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ON ONE ESTIMATE OF DIVIDED DIFFERENCES AND ITS APPLICATIONS*

ПРО ОДНУ ОЦІНКУ ДЛЯ ПОДІЛЕНИХ РІЗНИЦЬ ТА ЇЇ ЗАСТОСУВАННЯ

We give an estimate of the general divided differences $[x_0,\ldots,x_m;f]$, where some points x_i are allowed to coalesce (in this case, f is assumed to be sufficiently smooth). This estimate is then applied to significantly strengthen the celebrated Whitney and Marchaud inequalities and their generalization to the Hermite interpolation.

For example, one of the numerous corollaries of this estimate is the fact that, given a function $f\in C^{(r)}(I)$ and a set $Z=\{z_j\}_{j=0}^{\mu}$ such that $z_{j+1}-z_j\ge\lambda|I|$ for all $0\le j\le\mu-1$, where $I:=[z_0,z_\mu]$, $|I|$ is the length of I, and λ is a positive number, the Hermite polynomial $\mathcal{L}(\cdot;f;Z)$ of degree $\le r\mu+\mu+r$ satisfying the equality $\mathcal{L}^{(j)}(z_\nu;f;Z)=f^{(j)}(z_\nu)$ for all $0\le\nu\le\mu$ and $0\le j\le r$ approximates f so that, for all $x\in I$,

$$|f(x)-\mathcal{L}(x;f;Z)|\le C\,(\mathrm{dist}\,(x,Z))^{r+1}\int\limits_{\mathrm{dist}\,(x,Z)}^{2|I|}\frac{\omega_{m-r}(f^{(r)},t,I)}{t^2}\,dt,$$

where $m:=(r+1)(\mu+1)$, $C=C(m,\lambda)$ and $\mathrm{dist}\,(x,Z):=\min_{0\le j\le\mu}|x-z_j|$.

Наведено оцінку узагальненої поділеної різниці $[x_0,\ldots,x_m;f]$, де деякі з точок x_i можуть збігатися (в цьому випадку f вважається досить гладкою). Цю оцінку потім застосовано для суттєвого посилення відомих нерівностей Уітні і Маршу та узагальнення їх для поліноміальної інтерполяції Ерміта.

Наприклад, одним із численних наслідків цієї оцінки є той факт, що для заданої функції $f\in C^{(r)}(I)$ та набору точок $Z=\{z_j\}_{j=0}^{\mu}$ таких, що $z_{j+1}-z_j\ge\lambda|I|$ для всіх $0\le j\le\mu-1$, де $I:=[z_0,z_\mu]$, $|I|$ — довжина I, λ — деяке додатне число, поліном Ерміта $\mathcal{L}(\cdot;f;Z)$ степеня $\le r\mu+\mu+r$, який задовольняє $\mathcal{L}^{(j)}(z_\nu;f;Z)=f^{(j)}(z_\nu)$ для $0\le\nu\le\mu$ і $0\le j\le r$, наближає f так, що для всіх $x\in I$

$$|f(x)-\mathcal{L}(x;f;Z)|\le C\,(\mathrm{dist}\,(x,Z))^{r+1}\int\limits_{\mathrm{dist}\,(x,Z)}^{2|I|}\frac{\omega_{m-r}(f^{(r)},t,I)}{t^2}\,dt,$$

де $m:=(r+1)(\mu+1)$, $C=C(m,\lambda)$ і $\mathrm{dist}\,(x,Z):=\min_{0\le j\le\mu}|x-z_j|$.

Introduction. V. K. Dzyadyk had a significant impact on the theory of extension of functions, and we start this note with recalling three of his most significant results (in our opinion) in this direction.

First, in 1956 (see [4]), he solved a problem posed by S. M. Nikolskii on extending a function $f\in\mathrm{Lip}_M(\alpha,p)$, $0<\alpha\le1$, $p\ge1$, on a finite interval $[a,b]$, to a function $F\in\mathrm{Lip}_{M_1}(\alpha,p)$ on the whole real line, i.e., $F|_{[a,b]}=f$.

Then, in 1958 (see [5] or [6, p. 171, 172]), he showed that if $f\in C[0,1]$ then this function may be extended to a function $F\in C[-1,1]$ with a controlled second modulus of smoothness on $[-1,1]$, i.e., $F|_{[0,1]}=f$, and the second moduli of smoothness of f and F satisfy $\omega_2(F,\delta;[-1,1])\le 5\omega_2(f,\delta;[0,1])$, $0<\delta\le1$. (This result was independently proved by Frey [9] the same year.)

In this note, we mostly deal with results related to Dzyadyk's third result which we will now describe.

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Given a function $f \in C[a,b]$ and $a \le x_0 < x_1 < x_2 \le b$, the second divided difference $[x_0, x_1, x_2; f]$ can be estimated as follows (see, e.g., [16, p. 176] and [8, p. 237]):

$$|[x_0, x_1, x_2; f]| \le \frac{c}{x_2 - x_0} \int\limits_h^{x_2 - x_0} \frac{\omega_2(f,t)}{t^2}\, dt, \qquad (1.1)$$

where $c = \mathrm{const} < 18,\ h := \min\{x_1 - x_0, x_2 - x_1\}$.

Now, let ω_2 be an arbitrary function of the second modulus of smoothness type, i.e., $\omega_2 \in C[0,\infty]$ is nondecreasing and such that $\omega_2(0) = 0$ and $t_1^{-2}\omega_2(t_1) \le 4t_2^{-2}\omega_2(t_2),\ 0 < t_2 < t_1$.

In 1983, Dzyadyk and Shevchuk [7] proved that if f, defined on an arbitrary set $E \subset \mathbb{R}$, satisfies (1.1) with $\omega_2(t)$ instead of $\omega_2(f,t)$ for each triple of points $x_0, x_1, x_2 \in E$ satisfying $x_0 < x_1 < x_2$, then f may be extended from E to a function $F \in C(\mathbb{R})$ such that $\omega_2(F, t; \mathbb{R}) \le c\omega_2(t)$. In other words, (1.1) with $\omega_2(t)$ instead of $\omega_2(f,t)$ is necessary and sufficient for a function f to be the trace, on the set $E \subset \mathbb{R}$, of a function $F \in C(\mathbb{R})$ satisfying $\omega_2(F, t; \mathbb{R}) \le c\omega_2(t)$. This result was independently proved by Brudnyi and Shvartsman [2] in 1982 (see also Jonsson [14] for $\omega_2(t) = t$).

V. K. Dzyadyk posed the question to describe such traces for functions of the kth modulus of smoothness type with $k > 2$. He conjectured that an analog of (1.1) must be a corollary of Whitney and Marchaud inequalities. In 1984, this conjecture was confirmed by Shevchuk in [19], and a corresponding (exact) analog of (1.1) for $k > 2$ was found (see (2.7) below with $r = 0$). Earlier, the case $\omega(t) = t^{k-1}$ was proved by Jonsson whose paper [14] was submitted in 1981, revised in 1983 and published in 1985.

So what happens when we have differentiable functions? In 1934, Whitney [23] described the traces of r times continuously differentiable functions $F\colon \mathbb{R} \mapsto \mathbb{R}$ on arbitrary closed sets $E \subset \mathbb{R}$: this trace consists of all functions $f\colon E \mapsto \mathbb{R}$ whose rth differences converge on E (see [24] for the definition). In 1975, de Boor [1] described the traces of functions $F\colon \mathbb{R} \mapsto \mathbb{R}$ with bounded rth derivative on arbitrary sets $E \subset \mathbb{R}$ of isolated points: this trace consists of all functions whose rth divided differences are uniformly bounded on E (in 1965, Subbotin [22] obtained exact constants in the case when sets E consist of equidistant points).

Finally, given an arbitrary set $E \subset \mathbb{R}$, the necessary and sufficient condition for a function f to be a trace (on E) of a function $F \in C^{(r)}(\mathbb{R})$ with a prescribed kth modulus of continuity of the rth derivative was obtained by Shevchuk in 1984 in [19] (see also Theorems 11.1 and 12.3 in [20], Theorems 3.2 and 4.3 in Chapter 4 of [8], and [21], where a linear extension operator was given).

In fact, this necessary and sufficient condition is an analog of (1.1) for the kth modulus of continuity of the rth derivative of f which is inequality (2.7) in Theorem 2.1 below. However, the original proof of Theorem 2.1 was distributed among several publications (see [10, 18, 19] as well as [20] and [8]), and there was an unfortunate misprint in the formulation of Theorem 6.4 in Section 3 of [8]: in (3.6.36), "k" was written instead of "m". Hence, the main purpose of this note is to properly formulate this theorem (Theorem 2.1), provide its complete self-contained proof and discuss several important corollaries/applications that have been inadvertently overlooked in the past.

2. Definitions, notations and the main result. For $f \in C[a,b]$ and any $k \in \mathbb{N}$, set

$$\Delta_u^k(f,x;[a,b]) := \begin{cases} \sum_{i=0}^k (-1)^i \binom{k}{i} f(x+(k/2-i)u), & x \pm (k/2)u \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

and denote by

$$\omega_k(f,t;[a,b]) := \sup_{0<u\le t} \|\Delta_u^k(f,\cdot;[a,b])\|_{C[a,b]} \qquad (2.1)$$

the kth modulus of smoothness of f on $[a,b]$.

Now, we recall the definition of Lagrange – Hermite divided differences (see, e.g., [3, p. 118]). Let $X = \{x_j\}_{j=0}^m$ be a collection of $m+1$ points with possible repetitions. For each j, the multiplicity m_j of x_j is the number of x_i such that $x_i = x_j$, and let l_j be the number of $x_i = x_j$ with $i \le j$. We say that a point x_j is a simple knot if its multiplicity is 1. Suppose that a real valued function f is defined at all points in X and, moreover, for each $x_j \in X$, $f^{(l_j-1)}(x_j)$ is defined as well (i.e., f has $m_j - 1$ derivatives at each point that has multiplicity m_j).

Denote

$$[x_0; f] := f(x_0),$$

the divided difference of f of order 0 at the point x_0.

Definition 2.1. *Let $m \in \mathbb{N}$. If $x_0 = \ldots = x_m$, then we denote*

$$[x_0,\ldots,x_m;f] = [\underbrace{x_0,\ldots,x_0}_{m+1};f] := \frac{f^{(m)}(x_0)}{m!}.$$

Otherwise, $x_0 \ne x_{j^*}$, for some number j^*, and we denote

$$[x_0,\ldots,x_m;f] := \frac{1}{x_{j^*}-x_0}\left([x_1,\ldots,x_m;f] - [x_0,\ldots,x_{j^*-1},x_{j^*+1},\ldots,x_m;f]\right),$$

the divided (*Lagrange – Hermite*) *difference of f of order m at the knots $X = \{x_j\}_{j=0}^m$.*

Note that $[x_0,\ldots,x_m;f]$ is symmetric in $x_0,\ldots,x_m$ (i.e., it does not depend on how the points from X are numbered), and recall that

$$L_m(x;f) := L_m(x;f;x_0,\ldots,x_m) := f(x_0) + \sum_{j=1}^m [x_0,\ldots,x_j;f](x-x_0)\ldots(x-x_{j-1}) \qquad (2.2)$$

is the (Hermite) polynomial of degree $\le m$ that satisfies

$$L_m^{(l_j-1)}(x_j;f) = f^{(l_j-1)}(x_j), \quad \text{for all} \quad 0 \le j \le m. \qquad (2.3)$$

Hence, in particular, if x_{j_*} is a simple knot, then we can write

$$[x_0,\ldots,x_m;f] := \frac{f(x_{j_*}) - L_{m-1}(x_{j_*};f;x_0,\ldots,x_{j_*-1},x_{j_*+1},\ldots,x_m)}{\prod_{j=0,j\ne j_*}^m (x_{j_*}-x_j)}. \qquad (2.4)$$

From now on, for convenience, we assume that all interpolation points are numbered from left to right, i.e., the set of interpolation points $X = \{x_j\}_{j=0}^m$ is such that $x_0 \le x_1 \le \ldots \le x_m$. We also assume that the maximum multiplicity of each point is $r+1$ with $r \in \mathbb{N}_0$, so that

$$x_j < x_{j+r+1}, \quad \text{for all} \quad 0 \le j \le m-r-1. \tag{2.5}$$

Also, let

$$\mathcal{Q}_{m,r} := \{(p,q) \mid 0 \le p,q \le m \text{ and } q-p \ge r+1\} =$$
$$= \{(p,q) \mid 0 \le p \le m-r-1 \text{ and } p+r+1 \le q \le m\}, \tag{2.6}$$

and note that $\mathcal{Q}_{m,r} = \varnothing$ if $m \le r$.

Now, for all $(p,q) \in \mathcal{Q}_{m,r}$, put

$$d(p,q) := d(p,q;X) := \min\{x_{q+1} - x_p, x_q - x_{p-1}\},$$

where $x_{-1} := x_0 - (x_m - x_0)$ and $x_{m+1} := x_m + (x_m - x_0)$. Note, in particular, that

$$d := d(X) := d(0,m;X) = 2(x_m - x_0).$$

Everywhere below, Φ is the set of nondecreasing functions $\varphi \in C[0,\infty]$ satisfying $\varphi(0) = 0$. We also denote

$$\Lambda_{p,q,r}(x_0,\ldots,x_m;\varphi) := \frac{\int_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\varphi(u)du}{\prod_{i=0}^{p-1}(x_q - x_i)\prod_{i=q+1}^{m}(x_i - x_p)}, \quad (p,q) \in \mathcal{Q}_{m,r},$$

and

$$\Lambda_r(x_0,\ldots,x_m;\varphi) := \max_{(p,q)\in\mathcal{Q}_{m,r}} \Lambda_{p,q,r}(x_0,\ldots,x_m;\varphi).$$

Here, we use the usual convention that $\prod_{i=0}^{-1} := 1$ and $\prod_{i=m+1}^{m} := 1$.

The following theorem is the main result of this paper.

Theorem 2.1. *Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$ be such that $m \ge r+1$, and suppose that a set $X = \{x_j\}_{j=0}^m$ is such that $x_0 \le x_1 \le \ldots \le x_m$ and* (2.5) *is satisfied. If $f \in C^{(r)}[x_0,x_m]$, then*

$$|[x_0,\ldots,x_m;f]| \le c\Lambda_r(x_0,\ldots,x_m;\omega_k), \tag{2.7}$$

where $k := m-r$ and $\omega_k(t) := \omega_k(f^{(r)},t;[x_0,x_m])$, and the constant c depends only on m.

3. Auxiliary lemmas. Throughout this section, we assume that $r \in \mathbb{N}_0$, $m \in \mathbb{N}$, $m \ge r+1$, the set $X = \{x_j\}_{j=0}^m$ is such that $x_0 \le x_1 \le \ldots \le x_m$ and (2.5) is satisfied, and that $(p,q) \in \mathcal{Q}_{m,r}$. For convenience, we also denote $k := m-r$.

We first show that Theorem 2.1 is valid in the case $m = r+1$ (i.e., $k=1$).

Lemma 3.1. *Theorem* 2.1 *holds if* $m = r+1$.

Proof. If $m = r+1$, then $\mathcal{Q}_{m,r} = \{(0,r+1)\}$, and so

$$\Lambda_r(x_0,\ldots,x_m;\varphi) = \Lambda_{0,r+1,r}(x_0,\ldots,x_m;\varphi) = \int_{d/2}^{d} u^{-2}\varphi(u)du.$$

Hence, since $x_0 \ne x_m$ by assumption (2.5), (2.7) follows from the identity

$$[x_0,\ldots,x_m;f] = \frac{[x_1,\ldots,x_{r+1};f] - [x_0,\ldots,x_r;f]}{x_m - x_0} = \frac{f^{(r)}(\theta_1) - f^{(r)}(\theta_2)}{r!d/2},$$

where $\theta_1 \in (x_1, x_{r+1})$ and $\theta_2 \in (x_0, x_r)$, and the estimate

$$\frac{\left|f^{(r)}(\theta_1) - f^{(r)}(\theta_2)\right|}{d} \le \frac{\omega_1(d/2)}{d} \le \int\limits_{d/2}^{d} \frac{\omega_1(u)}{u^2}\,dt = \Lambda_r(x_0, \ldots, x_m; \omega_1).$$

Lemma 3.1 is proved.

For $k > 2$, we need the following lemma.

Lemma 3.2. *Let $(p, q) \in \mathcal{Q}_{m,r}$ be such that $q - p + 2 \le m$. If $\varphi \in \Phi$ and $\omega \in \Phi$ are such that*

$$\varphi(t) \le t^{k-1} \int\limits_{t}^{d} u^{-k}\omega(u)du, \quad t \in (0, d/2], \tag{3.1}$$

then

$$\Lambda_{p,q,r}(x_0, \ldots, x_m; \varphi) \le 2^{k^2}\Lambda_r(x_0, \ldots, x_m; \omega). \tag{3.2}$$

Proof. Let $(p, q) \in \mathcal{Q}_{m,r}$ such that $q - p + 2 \le m$ be fixed, and consider the collection $\{(p_\nu, q_\nu)\}_{\nu=0}^{m-q+p}$ which we define as follows. Let $(p_0, q_0) := (p, q)$ and, for $\nu \ge 1$,

$$(p_\nu, q_\nu) := \begin{cases} (p_{\nu-1} - 1, q_{\nu-1}), & \text{if} \quad x_{q_{\nu-1}} - x_{p_{\nu-1}-1} \le x_{q_{\nu-1}+1} - x_{p_{\nu-1}}, \\ (p_{\nu-1}, q_{\nu-1} + 1), & \text{otherwise.} \end{cases}$$

It is clear that $q_\nu - p_\nu = q_{\nu-1} - p_{\nu-1} + 1$, and so

$$q_\nu - p_\nu = q - p + \nu, \tag{3.3}$$

and one can easily check (for example, by induction) that, for all $1 \le \nu \le m - q + p$,

$$0 \le p_\nu \le p_{\nu-1} < q_{\nu-1} \le q_\nu \le m.$$

Hence, in particular,

$$(p_{m-q+p}, q_{m-q+p}) = (0, m).$$

In the rest of this proof, we use the notation

$$d_\nu := d(p_\nu, q_\nu), \quad 0 \le \nu \le m - q + p.$$

Also, observe that

$$d_\nu \ge d_{\nu-1} = x_{q_\nu} - x_{p_\nu}, \quad 1 \le \nu \le m + q - p,$$

and

$$d_{m-q+p-1} = x_m - x_0 = d/2.$$

We now show that, for all $1 \le \nu \le m - q + p$,

$$\frac{d_{\nu-1}}{\prod_{i=0}^{p_{\nu-1}-1}(x_{q_{\nu-1}} - x_i)\prod_{i=q_{\nu-1}+1}^{m}(x_i - x_{p_{\nu-1}})} \le$$

$$\le \frac{2^k}{\prod_{i=0}^{p_\nu-1}(x_{q_\nu} - x_i)\prod_{i=q_\nu+1}^{m}(x_i - x_{p_\nu})}. \tag{3.4}$$

Indeed, if $x_{q_{\nu-1}} - x_{p_{\nu-1}-1} \le x_{q_{\nu-1}+1} - x_{p_{\nu-1}}$, then $(p_\nu, q_\nu) = (p_{\nu-1} - 1, q_{\nu-1})$, $d_{\nu-1} = x_{q_{\nu-1}} -$

$-\, x_{p_{\nu-1}-1}$ and, for $q_{\nu-1}+1 \le j \le m$,

$$x_j - x_{p_\nu} = (x_j - x_{q_{\nu-1}}) + (x_{q_{\nu-1}} - x_{p_{\nu-1}-1}) \le$$
$$\le (x_j - x_{p_{\nu-1}}) + (x_{q_{\nu-1}+1} - x_{p_{\nu-1}}) \le 2(x_j - x_{p_{\nu-1}}),$$

whence

$$\prod_{i=q_{\nu-1}+1}^{m} (x_i - x_{p_{\nu-1}}) \ge 2^{q_{\nu-1}-m} \prod_{i=q_\nu+1}^{m} (x_i - x_{p_\nu}),$$

that yields (3.4) because $m - q_{\nu-1} \le m - q \le k$.

Similarly, if $x_{q_{\nu-1}} - x_{p_{\nu-1}-1} > x_{q_{\nu-1}+1} - x_{p_{\nu-1}}$, then $(p_\nu, q_\nu) = (p_{\nu-1}, q_{\nu-1}+1)$, $d_{\nu-1} = = x_{q_{\nu-1}+1} - x_{p_{\nu-1}}$, and, for $0 \le j \le p_{\nu-1}-1$,

$$x_{q_\nu} - x_j = (x_{q_{\nu-1}+1} - x_{p_{\nu-1}}) + (x_{p_{\nu-1}} - x_j) <$$
$$< (x_{q_{\nu-1}} - x_{p_{\nu-1}-1}) + (x_{q_{\nu-1}} - x_j) \le 2(x_{q_{\nu-1}} - x_j),$$

and whence

$$\prod_{i=0}^{p_{\nu-1}-1} (x_{q_{\nu-1}} - x_i) \ge 2^{-p_{\nu-1}} \prod_{i=0}^{p_\nu-1} (x_{q_\nu} - x_i),$$

that also yields (3.4) because $p_{\nu-1} \le p < k$.

Inequality (3.4) implies that, for all $1 \le \nu \le m - q + p$,

$$\frac{\prod_{i=0}^{\nu-1} d_i}{\prod_{i=0}^{p-1}(x_q - x_i)\prod_{i=q+1}^{m}(x_i - x_p)} \le \frac{2^{k\nu}}{\prod_{i=0}^{p_\nu-1}(x_{q_\nu} - x_i)\prod_{i=q_\nu+1}^{m}(x_i - x_{p_\nu})}. \tag{3.5}$$

It is clear that $d(p,q) \le x_m - x_0 = d/2$, and so condition (3.1) implies that

$$\int_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\varphi(u)du \le \int_{x_q-x_p}^{d(p,q)} u^{p+m-q-2}\left(\int_u^d v^{-k}\omega(v)dv\right)du.$$

Using integration by parts we write

$$(m-q+p-1)\int_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\varphi(u)du - \int_{x_q-x_p}^{d(p,q)} u^{p+r-q-1}\omega(u)du \le$$
$$\le d^{m-q+p-1}(p,q)\int_{d(p,q)}^{d} \frac{\omega(u)}{u^k}\,du = d^{m-q+p-1}(p,q)\sum_{\nu=1}^{m-q+p}\int_{d_{\nu-1}}^{d_\nu}\frac{\omega(u)}{u^k}\,du \le$$
$$\le 2\sum_{\nu=1}^{m-q+p}\prod_{i=0}^{\nu-1} d_i \int_{d_{\nu-1}}^{d_\nu} u^{p+r-q-1-\nu}\omega(u)du.$$

The last estimate is obvious for $1 \le \nu \le m-q+p-1$ and, for $\mu = m-q+p$, it follows from

$$d_0^{m-q+p-1} d_{m-q-p} \le 2 \prod_{i=0}^{m-q+p-1} d_i$$

which is valid because

$$d_0^{m-q+p-1} \le \prod_{i=0}^{m-q+p-2} d_i \quad \text{and} \quad d_{m-q-p} = d(0,m) = d = 2d_{m-q+p-1}.$$

Finally, taking into account (3.3), (3.5) and recalling that $d_{\nu-1} = x_{q_\nu} - x_{p_\nu}$, $1 \le \nu \le m-q+p$, we obtain

$$(m-q+p-1)\Lambda_{p,q,r}(x_0,\ldots,x_m;\varphi) \le$$

$$\le \Lambda_{p,q,r}(x_0,\ldots,x_m;\omega) + 2\sum_{\nu=1}^{m-q+p} 2^{k\nu}\Lambda_{p_\nu,q_\nu,r}(x_0,\ldots,x_m;\omega)$$

that implies (3.2).

Lemma 3.2 is proved.

Lemma 3.3. *If $k = m-r \ge 2$ and $\varphi \in \Phi$ and $\omega \in \Phi$ are such that*

$$\varphi(t) \le t^{k-1} \int_t^d u^{-k}\omega(u)du, \quad t \in (0, d/2], \tag{3.6}$$

and $\varphi(t) \le \omega(t)$, $t \in [d/2, d]$, then

$$\Lambda_r(x_0,\ldots,x_{m-1};\varphi) \le c(x_m - x_0)\Lambda_r(x_0,\ldots,x_m;\omega) \tag{3.7}$$

and

$$\Lambda_r(x_1,\ldots,x_m;\varphi) \le c(x_m - x_0)\Lambda_r(x_0,\ldots,x_m;\omega), \tag{3.8}$$

where constants c depend only on k.

Proof. We first note that (3.8) is a consequence of (3.7). Indeed, given $X = \{x_i\}_{i=0}^m$, define the set $Y = \{y_i\}_{i=0}^m$ by letting $y_i := -x_{m-i}$, $0 \le i \le m$. Then $y_0 \le y_1 \le \ldots \le y_m$, $y_m - y_0 = x_m - x_0$ (and so, in particular, $d(Y) = d(X) = d$),

$$d(p,q;Y) = \min\{y_{q+1} - y_p, y_q - y_{p-1}\} = \min\{x_{m-p} - x_{m-q-1}, x_{m-p+1} - x_{m-q}\} =$$

$$= d(m-q, m-p; X) = d(m-q, m-p),$$

and it is not difficult to check that, for any $\psi \in \Phi$,

$$\Lambda_{p,q,r}(y_0,\ldots,y_m;\psi) = \Lambda_{m-q,m-p,r}(x_0,\ldots,x_m;\psi)$$

and

$$\Lambda_{p,q,r}(y_0,\ldots,y_{m-1};\psi) = \Lambda_{m-q-1,m-p-1,r}(x_1,\ldots,x_m;\psi).$$

Hence, using the fact that $(p,q) \in \mathcal{Q}_{\mu,r}$ iff $(\mu-q,\mu-p) \in \mathcal{Q}_{\mu,r}$, $\mu = m-1, m$, we have

$$\Lambda_r(x_0,\ldots,x_m;\omega) = \max_{(p,q)\in\mathcal{Q}_{m,r}} \Lambda_{p,q,r}(x_0,\ldots,x_m;\omega) =$$

$$= \max_{(m-q,m-p)\in\mathcal{Q}_{m,r}} \Lambda_{m-q,m-p,r}(y_0,\ldots,y_m;\omega) = \Lambda_r(y_0,\ldots,y_m;\omega)$$

and

$$\Lambda_r(x_1,\ldots,x_m;\varphi) = \max_{(p,q)\in\mathcal{Q}_{m-1,r}} \Lambda_{p,q,r}(x_1,\ldots,x_m;\varphi) =$$

$$= \max_{(m-q-1,m-p-1)\in\mathcal{Q}_{m-1,r}} \Lambda_{m-q-1,m-p-1,r}(y_0,\ldots,y_{m-1};\varphi) = \Lambda_r(y_0,\ldots,y_{m-1};\varphi),$$

and so (3.8) follows from (3.7) applied to the set Y.

We are now ready to prove (3.7). Let $(p^*,q^*) \in \mathcal{Q}_{m-1,r}$ be such that

$$\Lambda^* := \Lambda_{p^*,q^*,r}(x_0,\ldots,x_{m-1};\varphi) = \Lambda_r(x_0,\ldots,x_{m-1};\varphi),$$

and denote, for convenience, $X_m := \{x_0,\ldots,x_m\}$ and $X_{m-1} := \{x_0,\ldots,x_{m-1}\}$.

We consider four cases.

Case I: $(p^*,q^*) = (0,m-1)$.

We put $h := x_{m-1} - x_0$ and note that $\Lambda^* = \displaystyle\int_h^{2h} u^{-k}\varphi(u)du$.

If $h \le d/4$, then

$$2^{1-k}\Lambda^* \le (2h)^{1-k}\varphi(2h) \le \int\limits_{2h}^{d} u^{-k}\omega(u)du \le \int\limits_{h}^{d/2} u^{-k}\omega(u)du + \int\limits_{d/2}^{d} u^{-k}\omega(u)du \le$$

$$\le \int\limits_{h}^{d/2} u^{-k}\omega(u)du + d\int\limits_{d/2}^{d} u^{-k-1}\omega(u)du =$$

$$= (x_m - x_0)\left(\Lambda_{0,m-1,r}(x_0,\ldots,x_m;\omega) + 2\Lambda_{0,m,r}(x_0,\ldots,x_m;\omega)\right) \le$$

$$\le 3(x_m - x_0)\Lambda_r(x_0,\ldots,x_m;\omega).$$

If $h > d/4$, then

$$\Lambda^* = \int\limits_{h}^{d/2} u^{-k}\varphi(u)du + \int\limits_{d/2}^{2h} u^{-k}\varphi(u)du \le (4/d)^{k-1}\varphi(d/2) + \int\limits_{d/2}^{2h} u^{-k}\varphi(u)du <$$

$$< 4^k\int\limits_{d/2}^{d} u^{-k}\varphi(u)du \le 4^k\int\limits_{d/2}^{d} u^{-k}\omega(u)du \le 4^k d\int\limits_{d/2}^{d} u^{-k-1}\omega(u)du =$$

$$= 2\cdot 4^k(x_m - x_0)\Lambda_{0,m,r}(x_0,\ldots,x_m;\omega) \le 2\cdot 4^k(x_m - x_0)\Lambda_r(x_0,\ldots,x_m;\omega).$$

Case II: either (i) $q^* \neq m-1$, or (ii) $q^* = m-1$, $p^* > 0$, and $x_m - x_{p^*} > x_{m-1} - x_{p^*-1}$.

In this case, $d\,(p^*, q^*; X_{m-1}) = d\,(p^*, q^*; X_m) = x_{m-1} - x_{p^*-1}$, and so

$$\Lambda^* = (x_m - x_{p^*})\,\Lambda_{p^*,q^*,r}(x_0,\ldots,x_m;\varphi) \le (x_m - x_0)\Lambda_{p^*,q^*,r}(x_0,\ldots,x_m;\varphi).$$

Since $q^* - p^* + 2 \le m$, we may apply Lemma 3.2 and obtain (3.7).

Case III: $q^* = m-1$, $p^* \ge 2$ and $x_m - x_{p^*} \le x_{m-1} - x_{p^*-1}$.

In this case, $d\,(p^*, q^*; X_{m-1}) = x_{m-1} - x_{p^*-1}$ and $d\,(p^*, q^*; X_m) = x_m - x_{p^*}$. Hence, taking into account that, for $0 \le i \le p^* - 1$,

$$x_m - x_i = x_m - x_{p^*} + x_{p^*} - x_i \le x_{m-1} - x_{p^*-1} + x_{p^*} - x_i \le 2(x_{m-1} - x_i),$$

we get

$$\Lambda_{p^*,m-1,r}(x_0,\ldots,x_{m-1};\varphi) - (x_m - x_{p^*})\,\Lambda_{p^*,m-1,r}(x_0,\ldots,x_m;\varphi) =$$

$$= \prod_{i=0}^{p^*-1}(x_{m-1} - x_i)^{-1} \int\limits_{x_m - x_{p^*}}^{x_{m-1} - x_{p^*-1}} u^{p^*+r-m}\varphi(u)du \le$$

$$\le 2^{p^*} \prod_{i=0}^{p^*-1}(x_m - x_i)^{-1}(x_m - x_{p^*-1}) \int\limits_{x_m - x_{p^*}}^{x_m - x_{p^*-1}} u^{p^*+r-m-1}\varphi(u)du =$$

$$= 2^{p^*}\,(x_m - x_{p^*-1})\,\Lambda_{p^*,m,r}(x_0,\ldots,x_m;\varphi).$$

Since $m - p^* + 2 \le m$, we may apply Lemma 3.2 to obtain (3.7).

Case IV: $(p^*, q^*) = (1, m-1)$ and $x_m - x_1 \le x_{m-1} - x_0$.

In this case, we have

$$\Lambda^* = \frac{1}{x_{m-1} - x_0} \int\limits_{x_{m-1} - x_1}^{x_{m-1} - x_0} u^{1-k}\varphi(u)du \le$$

$$\le \frac{1}{x_{m-1} - x_0} \int\limits_{x_{m-1} - x_1}^{x_{m-1} - x_0} \left(\int\limits_u^d v^{-k}\omega(v)dv \right) du \le$$

$$\le \int\limits_{x_{m-1} - x_0}^{d} u^{-k}\omega(u)\,du + \frac{1}{x_{m-1} - x_0} \int\limits_{x_{m-1} - x_1}^{x_{m-1} - x_0} u^{1-k}\omega(u)du =: \mathcal{A}_1 + \mathcal{A}_2.$$

Now,

$$\mathcal{A}_1 = \int\limits_{x_{m-1} - x_0}^{d/2} u^{-k}\omega(u)du + \int\limits_{d/2}^{d} u^{-k}\omega(u)du \le$$

$$\leq \int\limits_{x_{m-1}-x_0}^{d/2} u^{-k}\omega(u)du + d\int\limits_{d/2}^{d} u^{-k-1}\omega(u)du =$$

$$= (x_m - x_0)\left(\Lambda_{0,m-1,r}(x_0,\ldots,x_m;\omega) + 2\Lambda_{0,m,r}(x_0,\ldots,x_m;\omega)\right) \leq$$

$$\leq 3(x_m - x_0)\Lambda_r(x_0,\ldots,x_m;\omega)$$

and

$$\mathcal{A}_2 = \frac{1}{x_{m-1}-x_0}\int\limits_{x_{m-1}-x_1}^{x_m-x_1} u^{1-k}\omega(u)du + \frac{1}{x_{m-1}-x_0}\int\limits_{x_m-x_1}^{x_{m-1}-x_0} u^{1-k}\omega(u)du \leq$$

$$\leq (x_m - x_1)\Lambda_{1,m-1,r}(x_0,\ldots,x_m;\omega) + \int\limits_{x_m-x_1}^{x_{m-1}-x_0} u^{-k}\omega(u)du \leq$$

$$\leq (x_m - x_0)\Lambda_{1,m-1,r}(x_0,\ldots,x_m;\omega) + \int\limits_{x_m-x_1}^{x_m-x_0} u^{-k}\omega(u)du =$$

$$= (x_m - x_0)\left(\Lambda_{1,m-1,r}(x_0,\ldots,x_m;\omega) + \Lambda_{1,m,r}(x_0,\ldots,x_m;\omega)\right) \leq$$

$$\leq 2(x_m - x_0)\Lambda_r(x_0,\ldots,x_m;\omega).$$

Lemma 3.3 is proved.

4. Proof of Theorem 2.1. We use induction on $k = m - r$. The base case $k = 1$ is addressed in Lemma 3.1. Suppose now that $k \geq 2$ is given, assume that Theorem 2.1 holds for $k-1$ and prove it for k.

Denote by P_{k-1} the polynomial of best uniform approximation of $f^{(r)}$ on $[x_0, x_m]$ of degree at most $k-1$, and let g be such that

$$g^{(r)} := f^{(r)} - P_{k-1}.$$

Then

$$\omega_k(g^{(r)}, t; [x_0, x_m]) = \omega_k(f^{(r)}, t; [x_0, x_m]) =: \omega_k^f(t),$$

and Whitney's inequality yields

$$\left\|g^{(r)}\right\|_{[x_0,x_m]} \leq c\omega_k\left(f^{(r)}, x_m - x_0; [x_0, x_m]\right) = c\omega_k^f(x_m - x_0). \tag{4.1}$$

Hence, the well known Marchaud inequality:

if $F \in C[a,b]$ and $1 \leq \ell < k$, then, for all $0 < t \leq b - a$,

$$\omega_\ell(F, t; [a,b]) \leq c(k)t^\ell\left(\int\limits_t^{b-a}\frac{\omega_k(F,u;[a,b])}{u^{\ell+1}}\,du + \frac{\|F\|_{[a,b]}}{(b-a)^\ell}\right),$$

implies, for $0 < t \leq x_m - x_0$,

$$\omega^g_{k-1}(t) := \omega_{k-1}\left(g^{(r)}, t; [x_0, x_m]\right) \le$$

$$\le ct^{k-1}\left(\int\limits_t^{x_m-x_0} \frac{\omega^f_k(u)}{u^k}\,du + \frac{\omega^f_k(x_m-x_0)}{(x_m-x_0)^{k-1}}\right) \le$$

$$\le ct^{k-1}\int\limits_t^{2(x_m-x_0)} \frac{\omega^f_k(u)}{u^k}\,du. \tag{4.2}$$

We also note that (4.1) implies, in particular, that, for all $t \in [x_m - x_0, 2(x_m - x_0)]$,

$$\omega^g_{k-1}(t) \le c\|g^{(r)}\|_{[x_0,x_m]} \le c\omega^f_k(x_m - x_0) \le c\omega^f_k(t). \tag{4.3}$$

We now represent the divided difference in the form

$$(x_m - x_0)[x_0, \ldots, x_m; f] = (x_m - x_0)[x_0, \ldots, x_m; g] =$$

$$= [x_1, \ldots, x_m; g] - [x_0, \ldots, x_{m-1}; g] = [y_0, \ldots, y_{m-1}; g] - [x_0, \ldots, x_{m-1}; g],$$

where $y_j := x_{j+1},\ 0 \le j \le m-1$. By the induction hypothesis,

$$|[x_0, \ldots, x_{m-1}; g]| \le c\Lambda_r(x_0, \ldots, x_{m-1}; \omega^g_{k-1})$$

and

$$|[y_0, \ldots, y_{m-1}; g]| \le c\Lambda_r\left(y_0, \ldots, y_{m-1}; \omega^g_{k-1}\right).$$

Now, taking into account (4.2), (4.3) and homogeneity of $\Lambda_r(z_0, \ldots, z_m; \psi)$ with respect to ψ, Lemma 3.3 with $\varphi := \omega^g_{k-1}$ and $\omega := K\omega^f_k$, where K is the maximum of constants c in (4.2) and (4.3), implies that

$$\Lambda_r\left(x_0, \ldots, x_{m-1}; \omega^g_{k-1}\right) \le c(x_m - x_0)\Lambda_r\left(x_0, \ldots, x_m; \omega^f_k\right)$$

and

$$\Lambda_r\left(y_0, \ldots, y_{m-1}; \omega^g_{k-1}\right) = \Lambda_r\left(x_1, \ldots, x_m; \omega^g_{k-1}\right) \le c(x_m - x_0)\Lambda_r\left(x_0, \ldots, x_m; \omega^f_k\right),$$

which yields (2.7).

Theorem 2.1 is proved.

5. Applications. Throughout this section, the set $X = \{x_j\}_{j=0}^{m-1}$ is assumed to be such that $x_0 \le x_1 \le \ldots \le x_{m-1}$ (unless stated otherwise), and denote $I := [x_0, x_{m-1}]$ and $|I| = x_{m-1} - x_0$. Also, all constants written in the form $C(\mu_1, \mu_2, \ldots)$ may depend only on parameters $\mu_1, \mu_2, \ldots$ and not on anything else.

We first recall that the classical Whitney interpolation inequality can be written in the following form.

Theorem 5.1 (Whitney inequality, [25]). *Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$ be such that $m \ge \max\{r + 1, 2\}$, and suppose that a set $X = \{x_j\}_{j=0}^{m-1}$ is such that*

$$x_{j+1} - x_j \ge \lambda|I|, \quad \textit{for all} \quad 0 \le j \le m-2, \tag{5.1}$$

where $0 < \lambda \le 1$. *If* $f \in C^{(r)}(I)$, *then*

$$|f(x) - L_{m-1}(x; f; x_0, \ldots, x_{m-1})| \le C(m, \lambda)|I|^r \omega_{m-r}(f^{(r)}, |I|, I), \quad x \in I,$$

where $L_{m-1}(\cdot; f; x_0, \ldots, x_{m-1})$ *is the (Lagrange) polynomial of degree* $\le m - 1$ *interpolating* f *at the points in* X.

We emphasize that condition (5.1) implies that the points in the set X in the above theorem are assumed to be sufficiently separated from one another. A natural question is what happens if condition (5.1) is not satisfied and, moreover, if some of the points in X are allowed to coalesce. In that case, $L_{m-1}(\cdot; f; x_0, \ldots, x_{m-1})$ is the Hermite polynomial whose derivatives interpolate corresponding derivatives of f at points that have multiplicities more than 1, and Theorem 5.1 provides no information on its error of approximation of f.

It turns out that one can use Theorem 2.1 to provide an answer to this question and significantly strengthen Theorem 5.1. As far as we know the formulation of the following theorem (which is itself a corollary of a more general Theorem 5.3 below) is new and has not appeared anywhere in the literature.

Theorem 5.2. *Let* $r \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$ *be such that* $m \ge r + 2$, *and suppose that a set* $X = \{x_j\}_{j=0}^{m-1}$ *is such that*

$$x_{j+r+1} - x_j \ge \lambda|I|, \quad \textit{for all} \ \ 0 \le j \le m - r - 2, \tag{5.2}$$

where $0 < \lambda \le 1$. *If* $f \in C^{(r)}(I)$, *then*

$$\big|f(x) - L_{m-1}(x; f; x_0, \ldots, x_{m-1})\big| \le C(m, \lambda)|I|^r \omega_{m-r}(f^{(r)}, |I|, I), \quad x \in I,$$

where $L_{m-1}(\cdot; f; x_0, \ldots, x_{m-1})$ *is the Hermite polynomial defined in* (2.2) *and* (2.3).

Theorem 5.2 is an immediate corollary of the following more general theorem. Before we state it, we need to introduce the following notation. Given $X = \{x_j\}_{j=0}^{m-1}$ with $x_0 \le x_1 \le \ldots \le x_{m-1}$ and $x \in [x_0, x_{m-1}]$, we renumber all points x_j's so that their distance from x is nondecreasing. In other words, let $\sigma = (\sigma_0, \ldots, \sigma_{m-1})$ be a permutation of $(0, \ldots, m - 1)$ such that

$$|x - x_{\sigma_{\nu-1}}| \le |x - x_{\sigma_\nu}|, \quad \text{for all} \ \ 1 \le \nu \le m - 1. \tag{5.3}$$

Note that this permutation σ depends on x and is not unique if there are at least two points from X which are equidistant from x. Denote also

$$\mathcal{D}_r(x, X) := \prod_{\nu=0}^{r} |x - x_{\sigma_\nu}|, \quad 0 \le r \le m - 1. \tag{5.4}$$

Theorem 5.3. *Let* $r \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$ *be such that* $m \ge r + 2$, *and suppose that a set* $X = \{x_j\}_{j=0}^{m-1}$ *is such that*

$$x_{j+r+1} - x_j \ge \lambda|I|, \quad \textit{for all} \ \ 0 \le j \le m - r - 2, \tag{5.5}$$

where $0 < \lambda \le 1$. *If* $f \in C^{(r)}(I)$, *then, for each* $x \in I$,

$$\big|f(x) - L_{m-1}(x; f; x_0, \ldots, x_{m-1})\big| \le$$

$$\leq C(m,\lambda)\mathcal{D}_r(x,X)\int\limits_{|x-x_{\sigma_r}|}^{2|I|}\frac{\omega_{m-r}(f^{(r)},t,I)}{t^2}\,dt, \tag{5.6}$$

where $\mathcal{D}_r(x,X)$ *is defined in* (5.4)*, and* $L_{m-1}(\cdot;f;x_0,\ldots,x_{m-1})$ *is the Hermite polynomial defined in* (2.2) *and* (2.3)*.*

Before proving Theorem 5.3 we state another corollary. First, if $k\in\mathbb{N}$ and $\mathrm{w}(t):=\omega_k(f^{(r)},t;I)$, then $t_2^{-k}\mathrm{w}(t_2)\leq 2^k t_1^{-k}\mathrm{w}(t_1)$, for $0<t_1<t_2$. Hence, denoting $\lambda_x:=|I|\sqrt[k]{|x-x_{\sigma_r}|/|I|}$ and noting that $|x-x_{\sigma_r}|\leq\lambda_x\leq|I|$, we have, for $k\geq 2$,

$$\int\limits_{|x-x_{\sigma_r}|}^{2|I|}\frac{\mathrm{w}(t)}{t^2}\,dt=\left(\int\limits_{|x-x_{\sigma_r}|}^{\lambda_x}+\int\limits_{\lambda_x}^{2|I|}\right)\frac{\mathrm{w}(t)}{t^2}\,dt\leq$$

$$\leq\mathrm{w}(\lambda_x)\int\limits_{|x-x_{\sigma_r}|}^{\infty}t^{-2}\,dt+2^k\lambda_x^{-k}\mathrm{w}(\lambda_x)\int\limits_0^{2|I|}t^{k-2}\,dt=$$

$$=\frac{\mathrm{w}(\lambda_x)}{|x-x_{\sigma_r}|}\left(1+\frac{2^{2k-1}}{k-1}\right).$$

Therefore, we immediately get the following consequence of Theorem 5.3.

***Corollary* 5.1.** *Let* $r\in\mathbb{N}_0$ *and* $m\in\mathbb{N}$ *be such that* $m\geq r+2$*, and suppose that a set* $X=\{x_j\}_{j=0}^{m-1}$ *is such that condition* (5.5) *is satisfied.*

If $f\in C^{(r)}(I)$*, then, for each* $x\in I$*,*

$$\bigl|f(x)-L_{m-1}(x;f;x_0,\ldots,x_{m-1})\bigr|\leq C(m,\lambda)\mathcal{D}_{r-1}(x,X)\omega_{m-r}(f^{(r)},\lambda_x,I)\leq$$

$$\leq C(m,\lambda)\mathcal{D}_{r-1}(x,X)\omega_{m-r}(f^{(r)},|I|,I), \tag{5.7}$$

where $\lambda_x:=|I|\bigl(|x-x_{\sigma_r}|/|I|\bigr)^{1/(m-r)}$*.*

We are now ready to prove Theorem 5.3.

***Proof of Theorem* 5.3.** We note that all constants C below may depend only on m and λ and are different even if they appear in the same line. It is clear that we can assume that x is different from all x_j's. So we let $1\leq i\leq m-1$ and $x\in(x_{i-1},x_i)$ be fixed, and denote

$$y_j:=\begin{cases}x_j, & \text{if}\quad 0\leq j\leq i-1,\\ x, & \text{if}\quad j=i,\\ x_{j-1}, & \text{if}\quad i+1\leq j\leq m,\end{cases}$$

$Y:=\{y_j\}_{j=0}^m$, $d(Y):=2(y_m-y_0)=2(x_{m-1}-x_0)=2|I|$, $k:=m-r$, and $\omega_k(t):=$ $:=\omega_k\left(f^{(r)},t,[y_0,y_m]\right)=\omega_k\left(f^{(r)},t,I\right)$.

Condition (5.5) implies that $y_j<y_{j+r+1}$, for all $0\leq j\leq m-r-1$, and so we can use Theorem 2.1 to estimate $|[y_0,\ldots,y_m;f]|$. Now, identity (2.4) with $j_*:=i$ that yields $y_{j_*}=x$ implies

$$|f(x)-L_{m-1}(x;f;x_0,\ldots,x_{m-1})|=$$

$$= |f(x) - L_{m-1}(x; f; y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m)| =$$

$$= |[y_0, \ldots, y_m; f]| \prod_{j=0, j\neq i}^{m} |x - y_j| \leq$$

$$\leq c\Lambda_r(y_0, \ldots, y_m; \omega_k) \prod_{j=0}^{m-1} |x - x_j| \leq c\mathcal{D}_r(x, X)|I|^{k-1}\Lambda_r(y_0, \ldots, y_m; \omega_k). \tag{5.8}$$

We also note that it is possible to show that $\prod_{j=0}^{m-1} |x - x_j| \geq (\lambda/2)^{k-1}\mathcal{D}_r(x, X)|I|^{k-1}$, and so the above estimate cannot be improved.

In order to estimate Λ_r, we suppose that $(p, q) \in \mathcal{Q}_{m,r}$ and estimate $\Lambda_{p,q,r}$. Since $q - p \geq r + 1$, we have

$$y_q - y_i \geq y_q - y_{p-1} \geq y_{p+r+1} - y_{p-1} \geq \lambda|I|, \quad \text{for} \quad 0 \leq i \leq p - 1,$$

and

$$y_i - y_p \geq y_{q+1} - y_p \geq y_{p+r+2} - y_p \geq \lambda|I|, \quad \text{for} \quad q + 1 \leq i \leq m.$$

Hence,

$$\Lambda_{p,q,r}(y_0, \ldots, y_m; \omega_k) \leq C|I|^{q-m-p} \int\limits_{y_q - y_p}^{2|I|} u^{p+r-q-1}\omega_k(u)du. \tag{5.9}$$

We consider the two cases.

Case I: $q \geq p + r + 2$, or $q = p + r + 1$ and $x \notin [y_p, y_q]$.

It is clear that $y_q - y_p \geq \lambda|I|$, and so it follows from (5.9) that

$$\Lambda_{p,q,r}(y_0, \ldots, y_m; \omega_k) \leq C|I|^{-k}\omega_k(|I|) \leq C|I|^{1-k} \int\limits_{|I|}^{2|I|} \frac{\omega_k(u)}{u^2}\, du.$$

Case II: $q = p + r + 1$ and $x \in [y_p, y_q]$.

If $x = y_p$, then $p = i$, $q = i + r + 1$, and $y_q - y_p = x_{i+r} - x \geq |x - x_{\sigma_r}|$.

If $x = y_q$, then $q = i$, $p = i - r - 1$, and $y_q - y_p = x - x_{i-r-1} \geq |x - x_{\sigma_r}|$.

If $x \in (y_p, y_q)$, then $y_q - y_p = x_{p+r} - x_p$. Since it is impossible that $|x - x_{\sigma_r}| > \max\{x - x_p, x_{p+r} - x\}$, for this would imply that $\{p, \ldots, p + r\} \subset \{\sigma_0, \ldots, \sigma_{r-1}\}$ which cannot happen since these sets have cardinalities $r + 1$ and r, respectively, we conclude that $|x - x_{\sigma_r}| \leq \max\{x - x_p, x_{p+r} - x\} \leq x_{p+r} - x_p$. Thus, in this case, (5.9) implies that

$$\Lambda_{p,q,r}(y_0, \ldots, y_m; \omega_k) \leq C|I|^{1-k} \int\limits_{|x - x_{\sigma_r}|}^{2|I|} \frac{\omega_k(u)}{u^2}du.$$

Hence,

$$\Lambda_r(y_0, \ldots, y_m; \omega_k) \leq C|I|^{1-k} \int\limits_{|x - x_{\sigma_r}|}^{2|I|} \frac{\omega_k(u)}{u^2}du,$$

which together with (5.8) implies (5.6).

Theorem 5.3 is proved.

We state one more corollary to illustrate the power of Theorem 5.3. Suppose that $Z = \{z_j\}_{j=0}^{\mu}$ with $z_0 < z_1 < \ldots < z_\mu$, and let $X = \{x_j\}_{j=0}^{m-1}$ with $m := (r+1)(\mu+1)$ be such that $x_{\nu(r+1)+j} := z_\nu$, for all $0 \le \nu \le \mu$ and $0 \le j \le r$. In other words,

$$X = \Bigg\{ \underbrace{z_0, \ldots, z_0}_{r+1}, \underbrace{z_1, \ldots, z_1}_{r+1}, \ldots, \underbrace{z_\mu, \ldots, z_\mu}_{r+1} \Bigg\}.$$

Now, given $f \in C^{(r)}[z_0, z_\mu]$, let $\mathcal{L}(x; f; Z) := L_{m-1}(x, f; x_0, \ldots, x_{m-1})$ be the Hermite polynomial of degree $\le m-1 = r\mu + \mu + r$ satisfying

$$\mathcal{L}^{(j)}(z_\nu; f; Z) = f^{(j)}(z_\nu), \quad \text{for all} \quad 0 \le \nu \le \mu \quad \text{and} \quad 0 \le j \le r. \tag{5.10}$$

Also,

$$\operatorname{dist}(x, Z) := \min_{0 \le j \le \mu} |x - z_j|, \quad x \in \mathbb{R}.$$

***Corollary* 5.2.** *Let $r \in \mathbb{N}_0$ and $\mu \in \mathbb{N}$, and suppose that a set $Z = \{z_j\}_{j=0}^{\mu}$ is such that*

$$z_{j+1} - z_j \ge \lambda |I|, \quad \text{for all } \ 0 \le j \le \mu - 1,$$

where $0 < \lambda \le 1$, $I := [z_0, z_\mu]$ and $|I| := z_\mu - z_0$. If $f \in C^{(r)}(I)$, then, for each $x \in I$,

$$\left| f(x) - \mathcal{L}(x; f; Z) \right| \le C \left(\operatorname{dist}(x, Z)\right)^{r+1} \int\limits_{\operatorname{dist}(x,Z)}^{2|I|} \frac{\omega_{m-r}(f^{(r)}, t, I)}{t^2} dt \le$$

$$\le C \left(\operatorname{dist}(x, Z)\right)^{r} \omega_{m-r} \left(f^{(r)}, |I| \left(\operatorname{dist}(x, Z)/|I|\right)^{1/(m-r)}, I \right) \le$$

$$\le C \left(\operatorname{dist}(x, Z)\right)^{r} \omega_{m-r}(f^{(r)}, |I|, I),$$

where $m := (r+1)(\mu+1)$, $C = C(m, \lambda)$ and the polynomial $\mathcal{L}(\cdot; f; Z)$ of degree $\le m-1$ satisfies (5.10).

As a final note, we remark that some of the results that appeared in the literature follow from the results in this note. For example, (i) the main theorem in [12] immediately follows from Corollary 5.2 with $\mu = 1$, $z_0 = -1$ and $z_1 = 1$, (ii) Corollary 5.1 is much stronger than the main theorem in [13], (iii) a particular case in Lemmas 8 and 9 of [15] for $k = 0$ follows from Corollary 5.1, (iv) several propositions in the unconstrained case in [11] follow from Corollary 5.1, (v) Lemma 3.3 and Corollaries 3.4 – 3.6 of [17] follow from Corollary 5.1 and (vi) the proof of Lemma 3.1 of [16] may be simplified if Corollary 5.1 is used.

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