0,$$

where $1 < \alpha \leq 2$ and f is a Borel measurable function in $(0, \infty) \times (0, \infty) \times (0, \infty)$ satisfying some appropriate conditions.

Our aim in this paper is to extend the above results to the coupled system of nonlinear fractional differential equations on an unbounded domain

$$\begin{gathered} D^{\alpha}u(t) + f\big(t, v(t), D^{\beta-1}v(t)\big) = 0, \quad t > 0, \\ D^{\beta}v(t) + g\big(t, u(t), D^{\alpha-1}u(t)\big) = 0, \quad t > 0, \\ u > 0 \quad \text{and} \quad v > 0 \quad \text{in} \quad (0, \infty), \\ \lim_{t \to 0^+} u(t) = \lim_{t \to 0^+} v(t) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$, $1 < \beta \leq 2$, f and g are Borel measurable functions in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ satisfying the following assumptions:

(H$_1$**)** f and g are continuous with respect to the second and third variable.

(H$_2$**)** There exist nonnegative measurable functions h_1, h_2, k_1, and k_2 on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that

(i) for all $x, y, z \in \mathbb{R}^+$ we have

$$|f(x, y, z)| \leq h(x, y, z),$$

$$|g(x, y, z)| \leq k(x, y, z),$$

where $h(x, y, z) := yh_1(x, y, z) + zh_2(x, y, z)$ and $k(x, y, z) := yk_1(x, y, z) + zk_2(x, y, z)$;

(ii) for $j = 1, 2,$ the functions h_j and k_j are nondecreasing with respect to the second and the third variables and satisfying for all $x \in \mathbb{R}^+$

$$\lim_{(y,z)\to(0,0)} h_j(x,y,z) = \lim_{(y,z)\to(0,0)} k_j(x,y,z) = 0 \qquad \text{for} \quad j = 1, 2;$$

(iii) the integrals $\int_0^\infty h(t, \omega_\beta(t), 1)\, dt$ and $\int_0^\infty k(t, \omega_\alpha(t), 1)\, dt$ converge, where $\omega_\nu = \dfrac{t^\nu}{\Gamma(\nu)}$ for $1 < \nu \le 2$.

Our main result is the following.

Theorem 1.1. *Assume* ($\boldsymbol{H}_1$) *and* ($\boldsymbol{H}_2$). *Then problem* (1.1) *has infinitely many solutions. More precisely, there exists a number* $b > 0$ *such that for each* $c \in (0, b]$, *problem* (1.1) *has a continuous solution* (u, v) *satisfying*

$$u(t) = c\omega_\alpha(t) + \int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\alpha-1}\right) f(s, v(s), D^{\beta-1}v(s))\, ds,$$

$$v(t) = c\omega_\beta(t) + \int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\beta-1}\right) g(s, u(s), D^{\alpha-1}u(s))\, ds$$

and

$$\lim_{t\to\infty} \frac{u(t)}{\omega_\alpha(t)} = \lim_{t\to\infty} D^{\alpha-1}u(t) = c,$$

$$\lim_{t\to\infty} \frac{v(t)}{\omega_\beta(t)} = \lim_{t\to\infty} D^{\beta-1}v(t) = c,$$

where, for every $x \in \mathbb{R}$, $x^+ = \max(x, 0)$.

This paper is organized as follows. In Section 2, some facts and results about fractional calculus are given. We prove the main result in Section 3. Finally, we conclude this paper by considering an example in Section 4.

2. Preliminaries. In this section, we introduce some necessary definitions and results which will used throughout this paper.

Definition 2.1. *The Riemann – Liouville fractional integral of order* $\theta > 0$ *of any function* $u\colon \mathbb{R}^+ \to \mathbb{R}$ *is defined by*

$$I^\theta u(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} u(s)\, ds$$

provided the right-hand side is point-wise defined on $\mathbb{R}^+$.

Definition 2.2. *The Riemann – Liouville fractional derivative of order* $\theta > 0$ *of a continuous function* $u\colon (0,\infty) \longrightarrow \mathbb{R}$ *is given by*

$$D^\theta u(t) = \frac{1}{\Gamma(n-\theta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\theta-n+1}}\, ds,$$

where Γ *is the Gamma function and* $n = [\theta] + 1$, *provided that the right-hand side is point-wise defined.*

***Remark* 2.1.** The following properties are well known (see [12, 15]):

(i) $D^\theta I^\theta u(t) = u(t)$, a.e. in $\mathbb{R}^+$, $\theta > 0$, $u \in L^1_{\text{loc}}([0,\infty))$.

(ii) $I^\delta I^\theta u(t) = I^{\theta+\delta} u(t)$, a.e. in $\mathbb{R}^+$, $\theta + \delta \geq 1$, $u \in L^1_{\text{loc}}([0,\infty))$.

(iii) Let $\theta > 0$, then $D^\theta u(t) = 0$ if and only if $u(t) = \sum_{j=1}^{n} c_j t^{\theta-j}$, where $n = -[-\theta]$ the smallest integer greater than or equal to θ and $(c_1, \ldots, c_n) \in \mathbb{R}^n$.

(iv) Let $1 < \theta \leq 2$, and $t \geq 0$, then we have

$$I^{\theta-1}(1)(t) = \omega_\theta(t) = \frac{t^{\theta-1}}{\Gamma(\theta)}.$$

In the sequel, we denote by $C([0,\infty])$ the set of continuous functions u on $\mathbb{R}^+$ such that $\lim_{t\to 0^+} u(t)$ and $\lim\limits_{t\to\infty} u(t)$ exist. It is easy to see that $C([0,\infty])$ is a Banach space with the norm $\|u\|_\infty = \sup_{t\geq 0} |u(t)|$. For $1 < \theta \leq 2$, we define

$$E_\theta = \Big\{ u \in C([0,\infty]) : D^{\theta-1}(\omega_\theta u) \in C([0,\infty]) \Big\} \tag{2.1}$$

endowed with the norm $\|u\|_\theta = \|D^{\theta-1}(\omega_\theta u)\|_\infty$. Then, it's easy to see that the map

$$(E_\theta, \|.\|_\theta) \longrightarrow \big(C([0,\infty]), \|.\|_\infty\big),$$

$$u \longmapsto D^{\theta-1}\omega_\theta(u)$$

is an isometry. It follows that $(E_\theta, \|.\|_\theta)$ is a Banach space.

Let $E = E_\alpha \times E_\beta$ endowed with the norm

$$\|(u,v)\| = \max\big(\|u\|_\alpha, \|u\|_\beta\big),$$

then $(E, \|.\|)$ is a Banach space.

Next, we quote some results in the following lemmas that will be used later.

Lemma 2.1 (see [7]). *Let $1 < \theta \leq 2$ and let f be a function in $C([0,\infty))$ such that $f(0) = 0$ and $D^{\theta-1}f$ belongs to $C([0,\infty))$. Then, for $t \geq 0$, we have*

$$I^{\theta-1}D^{\theta-1}f(t) = f(t).$$

Lemma 2.2 (see [14]). *Let $m_1, m_2 \in \mathbb{R}$ and $u \in C([0,\infty))$ such that $D^{\theta-1}(\omega_\theta u) \in C([0,\infty))$ and $m_1 \leq D^{\theta-1}(\omega_\theta u)(t) \leq m_2$ for all $t \geq 0$. Then, for each $t \geq 0$,*

$$m_1 \leq u(t) \leq m_2.$$

In particular, $\|u\|_\infty \leq \|D^{\theta-1}(\omega_\theta u)\|_\infty$ and $E_\theta \subset C([0,\infty])$.

Let $\mathcal{F}_\mu = \big\{ u \in E_\mu : 0 \leq D^{\mu-1}(\omega_\mu u) \leq 1 \big\}$. Then we have the following result.

Lemma 2.3 (*Assume* $(\mathbf{H}_2)$). *Then the family of functions*

$$\left\{ \int_0^t \Big(1 - \frac{s}{t}\Big)^{\alpha-1} f\left(s, \omega_\beta(s)v(s), D^{\alpha-1}(\omega_\beta v)(s)\right) ds,\ v \in \mathcal{F}_\beta \right\}$$

and

$$\left\{ \int_0^t \Big(1 - \frac{s}{t}\Big)^{\beta-1} g\left(s, \omega_\alpha(s)u(s), D^{\beta-1}(\omega_\alpha u)(s)\right) ds,\ u \in \mathcal{F}_\alpha \right\}$$

are relatively compact in $C([0,\infty])$.

Proof. The proof is very similar to and based on the technique used in the proofs of [13] (Lemma 1.5); hence we omit it.

3. Proof of Theorem 1.1. Let $BC([0,\infty))$ be the Banach space of all bounded continuous real-valued functions on $[0,\infty)$, endowed with the sup-norm $\|.\|_\infty$. In order to prove Theorem 1.1, we need the following compactness criterion for a subset of $BC([0,\infty))$, which is a consequence of the well-known Arzela – Ascoli theorem. This compactness criterion is an adaptation of a lemma due to Avramescu [3]. In order to formulate this criterion, we note that a set U of real-valued functions defined on $[0,\infty)$ is said to be equiconvergent at ∞ if all the functions in U are convergent in $\mathbb{R}$ at the point ∞ and, in addition, for each $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that, for any function $\psi \in U$, we have $|\psi(t) - \lim_{s\to\infty}\psi(s)| < \epsilon$ for $t > T$.

Theorem 3.1 (see [3])**.** *Let U be an equicontinuous and uniformly bounded subset of the Banach space $BC([0,\infty))$. If U is equiconvergent at ∞, it is also relatively compact.*

In the sequel, for $x, y, z \in \mathbb{R}^+$, we denote

$$F(x,y,z) = \omega_\beta(x)h_1(x,y,z) + h_2(x,y,z) \qquad \text{and} \qquad G(x,y,z) = \omega_\alpha(x)k_1(x,y,z) + k_2(x,y,z).$$

It follows from (**H**$_2$) and Lebesgue's theorem that

$$\lim_{s\to 0}\int_0^\infty F(t, s\omega_\beta(t), s)dt = 0 \qquad \text{and} \qquad \lim_{s\to 0}\int_0^\infty G(t, s\omega_\alpha(t), s)dt = 0.$$

Hence, we can fix a number $0 < \lambda < 1$ such that

$$\max\left(\int_0^\infty F(t,\lambda\omega_\beta(t),\lambda)\,dt, \int_0^\infty G(t,\lambda\omega_\alpha(t),\lambda)\,dt\right) \le \frac{1}{3}.$$

Let $b = \dfrac{2\lambda}{3}$ and $c \in (0,b]$. To apply a fixed point argument, set

$$\Lambda := \Lambda_\alpha \times \Lambda_\beta,$$

where

$$\Lambda_\alpha = \left\{u \in E_\alpha : \frac{c}{2} \le D^{\alpha-1}(\omega_\alpha u) \le \frac{3c}{2}\right\} \qquad \text{and} \qquad \Lambda_\beta = \left\{v \in E_\beta : \frac{c}{2} \le D^{\beta-1}(\omega_\beta v) \le \frac{3c}{2}\right\}.$$

Then Λ is a nonempty closed bounded and convex set in E. Now, we define the operator T on Λ by

$$T(u,v) := (A_1 v, A_2 u),$$

where, for a given $t > 0$,

$$A_1 v(t) = c + \int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\alpha-1}\right) f\left(s, \omega_\beta(s)v(s), D^{\alpha-1}(\omega_\beta v)(s)\right) ds, \quad v \in \Lambda_\beta,$$

and

$$A_2u(t) = c + \int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\beta-1}\right) f\Big(s, \omega_\alpha(s)u(s), D^{\beta-1}(\omega_\alpha u)(s)\Big)\, ds, \quad u \in \Lambda_\alpha.$$

First, we shall prove that the operator T maps Λ into itself. Let $v \in \Lambda_\beta$. Using Lemma 2.3, we deduce that the function A_1v is in $C([0,\infty])$. On the other hand, for $t \geq 0$, we obtain

$$\omega_\alpha(t)A_1v = \omega_\alpha(t)\left(c + \int_0^\infty f\left(s, \omega_\beta(s)v(s), D^{\alpha-1}(\omega_\beta v)(s)\right) ds\right) - I^\alpha(f(., \omega_\beta v, D^{\alpha-1}(\omega_\beta v)))(t).$$

Hence, applying $D^{\alpha-1}$ on both sides of this equality, we conclude that, for each $t \geq 0$,

$$D^{\alpha-1}(\omega_\alpha A_1v)(t) = c + \int_t^\infty f\left(s, \omega_\beta(s)v(s), D^{\alpha-1}(\omega_\beta v)(s)\right) ds. \tag{3.1}$$

This implies that $D^{\alpha-1}(\omega_\alpha A_1v)$ is in $C([0,\infty])$ and $A_1\Lambda_\alpha \subset E_\alpha$. Furthermore, for $v \in \Lambda_\beta$ and $t \geq 0$, we have

$$|D^{\alpha-1}(\omega_\alpha A_1v)(t) - c| \leq \int_0^\infty \left|f\left(s, \omega_\beta(s)v(s), D^{\alpha-1}(\omega_\beta v)(s)\right|\right) ds \leq$$

$$\leq \int_0^\infty h\left(s, \omega_\beta(s)v(s), D^{\alpha-1}(\omega_\beta v)(s)\right) ds \leq$$

$$\leq \int_0^\infty h\left(s, \frac{3c}{2}\omega_\alpha(s), \frac{3c}{2}\right) ds =$$

$$= \frac{3c}{2}\int_0^\infty F\left(s, \frac{3c}{2}\omega_\beta(s), \frac{3c}{2}\right) ds \leq$$

$$\leq \frac{3c}{2}\int_0^\infty F\left(s, \lambda\omega_\beta(s), \lambda\right) ds \leq \frac{c}{2}.$$

It follows that for each $t \geq 0$

$$\frac{c}{2} \leq D^{\alpha-1}(\omega_\alpha A_1v)(t) \leq \frac{3c}{2}.$$

So, since from Lemma 2.3 $A_1\Lambda_\alpha \subset C([0,\infty])$, we conclude that Λ_α is invariant under A_1. Similarly we prove that Λ_β is invariant under A_2 and hence Λ is invariant under T.

Next, we prove that $T\Lambda$ is relatively compact in $(E, \|\cdot\|)$. For any $v \in \Lambda_\beta$ and $t > 0$, we have

$$\frac{d}{dt}D^{\alpha-1}(\omega_\alpha A_1v)(t) = -f\big(x, \omega_\beta(t)v(t), D^{\beta-1}(\omega_\beta v)(t)\big) \quad \text{a.e. in} \quad \mathbb{R}^+.$$

Since

$$\left|f(t,\omega_\beta(t)v(t),D^{\beta-1}(\omega_\beta v)(t))\right| \le h\big(t,\omega_\beta(t)v(t),D^{\beta-1}(\omega_\beta v)(t)\big) \le h(t,\omega_\beta(t),1) \tag{3.2}$$

and

$$\int\limits_0^\infty h(x,\omega_\beta(t),1)\,dt < \infty, \tag{3.3}$$

it follows that the family $\left\{D^{\alpha-1}(\omega_\alpha A_1 v),\ v\in\Lambda_\beta\right\}$ is equicontinuous on $[0,\infty]$. Moreover, $\left\{D^{\alpha-1}(\omega_\alpha A_1 v),\ v\in\Lambda_\beta\right\}$ is uniformly bounded. Thus, by Theorem 3.1, to prove that $\left\{D^{\alpha-1}(\omega_\alpha u),\ u\in\Lambda_\beta\right\}$ is relatively compact, it suffice to prove that all elements of $\left\{D^{\alpha-1}(\omega_\alpha v),\ v\in\Lambda_\beta\right\}$ are equiconvergent at infinity. Endeed, since for all $v\in\Lambda_\beta$, $D^{\alpha-1}(\omega_\alpha A_1 v)\subset C([0,\infty])$, it follows that $\lim_{t\longrightarrow\infty} D^{\alpha-1}(\omega_\alpha v)(t)$ exists. On the other hand, it follows from (3.2), (3.3) and the dominated convergence theorem that

$$\lim_{t\longrightarrow\infty}\int\limits_t^\infty f\left(s,\omega_\beta(s)v(s),D^{\alpha-1}(\omega_\beta v)(s)\right)ds = 0.$$

So, using (3.1), we obtain

$$\lim_{t\longrightarrow\infty}\left|D^{\beta-1}(\omega_\alpha v)(t)-\lim_{t\longrightarrow\infty}D^{\beta-1}(\omega_\alpha v)(t)\right| =$$

$$=\lim_{t\longrightarrow\infty}\int\limits_t^\infty f\left(s,\omega_\beta(s)v(s),D^{\alpha-1}(\omega_\beta v)(s)\right)ds = 0.$$

That is $\left\{D^{\beta-1}(\omega_\alpha u),\ u\in\Lambda_\beta\right\}$ is relatively compact in $(C([0,\infty]),\|\cdot\|_\infty)$. This implies that $A_1\Lambda_\alpha$ is relatively compact in $(E_\alpha,\|\cdot\|_\alpha)$.

Similar process can be repeated to prove that $A_2\Lambda_\beta$ is relatively compact in $(E_\beta,\|\cdot\|_\beta)$. Thus $T\Lambda$ is relatively compact in $(E,\|\cdot\|)$.

Now, we prove the continuity of T in Λ. Let (v_k) be a sequence in Λ_β such that

$$\|v_k-v\|_\beta = \left\|D^{\beta-1}(\omega_\beta v_k)-D^{\beta-1}(\omega_\beta v)\right\|_\infty \to 0 \quad \text{as} \quad k\to\infty.$$

Then, by Lemma 2.2, $\|v_k-v\|_\infty\to 0$ as $k\to\infty$, and, for any $t\in[0,\infty]$, we have

$$\left|D^{\alpha-1}(\omega_\alpha A_1 v_k)(t)-D^{\alpha-1}(\omega_\alpha A_1 v)(t)\right| =$$

$$=\left|\int\limits_t^\infty\left[f(s,\omega_\alpha(s)v_k(s),D^{\beta-1}(\omega_\beta v_k)(s))-f(s,\omega_\beta(s)v(s),D^{\beta-1}(\omega_\beta v)(s))\right]ds\right| \le$$

$$\le\int\limits_0^\infty\left|f(s,\omega_\beta(s)v_k(s),D^{\alpha-1}(\omega_\beta v_k)(s))-f(s,\omega_\beta(s)v(s),D^{\beta-1}(\omega_\beta v)(s))\right|ds.$$

Since

$$\left|f(s,\omega_\beta(s)v_k(s),D^{\alpha-1}(\omega_\beta v_k)(s))-f(s,\omega_\beta(s)v(s),D^{\alpha-1}(\omega_\beta v)(s))\right| \le 2h(s,\omega_\beta(s),1),$$

and, by (H_1) and Lebesgue's theorem, we get

$$\left\|A_1v_k - A_1v\right\|_\alpha = \left\|D^{\alpha-1}(\omega_\alpha A_1v_k) - D^{\alpha-1}(\omega_\alpha A_1v)\right\|_\infty \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, A_1 is continuous in Λ_α. In a similar way, A_2 is continuous in Λ_β and so T is continuous in Λ. Therefore, by Schauder fixed point theorem there exists $(x, y) \in \Lambda$ such that $T(x, y) = (x, y)$. That is, for $t > 0$,

$$x(t) = c + \int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\alpha-1}\right) f(s, \omega_\beta(s)y(s), D^{\beta-1}(\omega_\beta y)(s))\, ds$$

and

$$y(t) = c + \int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\beta-1}\right) g(s, \omega_\alpha(s)x(s), D^{\alpha-1}(\omega_\alpha x)(s))\, ds.$$

We put $u(t) = \omega_\alpha(t)x(t)$ and $v(t) = \omega_\beta(t)y(t)$. Then for any $t > 0$, we have

$$u(t) = c\omega_\alpha(t) + \omega_\alpha(t)\int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\alpha-1}\right) f(s, v(s), D^{\beta-1}(v)(s))\, ds \tag{3.4}$$

and

$$v(t) = c\omega_\beta(t) + \omega_\beta(t)\int_0^\infty \left(1 - \left(\left(1 - \frac{s}{t}\right)^+\right)^{\beta-1}\right) g(s, u(s), D^{\alpha-1}(u)(s))\, ds. \tag{3.5}$$

Moreover, for $t > 0$, we obtain

$$\frac{c}{2}\omega_\alpha(t) \le u(t) \le \frac{3c}{2}\omega_\alpha(t),$$

$$\lim_{t\to\infty} \frac{u(t)}{\omega_\alpha(t)} = \lim_{t\to\infty} D^{\alpha-1}u(t) = c$$

and

$$\frac{c}{2}\omega_\beta(t) \le v(t) \le \frac{3c}{2}\omega_\beta(t),$$

$$\lim_{t\to\infty} \frac{v(t)}{\omega_\beta(t)} = \lim_{t\to\infty} D^{\beta-1}v(t) = c.$$

It remains to show that u is a solution of problem (1.1). Indeed, applying D^α on both sides of (3.4) and D^β on both sides of (3.5), we obtain by Remark 2.1

$$D^\alpha u(x) = -f(x, u, D^{\beta-1}u) \quad \text{a.e. in} \quad \mathbb{R}^+$$

and

$$D^\beta v(x) = -g(x, u, D^{\alpha-1}u) \quad \text{a.e. in} \quad \mathbb{R}^+.$$

Theorem 1.1 is proved.

***Example* 3.1.** Let $p_1,\ p_2,\ q_1,\ q_2 \geq 0$ such that $\max(p_1, q_1) > 1,\ \max(p_2, q_2) > 1$ and let $k,\ h$ be measurable functions satisfying

$$\int_0^\infty t^{(\alpha-1)p_1}\big|k(t)\big|\,dt < \infty$$

and

$$\int_0^\infty t^{(\beta-1)p_2}\big|h(t)\big|\,dt < \infty.$$

Then, there exists a constant $b > 0$ such that for each $c \in (0, b]$, the problem

$$D^\alpha u + k(x)v^{p_1}(D^{\beta-1}v)^{q_1} = 0, \quad u > 0, \quad \text{in} \quad \mathbb{R}^+,$$

$$D^\beta v + h(x)u^{p_2}(D^{\alpha-1}u)^{q_2} = 0, \quad v > 0, \quad \text{in} \quad \mathbb{R}^+,$$

$$\lim_{x\to 0^+} u(x) = \lim_{x\to 0^+} v(x) = 0,$$

has a continuous solution (u, v) in $\mathbb{R}^+$ satisfying

$$\lim_{x\to 0^+} \frac{u(x)}{\omega_\alpha(x)} = \lim_{x\to\infty} D^{\alpha-1}u(x) = c$$

and

$$\lim_{x\to 0^+} \frac{v(x)}{\omega_\beta(x)} = \lim_{x\to\infty} D^{\beta-1}v(x) = c.$$

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Received 21.04.16,
after revision — 06.09.16