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ANTINORMAL COMPOSITION OPERATORS ON L^2-SPACE OF AN ATOMIC MEASURE SPACE

АНТИНОРМАЛЬНІ ОПЕРАТОРИ КОМПОЗИЦІЇ НА ПРОСТОРІ L^2, ЩО ВІДПОВІДАЄ ПРОСТОРУ З АТОМНОЮ МІРОЮ

Let $L^2(\mu)$ denotes the Hilbert space associated with a σ-finite atomic measure μ. We propose a characterization of antinormal composition operators on $L^2(\mu)$.

Позначимо гільбертів простір, асоційований з σ-скінченною атомною мірою μ, через $L^2(\mu)$. Наведено характеризацію антинормальних операторів композиції на $L^2(\mu)$.

1. Introduction. A classical problem in operator theory is to determine the distance of an operator from a given class of bounded linear operators on a Hilbert space. The distance between an operator and the set of Hermitian, positive, compact and unitary operators have been investigated in [2, 3, 5, 7], respectively. It is therefore natural to make analogous study about the set of normal operators. But most of the usual approximation criterion are not applicable since the set of normal operators is not convex. In 1974 Holmes [6] investigated those operators which admit a best normal approximation. He observed that there are operators for which its largest possible distance from the set of normal operators can be achieved. He named such operators as antinormal and showed that no invertible operator is antinormal and consequently, no compact operator is antinormal. Thus, the existence of antinormal operators is infinite dimensional phenomenon. Subsequently this class has been extensively studied by several authors in [4, 8 – 10]. In 2008 Tripathi and Lal [13] characterized antinormal composition operators on the Hilbert space l^2. In this paper we obtain a characterization of normal and antinormal composition operators on $L^2(\mu)$.

2. Preliminaries. In this section we give certain basic definitions and fix some notations.

Definition 2.1. *Let $(X, \mathcal{S}, \mu)$ be a measure space. A measurable set E is called an atom if $\mu(E) \neq 0$ and for each measurable subset F of E either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. A measure space $(X, \mathcal{S}, \mu)$ is called atomic measure space if each measurable subset of non-zero measure contains an atom.*

A trivial example of an atomic measure space is $(X, \mathcal{S}, \mu)$, where X is any non-empty set, $\mathcal{S}$ is a σ-algebra and μ is the counting measure.

Definition 2.2. *An atomic measure space $(X, \mathcal{S}, \mu)$ is called σ-finite atomic measure space if X is expressible as countable union of its atoms of finite measure.*

Let $(X, \mathcal{S}, \mu)$ be a σ-finite atomic measure space. Then $X = \bigcup_{n\in\mathbb{N}} A_n$, where A_n are disjoint atoms of finite measure. These atoms are unique in the sense that if $X = \bigcup_{n\in\mathbb{N}} B_n$, where B_n are disjoint atoms of finite measure, then $A_n = B_n$ upto a null set for each $n \in \mathbb{N}$. A measurable transformation $\varphi\colon X \to X$ is called non-singular if $\mu\phi^{-1}$ is absolutely continuous with respect to measure μ. If a non-singular transformation on X takes some part of an atom A_n to a subset of an atom A_m and the remaining part of A_n to a subset of another atom A_r, then both part must be null sets as image of an atom under a non-singular transformation cannot be a null set. Therefore a non-singular measurable transformation takes atoms into atoms. A non-singular transformation φ

of X into X is called injective almost everywhere (a.e.) if the inverse image of every atom under φ contains at most one atom. It is called surjective a.e. if the inverse image of every atom under φ contains at least one atom. If φ is both injective and surjective a.e., then it is called bijective a.e.

Let $\mathbb{N}$ and $\mathbb{C}$ denote the set of all positive integers and the set of all complex numbers, respectively. Then for $n \in \mathbb{N}$, $\chi_{A_n} : X \to \{0, 1\}$ be defined as

$$\chi_{A_n}(x) = \begin{cases} 1, & \text{if} \quad x \in A_n, \\ 0, & \text{otherwise.} \end{cases}$$

Henceforth, we take $L^2(\mu)$ of a σ-finite atomic measure space $(X, \mathcal{S}, \mu)$, where $X = \bigcup_{n\in\mathbb{N}} A_n$ and A_n are disjoint atoms of finite measure. Further, each $f \in L^2(\mu)$ is constant a.e. on an atom. Hence, the span of the characteristic functions $\{\chi_{A_n} : n \in \mathbb{N}\}$ forms a dense subset of $L^2(\mu)$. Thus, $\left\{K_{A_n} = \dfrac{1}{\mu(A_n)}\chi_{A_n} : n \in \mathbb{N}\right\}$ forms an orthonormal basis for $L^2(\mu)$.

2.1. Composition operators. Let $(X, \mathcal{S}, \mu)$ be a σ-finite atomic measure space. A non-singular measurable transformation φ induces a linear transformation C_φ on $L^2(\mu)$ defined by

$$C_\varphi(f) = f \circ \varphi \quad \forall f \in L^2(\mu).$$

When C_φ is bounded, it is called composition operator. A necessary and sufficient condition for boundedness of C_φ is given below.

Theorem 2.1 [11]. *A composition transformation C_ϕ is bounded on $L^2(\mu)$ if and only if there exists a positive real number $M > 0$ such that $\mu(\varphi^{-1}(E)) \leq M\mu(E)$ $\forall E \in \mathcal{S}$.*

Theorem 2.2 [11]. *Let C_φ be a composition operator. Then C_φ is normal if and only if the range of C_φ is dense in $L^2(\mu)$ and $f_\varphi \circ \varphi = f_\varphi$ a.e., where f_φ is a Radon–Nikodym derivative of $\mu\varphi^{-1}$ w.r.t. μ.*

In 1983 Singh and Veluchamy computed the adjoint of a composition operator on $L^2(\mu)$ and gave following characterization for an operator to be a composition operator. Further, they obtained adjoint of the composition operator as follows.

Theorem 2.3 [12]. *Let C_φ be a composition operator. Then*

$$(C_\varphi^* f)(A_n) = \frac{1}{\mu(A_n)} \int\limits_{\varphi^{-1}(A_n)} f \, d\mu$$

a.e. for $f \in L^2(\mu)$ and for every atom A_n.

Moreover, they gave following characterization for an operator to be a composition operator.

Theorem 2.4 [12]. *Let T be a bounded linear operator on $L^2(\mu)$. Then T is a composition operator if and only if the set $\{K_{A_n} : n \in \mathbb{N}\}$ is invariant under T^*. In this case φ is determined by $T^*(K_{A_n}) = K_{\varphi(A_n)}$.*

2.2. Antinormal operators on Hilbert space. Suppose H is a complex Hilbert space and $B(H)$ is the algebra of all bounded linear operators on H. Further, for $T \in B(H)$, let $N(T)$ and $R(T)$ respectively denote the null space and the range space of T.

Definition 2.3 [1]. *An operator $T \in B(H)$ is said to be Fredholm operator if dimension of $N(T)$ and the dimension of the quotient space $H/R(T)$ are both finite.*

Equivalently, T is Fredholm if both $N(T)$ and $N(T^*)$ are finite dimensional.

Definition 2.4. *Essential spectrum of an operator* $T \in B(H)$ *is defined as* $\sigma_e(T) = \{\alpha \in \mathbb{C} : T - \alpha I$ *is not Fredholm*$\}$.

Since every invertible operator is Fredholm operator, $\sigma_e(T) \subseteq \sigma(T)$.

Definition 2.5. *Minimum modulus of an operator* $T \in B(H)$ *is defined as* $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$.

Definition 2.6. *Essential minimum modulus of an operator* $T \in B(H)$ *is defined as* $m_e(T) = \inf\{\alpha \geq 0 : \alpha \in \sigma_e(|T|)\}$, *where* $|T| = (T^*T)^{1/2}$.

Definition 2.7. *An operator* $T \in B(H)$ *is said to be antinormal if* $d(T,\mathcal{N}) = \inf_{N\in\mathcal{N}} \|T - N\| = \|T\|$, *where* $\mathcal{N}$ *is the class of all normal operators in* $B(H)$.

***Remark* 2.1.** An operator $T \in B(H)$ is antinormal if and only if its adjoint T^* is antinormal.

Definition 2.8. *For an operator* T *in* $B(H)$, *index of* T *is defined as*

$$\operatorname{index}(T) = \begin{cases} \dim(N(T)) - \dim(N(T^*)), & \textit{if} \quad \dim(N(T)) \quad \textit{or} \\ & \dim(N(T^*)) < \infty, \\ 0, & \textit{otherwise}. \end{cases}$$

***Remark* 2.2.** Observe that $\operatorname{index}(T) = -\operatorname{index}(T^*)$.

The following results will be used in the later part of the paper.

Theorem 2.5 [8]. *Let* $T \in B(H)$.

(i) *If* $\operatorname{index}(T) = 0$, *then* $d(T,\mathcal{N}) \leq \dfrac{\|T\| - m(T)}{2}$.

(ii) *If* $\operatorname{index}(T) < 0$, *then* $m_e(T) \leq d(T,\mathcal{N}) \leq \dfrac{\|T\| + m_e(T)}{2}$.

***Remark* 2.3.** If $\operatorname{index}(T) = 0$, then T can not be antinormal.

Theorem 2.6 [8]. *Let* $T \in B(H)$ *with* $\operatorname{index}(T) < 0$. *Then following conditions are equivalent*:

(i) *T is antinormal*;

(ii) $m_e(T) = \|T\|$;

(iii) $d(T,\mathcal{U}) = 1 + \|T\|$, *where* $\mathcal{U}$ *is the class of all unitary operators in* $B(H)$;

(iv) $T = \alpha W(1 - K)$ *for some* $\alpha > 0$, *isometry* W *and positive compact contraction* K.

3. Antinormal composition operators on $L^2(\mu)$. We begin with a characterization of normal composition operators on $L^2(\mu)$.

Theorem 3.1. C_φ *is an injective if and only if* φ *is surjective a.e.*

Proof. Suppose that φ is surjective a.e. Therefore, $\varphi(\varphi^{-1}(A_n)) = A_n$ $\forall n \in \mathbb{N}$. Suppose $f,\ g \in L^2(\mu)$ be such that

$$C_\varphi f = C_\varphi g.$$

Then

$$f(\varphi(\varphi^{-1}(A_n))) = g(\varphi(\varphi^{-1}(A_n))) \quad \forall n \in \mathbb{N}$$

$$\Rightarrow f(A_n) = g(A_n) \quad \forall n \in \mathbb{N}.$$

Thus, $f = g$. Therefore, C_φ is injective. Conversely, suppose that C_φ is injective. It follows that for each $n \in \mathbb{N}$

$$C_\varphi(\chi_{A_n}) \neq 0,$$

$$\chi_{\varphi^{-1}(A_n)} \neq 0.$$

Hence, $\mu(\varphi^{-1}(A_n)) \neq 0$ for each $n \in \mathbb{N}$. Consequently φ is surjective a.e.

Theorem 3.2. *C_φ is surjective if and only if φ is injective a.e.*

Proof. Suppose $\varphi(A_n) = \varphi(A_m)$ for $m, n \in \mathbb{N}$. Since $\chi_{A_n} \in L^2(\mu)$ and C_φ is surjective, there is a function $f \in L^2(\mu)$ such that $C_\varphi f = \chi_{A_n}$. Therefore,

$$f(\varphi(A_n)) = 1 \qquad \text{and} \qquad f(\varphi(A_m)) = \delta_{n,m},$$

where $\delta_{n,m}$ is Kronecker delta. In view of the fact that $\varphi(A_n) = \varphi(A_m)$ we have $\delta_{n,m} = 1$. Hence, $n = m$. Conversely, suppose that φ is injective a.e. Let $f \in L^2(\mu)$. Define a function $g\colon X \to \mathbb{C}$ as follows. For each $n \in \mathbb{N}$

$$g(A_n) = \begin{cases} f(A_m), & \text{if} \quad \varphi(A_m) = A_n \quad \text{for some} \quad n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Function g is well defined as φ is injective a.e. Also

$$\|g\|^2 = \int\limits_X |g|^2\, d\mu = \sum_{n\in\mathbb{N}} \int\limits_{A_n} |g|^2\, d\mu =$$

$$= \sum_{n\in\mathbb{N}} \int\limits_{\varphi^{-1}(A_n)} |f|^2\, d\mu = \int\limits_X |f|^2\, d\mu = \|f\|^2 < \infty.$$

Therefore, $g \in L^2(\mu)$. Now it is easy to see that $C_\varphi g = f$. Hence, C_φ is surjective.

Theorem 3.3. *C_φ is normal if and only if φ is bijective a.e.*

Proof. It is immediate from Theorems 2.2, 3.1 and 3.2.

***Remark* 3.1.** If φ is bijective a.e., then C_φ is a non-zero normal operator. Hence it is not antinormal.

Theorem 3.4. *Let $\varphi\colon X \to X$ be a non-singular transformation such that φ is injective a.e. but not surjective a.e. Then C_φ is antinormal.*

Proof. Since φ is not surjective a.e., there exists an atom A_{n_0} such that $\mu(\varphi^{-1}(A_{n_0})) = 0$. Therefore,

$$C_\varphi K_{A_{n_0}} = K_{\varphi^{-1}(A_{n_0})} = 0.$$

Consequently C_φ is not injective.

Let $f = \sum_{n\in\mathbb{N}} f|A_n K_{A_n} \in L^2\mu$. Then

$$C_\varphi^* f = \sum_{n\in\mathbb{N}} f|A_n K_{\varphi(A_n)}.$$

Since φ is injective, $\varphi(A_m) = \varphi(A_n)$ implies $A_m = A_n$. Thus, $C_\varphi^* f = 0$ asserts $f = 0$. Hence, C_φ^* is injective. Therefore, $\operatorname{index}(C_\varphi) < 0$. Now let $f = \sum_{n\in\mathbb{N}} f|A_n K_{A_n} \in L^2(\mu)$. We get

$$(C_\varphi C_\varphi^* - \alpha I) f = (1 - \alpha) f \quad \text{for each} \quad \alpha \in \mathbb{C}. \tag{3.1}$$

From equation (3.1) it follows that $C_\varphi C_\varphi^* - \alpha I$ is invertible whenever $\alpha \neq 1$. Consequently $C_\varphi C_\varphi^* - \alpha I$ is Fredholm for each $\alpha \neq 1$. Thus, $\sqrt{\alpha} \notin \sigma_e(|C_\varphi^*|)$ for each $\alpha \neq 1$. Again, by above equation, $\dim\, \ker(C_\varphi C_\varphi^* - I)$ is infinite. Therefore, $1 \in \sigma_e(|C_\varphi^*|)$. Thus, $m_e(C_\varphi^*) = 1 = \|C_\varphi\|$. Hence, C_φ^* is antinormal by Theorem 2.6. Since adjoint of an antinormal operator is antinormal, therefore C_φ is antinormal.

The following result gives a necessary and sufficient condition for antinormality of C_φ if φ is surjective a.e. but not injective a.e.

Theorem 3.5. *Suppose φ is surjective a.e. but is not injective a.e. Then C_φ is antinormal if and only if the following conditions hold*:

(a) *for each* $0 \leq \alpha < \|C_\varphi\|^2$, $\dfrac{\mu\varphi^{-1}(A_n)}{\mu(A_n)} \neq \alpha$ *except for finitely many* $n \in \mathbb{N}$;

(b) $\|C_\varphi\|^2 = \dfrac{\mu\varphi^{-1}(A_n)}{\mu(A_n)}$ *for infinitely many* $n \in \mathbb{N}$.

Proof. Let $f \in L^2(\mu)$ be such that $C_\varphi(f) = 0$. Then $(f \circ \varphi)(A_n) = 0\ \forall n \in \mathbb{N}$. Now surjectivity of φ implies $f = 0$. Further, since φ is not injective a.e., there exist $m,\ n \in \mathbb{N}$ with $m \neq n$ such that $\varphi(A_m) = \varphi(A_n)$. Now put $f = K_{A_n}$ and $g = K_{A_m}$. It is easy to see that $f,\ g \in L^2(\mu)$ and $f \neq g$ but $C_\varphi^*(f) = C_\varphi^*(g)$. Therefore, C_φ^* is not injective. Let $f = \sum_{n\in\mathbb{N}} f|A_n K_{A_n} \in L^2(\mu)$ and α be a real number. Then

$$(C_\varphi^* C_\varphi - \alpha I)f = \sum_{n\in\mathbb{N}} \left(\frac{\mu\varphi^{-1}(A_n)}{\mu(A_n)} - \alpha\right) f|A_n K_{A_n}. \tag{3.2}$$

It follows by condition (a) and equation (3.2) that $C_\varphi^* C_\varphi - \alpha I$ is Fredholm for each $0 \leq \alpha < \|C_\varphi\|^2$. Therefore, $\sqrt{\alpha} \notin \sigma_e(|C_\varphi|)$ for each $0 \leq \alpha < \|C_\varphi\|^2$. Condition (b) together with equation (3.2) asserts that $C_\varphi^* C_\varphi - \alpha I$ is not Fredholm for $\alpha = \|C_\phi\|^2$. Hence, $\|C_\phi\| \in \sigma_e(|C_\varphi|)$ Thus, $m_e(C_\varphi) = \|C_\varphi\|$. Consequently C_φ is antinormal by Theorem 2.6. Conversely, if either of the conditions is not true, then we claim that C_φ is not antinormal. If condition (a) fails, then there exists an α_0 with $0 \leq \sqrt{\alpha_0} < \|C_\varphi\|^2$, such that $\sqrt{\alpha_0} \in \sigma_e(|C_\varphi|)$. Therefore, $m_e(C_\varphi) \leq \sqrt{\alpha_0} < \|C_\varphi\|$ and so C_φ is not antinormal. Now suppose condition (b) fails. Then, by equation (3.2), $\|C_\varphi\| \notin \sigma_e(|C_\varphi|)$. Therefore, $m_e(C_\varphi) < \|C_\varphi\|$. Thus, C_φ is not antinormal in either case.

Following theorem characterize antinormal composition operator when ϕ is neither injective nor surjective.

Theorem 3.6. *Suppose φ is neither injective nor surjective a.e.*

(i) *If* $\operatorname{index}(C_\varphi) \geq 0$, C_φ *is not antinormal.*

(ii) *If* $\operatorname{index}(C_\varphi) < 0$, C_φ *is antinormal if and only if following conditions hold*:

(a) *for each* $0 \leq \alpha < \|C_\varphi\|^2$, $\dfrac{\mu\varphi^{-1}(A_n)}{\mu(A_n)} \neq \alpha$ *except for finitely many* $n \in \mathbb{N}$;

(b) $\|C_\varphi\|^2 = \dfrac{\mu\varphi^{-1}(A_n)}{\mu(A_n)}$ *for infinitely many* $n \in \mathbb{N}$.

Proof. Suppose that φ is neither injective a.e. nor surjective a.e. We split the proof of (i) into two case.

Case I. If $\operatorname{index}(C_\varphi) = 0$, then it is not antinormal by Remark 2.3.

Case II. If $\operatorname{index}(C_\varphi) > 0$, then $\operatorname{index}(C_\varphi^*) < 0$. Therefore $\dim\ker C_\varphi^*$ is finite. But by [12] we have

$$\dim\ker C_\varphi^* = \sum_{n\in\mathbb{N}} (\alpha_n - 1),$$

where α_n denotes the number of atoms in $\varphi^{-1}(A_n)$ for each $n \in \mathbb{N}$. Consequently $\alpha_n = 1$ for all but finitely many $n \in \mathbb{N}$. Now by

$$(C_\varphi C_\varphi^* - I)f = \sum_{n\in\mathbb{N}} f|A_n K_{\varphi^{-1}(\varphi(A_n))} - \sum_{n\in\mathbb{N}} f|A_n K_{A_n}.$$

Hence, $K_{\varphi^{-1}(A_n)} \in \ker(C_\varphi C_\varphi^* - I)$ for all but finitely many $n \in \mathbb{N}$. Therefore, $\ker((C_\varphi C_\varphi^* - I))$ is infinite dimensional. Thus, $1 \in \sigma_e(|C_\varphi^*|)$. Therefore, $m_e(C_\varphi^*) \leq 1 < \|C_\varphi\|$. Hence, C_φ is antinormal.

(ii) If $\operatorname{index}(C_\varphi) < 0$, then result follows using arguments used in Theorem 3.5.

Now we present some examples to illustrate obtained results.

***Example* 3.1.** Let $X = [0, \infty)$ with atoms $A_n = [n-1, n]$ and $\mu(A_n) = 1$ for each $n \in \mathbb{N}$. Let $\varphi(x) = 2x$. Let E be a measurable set such that $\mu(E) = 0$. Then $\mu(\varphi^{-1}(E)) = \mu\left(\frac{1}{2}E\right) = 0$. Hence, φ is non singular. It is easy to see that φ is injective but not surjective since $\mu(\varphi^{-1}(A_n)) = 0$ for each $n \in \mathbb{N}$. Hence, by Theorem 3.4, the composition operator C_φ induced by φ is antinormal.

***Example* 3.2.** Let $X = N$, μ = counting measure and $A_n = \{n\}$ for each $n \in \mathbb{N}$. Define a function φ on $\mathbb{N}$ such that

$$\varphi(n) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n = 2, 3, \\ 3, & \text{if } n = 4, \\ \dfrac{n+3}{2}, & \text{if } \ n\ (\geq 5) \ \text{ is odd}, \\ \dfrac{n+2}{2}, & \text{if } \ n\ (\geq 6) \ \text{ is even}. \end{cases}$$

It is easy to see that φ is surjective but not injective. Also $\|C_\varphi\|^2 = \sup\{|\varphi^{-1}(n)| : n \in \mathbb{N}\} = 2$, where $|\varphi^{-1}(n)|$ denotes cardinality of the set $\varphi^{-1}(n)$ for each $n \in \mathbb{N}$. Further,

$$\frac{\mu\varphi^{-1}(A_n)}{\mu(A_n)} = \begin{cases} 1, & \text{if } \ n = 1, 3, \\ 2, & \text{otherwise}. \end{cases}$$

Thus, both the conditions of Theorem 3.5 are satisfied. Therefore, C_φ is antinormal.

***Example* 3.3.** Let $X = N$ and μ = counting measure. Define a function φ on $\mathbb{N}$ such that

$$\varphi(n) = \begin{cases} 1, & \text{if } \ n = 1, 2, \\ n+2, & \text{otherwise}. \end{cases}$$

It is easy to see that φ is neither surjective but not injective. $\operatorname{Index}(C_\varphi) = \dim\ker(C_\varphi) - \dim\ker(C_\varphi^*) = 3 - 1 = 2 > 0$. Therefore, C_φ is not antinormal by Theorem 3.6 (a).

***Example* 3.4.** Let $X = N$, μ = counting measure and $A_n = \{n\}$ for each $n \in \mathbb{N}$. Define a function φ on $\mathbb{N}$ such that

$$\varphi(n) = \begin{cases} 1, & \text{if} \quad n = 1, \\ 2, & \text{if} \quad n = 2, 3, \\ 4 + m - 1, & \text{if} \quad n = 2m + 2, \ 2m + 3, \quad \text{where} \quad m \in \mathbb{N}. \end{cases}$$

It is easy to see that φ is neither surjective but not injective. Since $\ker(C_\varphi)$ is one dimensional and $\ker(C_\varphi^*)$ is infinite dimensional, hence $\operatorname{index}(C_\varphi) < 0$. Also $\|C_\varphi\|^2 = 2$. Further,

$$\frac{\mu\varphi^{-1}(A_n)}{\mu(A_n)} = \begin{cases} 1, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad n = 3, \\ 2, & \text{otherwise}. \end{cases}$$

Thus, both the conditions posed in Theorem 3.6 (b) are satisfied. Therefore, C_φ is antinormal.

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