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# STRONG SUMMABILITY OF TWO-DIMENSIONAL VILENKIN – FOURIER SERIES *

# СИЛЬНА СУМОВНІСТЬ ДВОВИМІРНИХ РЯДІВ ВІЛЕНКІНА – ФУР'Є

We study the exponential uniform strong summability of two-dimensional Vilenkin – Fourier series. In particular, it is proved that the two-dimensional Vilenkin – Fourier series of a continuous function $f$ is uniformly strongly summable to a function $f$ exponentially in the power $1/2$. Moreover, it is proved that this result is best possible.

Вивчається експоненціальна рівномірна сильна сумовність двовимірних рядів Віленкіна – Фур'є. Зокрема, доведено, що двовимірний ряд Віленкіна – Фур'є неперервної функції $f$ є рівномірно сильно сумовним до функції $f$ експоненціально в степені $1/2$. Крім того, доведено, що цей результат є найкращим із можливих.

**1. Introduction.** It is known that there exist continuous functions the trigonometric (Walsh) Fourier series of which do not converge. However, as it was proved by Fejér's [2] in 1905, the arithmetic means of the differences between the function and its Fourier partial sums converge uniformly to zero. The problem of strong summation was initiated by Hardy and Littlewood [16]. They generalized Fejér's result by showing that the strong means also converge uniformly to zero for any continuous function. The investigation of the rate of convergence of the strong means was started by Alexits [1]. Many papers have been published which are closely related with strong approximation and summability. We note that a number of signficant results are due to Leindler [17 – 19], Totik [26 – 28], Gogoladze [9], Goginava, Gogoladze, Karagulyan [13]. Leindler has also published the monograph [20].

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system see [3 – 7, 11 – 13, 21 – 24] and concerning the Ciselski system see Weisz [29, 30]. The summability of multiple Walsh – Fourier series have been investigated in [14, 15, 31].

Fridli and Schipp [5] proved that the following is true.

**Theorem FS.** *Let* $\Phi$ *stand for the trigonometric or the Walsh system, and let* $\psi$ *be a monotonically increasing function defined on* $[0,\infty)$ *for which* $\lim_{u\to 0+}\psi(u)=0$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\psi\left(\left|S_k^{\Phi}f(x)-f(x)\right|\right)=0,\quad f\in C(G_2),$$

*if and only if there exists* $A>0$ *such that* $\psi(t)\le\exp(At)$, $0\le t<\infty$. *Moreover, the convergence is uniform in* $x$, *where* $S_k^{\Phi}f$ *denotes the* $k$*th partial sums of Fourier series of* $f$ *by orthonormal sysstem* $\Phi$, *and* $G_2$ *refers to the Vilenkin group* $G_m$ *with* $m=(2,2,\ldots)$.

In this paper we study the exponential uniform strong summability of two-dimensional Vilenkin – Fourier series. In particular, it is proved that the two-dimensional Vilenkin – Fourier series of the continuous function $f$ is uniformly strong summable to the function $f$ exponentially in the power $1/2$. Moreover, it is proved that this result is best possible.

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Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo $m_k$. Define the group $G_m$ as the complete direct product of the groups $Z_{m_j}$, with the product of the discrete topologies of $Z_{m_j}$'s. The direct product $\mu$ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k}, \quad j \in Z_{m_k},$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$. If the sequence $m$ is bounded, then $G_m$ is called a bounded Vilenkin group. The elements of $G_m$ can be represented by sequences $x := (x_0, x_1, \ldots \ldots, x_j, \ldots)$, $x_j \in Z_{m_j}$. The group operation $+$ in $G_m$ is given by $x + y = (x_0 + y_0 (\operatorname{mod} m_0), \ldots \ldots, x_k + y_k (\operatorname{mod} m_k), \ldots)$, where $x = (x_0, \ldots, x_k, \ldots)$ and $y = (y_0, \ldots, y_k, \ldots) \in G_m$. The inverse of $+$ will be denoted by $-$.

It is easy to give a base for the neighborhoods of $G_m$ :

$$I_0(x) := G_m,$$

$$I_n(x) := \left\{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\right\}$$

for $x \in G_m$, $n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_+$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G_m$ the $n$th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

If we define the so-called generalized number system based on $m$ in the following way: $M_0 := := 1$, $M_{k+1} := m_k M_k$, $k \in \mathbb{N}$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$, $j \in \mathbb{N}_+$, and only a finite number of $n_j$'s differ from zero. We use the following notation. Let (for $n > 0$) $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$).

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions in this way

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k}, \qquad \imath^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}.$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.$$

Specifically, we call this system the Walsh – Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_1(G_m)$. It is well-known that $\psi_n(x)\psi_n(y) = = \psi_n(x + y)$, $|\psi_n(x)| = 1$, $n \in \mathbb{N}$, $\psi_n(-x) = \overline{\psi}_n(x)$ [25].

Now, introduce analogues of the usual definitions of the Fourier analysis. If $f \in L_1(G_m)$ we can establish the following definitions in the usual way:

Fourier coefficients:

$$\widehat{f}(k) := \int\limits_{G_m} f \overline{\psi}_k \, d\mu, \quad k \in \mathbb{N},$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \qquad n \in \mathbb{N}_+, \quad S_0 f := 0,$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+.$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if} \quad x \in I_n, \\ 0, & \text{if} \quad x \in G_m \backslash I_n, \end{cases} \tag{1}$$

$$D_n(x) = \psi_n(x) \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{q=m_j-n_j}^{m_j-1} r_j^q(x), \qquad f \in L_1(G_m), \quad n \in \mathbb{N}. \tag{2}$$

It is well known that

$$S_n f(x) = \int\limits_{G_m} f(t) D_n(x-t) d\mu(t).$$

Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin system. Let us fix $d \geq 1, d \in \mathbb{N}_+$. For Vilenkin group $G_m$ let $G_m^d$ be its Cartesian product $G_m \times \ldots \times G_m$ taken with itself $d$-times. Denote by $\mu$ the product measure $\mu \times \ldots \times \mu$. The rectangular partial sums of the two-dimensional Vilenkin – Fourier series are defined as follows:

$$S_{M,N}(f; x, y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j)\psi_i(x)\psi_j(y),$$

where the number

$$\widehat{f}(i,j) = \int\limits_{G_m \times G_m} f(x,y) \overline{\psi}_i(x) \overline{\psi}_j(y) d\mu(x,y)$$

is said to be the $(i,j)$th Vilenkin – Fourier coefficient of the function $f$.

Denote

$$S_n^{(1)}(f; x, y) := \sum_{l=0}^{n-1} \widehat{f}(l, y)\psi_l(x),$$

$$S_m^{(2)}(f; x, y) := \sum_{r=0}^{m-1} \widehat{f}(x, r)\psi_r(y),$$

where

$$\widehat{f}(l, y) = \int\limits_{G_m} f(x,y)\psi_l(x)\, d\mu(x)$$

and

$$\widehat{f}(x, r) = \int\limits_{G_m} f(x,y)\psi_r(y)\, d\mu(y).$$

**2. Best approximation.** Denote by $E_{l,r}(f)$ the best approximation of a function $f \in C\left(G_m^2\right)$ by Vilenkin polynomials of degree $\leq l$ of a variable $x$ and of degree $\leq r$ of a variable $y$ and let $E_l^{(1)}(f)$ be the partial best approximation of a function $f \in C\left(G_m^2\right)$ by Vilenkin polynomials of degree $\leq l$ of a variable $x,$ whose coefficients are continuous functions of the remaining variable $y.$ Analogously, we can define $E_r^{(2)}(f).$

Let $M_L \leq l < M_{L+1}, M_R \leq r < M_{R+1}$ and $E_{M_L,M_R}(f) := \|f - T_{M_L,M_R}\|_C$ , where $T_{M_L,M_R}$ is Vilenkin polynomial of best approximation of function $f$. Since (see (1))

$$\|S_{M_L,M_R}(f)\|_C \leq \|f\|_C ,$$

we can write

$$|S_{l,r}(f;x,y) - f(x,y)| \leq$$

$$\leq |S_{l,r}\left(f - S_{M_L,M_R}(f);x,y\right)| + \|S_{M_L,M_R}(f) - f\|_C \leq$$

$$\leq |S_{l,r}\left(f - S_{M_L,M_R}(f);x,y\right)| + \|S_{M_L,M_R}\left(f - T_{M_L,M_R}\right)\|_C +$$

$$+ \|f - T_{M_L,M_R} f\|_C \leq$$

$$\leq |S_{l,r}\left(f - S_{M_L,M_R}(f);x,y\right)| + 2E_{M_L,M_R}(f). \tag{3}$$

Now, we prove that the following inequality holds:

$$E_{M_L,M_R}(f) \leq 2E_{M_L}^{(1)}(f) + 2E_{M_R}^{(2)}(f). \tag{4}$$

Indeed, we have

$$E_{M_L,M_R}(f) \leq \|f - S_{M_L,M_R}(f)\|_C = \left\|f - S_{M_L}^{(1)}\left(S_{M_R}^{(2)}(f)\right)\right\|_C \leq$$

$$\leq \left\|f - S_{M_L}^{(1)}(f)\right\|_C + \left\|S_{M_L}^{(1)}\left(S_{M_R}^{(2)}(f) - f\right)\right\|_C \leq$$

$$\leq \left\|f - S_{M_L}^{(1)}(f)\right\|_C + \left\|S_{M_R}^{(2)}(f) - f\right\|_C . \tag{5}$$

Let $T_{M_L}^{(1)}(x,y)$ be a polynomial of the best approximation $E_{M_L}^{(1)}(f)$. Then

$$\left\|S_{M_L}^{(1)}(f) - f\right\|_C \leq \left\|f - T_{M_L}^{(1)}\right\|_C + \left\|S_{M_L}^{(1)}\left(f - T_{M_L}^{(1)}\right)\right\|_C \leq$$

$$\leq 2\left\|f - T_{M_L}^{(1)}\right\|_C = 2E_{M_L}^{(1)}(f). \tag{6}$$

Analogously, we can prove that

$$\left\|S_{M_R}^{(2)}(f) - f\right\|_C \leq 2E_{M_R}^{(2)}(f). \tag{7}$$

Combining (5) – (7) we obtain (4).

It is easy to show that

$$\|f - S_{M_L M_R}(f)\|_C \leq 2E_{M_L,M_R}(f). \tag{8}$$

**3. Main results.**

**Theorem 1.** *Let* $f \in C\left(G_m^2\right)$. *Then the inequality*

$$\left\|\frac{1}{nm}\sum_{l=1}^{n}\sum_{r=1}^{m}\left(e^{A\left|S_{l,r}(f)-f\right|^{1/2}}-1\right)\right\|_C \le$$

$$\le \frac{c(f,A)}{n}\sum_{l=1}^{n}\sqrt{E_l^{(1)}(f)}+\frac{c(f,A)}{m}\sum_{r=1}^{m}\sqrt{E_r^{(2)}(f)}$$

*is satisfied for any* $A>0$, *where* $c(f,A)$ *is a positive constant depend on* $A$ *and* $f$.

We say that the function $\psi$ belongs to the class $\Psi$ if it increases on $[0,+\infty)$ and

$$\lim_{u\to 0}\psi(u)=\psi(0)=0.$$

**Theorem 2.** (a) *Let* $\varphi\in\Psi$ *and the inequality*

$$\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{\sqrt{u}}<\infty \tag{9}$$

*holds. Then for any function* $f\in C\left(G_m^2\right)$ *the equality*

$$\lim_{n,m\to\infty}\left\|\frac{1}{nm}\sum_{l=1}^{n}\sum_{r=1}^{m}\left(e^{\varphi\left(\left|S_{l,r}(f)-f\right|\right)}-1\right)\right\|_C=0$$

*is satisfied.*

(b) *For any function* $\varphi\in\Psi$ *satisfying the condition*

$$\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{\sqrt{u}}=\infty \tag{10}$$

*there exists a function* $F\in C\left(G_m^2\right)$ *such that*

$$\overline{\lim_{u\to\infty}}\frac{1}{m^2}\sum_{l=1}^{m}\sum_{r=1}^{m}e^{\varphi\left(\left|S_{l,r}(F;0,0)-F(0,0)\right|\right)}=+\infty.$$

**4. Auxiliary results.**

**Lemma 1** [8]. *Let* $p\in\mathbb{N}_+$. *Then*

$$\sup_n\left(\int\limits_{G_m^p}\frac{1}{M_n}\left|\sum_{l=M_n}^{M_{n+1}-1}\prod_{k=1}^{p}D_l\left(s_k\right)\right|d\mu\left(s_1,\ldots,s_p\right)\right)^{1/p}\le cp,$$

*where* $c$ *is a positive constant.*

**Lemma 2** [9]. *Let* $\varphi, \psi \in \Psi$ *and the equality*

$$\lim_{n,m\to\infty} \frac{1}{nm} \sum_{l=1}^{n} \sum_{r=1}^{m} \psi\big( |S_{l,r}(f;x,y) - f(x,y)| \big) = 0$$

*be satisfied at the point* $(x_0, y_0)$ *or uniformly on a set* $E \subset I^2$. *If*

$$\overline{\lim_{u\to\infty}} \frac{\varphi(u)}{\psi(u)} < \infty,$$

*then the equality*

$$\lim_{n,m\to\infty} \frac{1}{nm} \sum_{l=1}^{n} \sum_{r=1}^{m} \varphi\left( |S_{l,r}(f;x,y) - f(x,y)| \right) = 0$$

*is satisfied at the point* $(x_0, y_0)$ *or uniformly on a set* $E \subset I^2$.

**Lemma 3.** *Let* $p > 0,\ A,\ B \in \mathbb{N}$. *Then*

$$\left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f;x,y)|^p \right\}^{1/p} \leq c\, \|f\|_C\, (p+1)^2. \tag{11}$$

***Proof.*** Since

$$\left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f;x,y)|^p \right\}^{1/p} \leq$$

$$\leq \left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f;x,y)|^{p+1} \right\}^{1/(p+1)}$$

without lost of generality we can suppose that $p = 2^m,\ m \in \mathbb{N}_+$. We can write

$$|S_{n,l}(f;x,y)|^2 = S_{n,l}(f;x,y)\overline{S}_{n,l}(f;x,y) =$$

$$= \int_{G_m^2} f(x - s_1, y - t_1) D_n(s_1) D_l(t_1) d\mu\,(s_1, t_1) \times$$

$$\times \int_{G_m^2} \overline{f}\,(x - s_2, y - t_2)\, \overline{D}_n(s_2) \overline{D}_l(t_2) d\mu\,(s_2, t_2) =$$

$$= \int_{G_m^2} f(x - s_1, y - t_1) D_n(s_1) D_l(t_1) d\mu\,(s_1, t_1) \times$$

$$\times \int_{G_m^2} \overline{f}(x + s_2, y + t_2) D_n(s_2) D_l(t_2) d\mu\,(s_2, t_2) =$$

$$= \int\limits_{G_m^4} f(x - s_1, y - t_1)\overline{f}\,(x + s_2, y + t_2) \times$$

$$\times D_n(s_1)D_n(s_2)D_l(t_1)D_l(t_2)d\mu(s_1, t_1, s_2, t_2).$$

Hence, we get

$$|S_{n,l}(f; x, y)|^p = \left(|S_{n,l}(f; x, y)|^2\right)^{p/2} =$$

$$= \left( \int\limits_{G_m^4} f(x - s_1, y - t_1)\overline{f}(x + s_2, y + t_2)\times \right.$$

$$\left. \times D_n(s_1)D_n(s_2)D_l(t_1)D_l(t_2)d\mu(s_1, t_1, s_2, t_2)\right)^{p/2} =$$

$$= \int\limits_{G_m^{2p}} \prod_{k=1}^{p/2} f\left(x - s_{2k-1}, y - t_{2k-1}\right) \prod_{r=1}^{p/2} \overline{f}(x + s_{2r}, y + t_{2r})\times$$

$$\times \prod_{i=1}^{p} D_n(s_i) \prod_{j=1}^{p} D_l(t_j) d\mu\left(s_1, t_1, \ldots, s_p, t_p\right),$$

$$\left\{ \frac{1}{M_A M_B} \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} |S_{n,l}(f; x, y)|^p \right\}^{1/p} \le$$

$$\le \left( \int\limits_{G_m^{2p}} \prod_{k=1}^{p/2} \left|f\left(x - s_{2k-1}, y - t_{2k-1}\right)\right| \prod_{r=1}^{p/2} \left|\overline{f}(x + s_{2r}, y + t_{2r})\right| \times \right.$$

$$\left. \times \frac{1}{M_A M_B} \left| \sum_{n=M_A}^{M_{A+1}-1} \sum_{l=M_B}^{M_{B+1}-1} \prod_{i=1}^{p} D_n(s_i) \prod_{j=1}^{p} D_l(t_j) \right| d\mu\left(s_1, t_1, \ldots, s_p, t_p\right) \right)^{1/p} \le$$

$$\le \|f\|_C \left( \int\limits_{G_m^p} \frac{1}{M_A} \left| \sum_{n=M_A}^{M_{A+1}-1} \prod_{i=1}^{p} D_n(s_i) \right| d\mu\left(s_1, \ldots, s_p\right) \right)^{1/p} \times$$

$$\times \left( \int\limits_{G_m^p} \frac{1}{M_B} \left| \sum_{l=M_B}^{M_{B+1}-1} \prod_{j=1}^{p} D_l(t_j) \right| d\mu\left(t_1, \ldots, t_p\right) \right)^{1/p} \le$$

$$\le cp^2 \left\|f\right\|_C.$$

Lemma 3 is proved.

**Lemma 4.** *Let $f \in C\left(G_m^2\right)$ and $p > 0$. Then*

$$\frac{1}{nk}\sum_{l=1}^{n}\sum_{r=1}^{k}\left|S_{l,r}(f;x,y)-f(x,y)\right|^p \leq$$

$$\leq c^p(p+1)^{2p}\left\{\frac{1}{n}\sum_{l=1}^{n}\left(E_l^{(1)}(f)\right)^p+\frac{1}{k}\sum_{r=1}^{k}\left(E_r^{(2)}(f)\right)^p\right\}. \tag{12}$$

***Proof.*** Since

$$(a+b)^\beta \leq 2^\beta\left(a^\beta+b^\beta\right), \qquad \beta > 0,$$

from (3), (4), (8) and using Lemma 3 we get

$$\frac{1}{M_AM_B}\sum_{n=M_A}^{M_{A+1}-1}\sum_{l=M_B}^{M_{B+1}-1}\left|S_{n,l}(f;x,y)-f(x,y)\right|^p \leq$$

$$\leq \frac{2^p}{M_AM_B}\sum_{n=M_A}^{M_{A+1}-1}\sum_{l=M_B}^{M_{B+1}-1}\left|S_{n,l}\left(f-S_{M_AM_B}(f);x,y\right)\right|^p+$$

$$+\frac{2^{2p}}{M_AM_B}\left(M_{A+1}-M_A\right)\left(M_{B+1}-M_B\right)E^p_{M_AM_B}(f) \leq$$

$$\leq c^p(p+1)^{2p}\left\|f-S_{M_AM_B}(f)\right\|_C^p+$$

$$+c^p\left(\left(E_{M_A}^{(1)}(f)\right)^p+\left(E_{M_B}^{(2)}(f)\right)^p\right) \leq$$

$$\leq c^p(p+1)^{2p}\left(\left(E_{M_A}^{(1)}(f)\right)^p+\left(E_{M_B}^{(2)}(f)\right)^p\right). \tag{13}$$

Let $M_L \leq n < M_{L+1}$ and $M_R \leq k < M_{R+1}$. Then from (13) we have

$$\frac{1}{nk}\sum_{l=1}^{n}\sum_{r=1}^{k}\left|S_{l,r}(f;x,y)-f(x,y)\right|^p \leq$$

$$\leq \frac{1}{nk}\sum_{l=1}^{M_{L+1}-1}\sum_{r=1}^{M_{R+1}-1}\left|S_{l,r}(f;x,y)-f(x,y)\right|^p =$$

$$=\frac{1}{nk}\sum_{A=0}^{L}\sum_{B=0}^{R}\sum_{l=M_A}^{M_{A+1}-1}\sum_{r=M_B}^{M_{B+1}-1}\left|S_{l,r}(f;x,y)-f(x,y)\right|^p \leq$$

$$\leq \frac{c^p(p+1)^{2p}}{nk}M_AM_B\sum_{A=0}^{L}\sum_{B=0}^{R}\left(\left(E_{M_A}^{(1)}(f)\right)^p+\left(E_{M_b}^{(2)}(f)\right)^p\right) \leq$$

$$\leq \frac{c^p(p+1)^{2p}}{nk}\sum_{A=0}^{L}\sum_{B=0}^{R}\sum_{l=M_{A-1}}^{M_A-1}\sum_{r=M_{B-1}}^{M_B-1}\left(\left(E_{M_A}^{(1)}(f)\right)^p+\left(E_{M_B}^{(2)}(f)\right)^p\right) \leq$$

$$\leq \frac{c^p(p+1)^{2p}}{nk}\sum_{A=0}^{L}\sum_{B=0}^{R}\sum_{l=M_{A-1}}^{M_A-1}\sum_{r=M_{B-1}}^{M_B-1}\left(\left(E_l^{(1)}(f)\right)^p+\left(E_r^{(2)}(f)\right)^p\right)\leq$$

$$\leq \frac{c^p(p+1)^{2p}}{nk}\sum_{l=1}^{n}\sum_{r=1}^{k}\left(\left(E_l^{(1)}(f)\right)^p+\left(E_r^{(2)}(f)\right)^p\right)\leq$$

$$\leq c^p(p+1)^{2p}\left\{\frac{1}{n}\sum_{l=1}^{n}\left(E_l^{(1)}(f)\right)^p+\frac{1}{k}\sum_{r=1}^{k}\left(E_r^{(2)}(f)\right)^p\right\}.$$

Lemma 4 is proved.

**5. Proofs of main results.** The Walsh – Paley version of Theorem 1 were proved in [12]. Based on inequality (12) the same construction works for the Vilenkin case. Therefore the proof of Theorem 1 will be omitted.

***Proof of Theorem* 2.** (a) It is easy to see that if $\varphi\in\Psi$, then $e^{\varphi}-1\in\Psi$. Besides, (9) implies the existence of a number $A$ such that

$$\overline{\lim_{u\to\infty}}\frac{e^{\varphi(u)}-1}{e^{Au^{1/2}}-1}<\infty.$$

Therefore, in view of Lemma 2, for the proof of Theorem 2 it is sufficient to prove that

$$\lim_{n,m\to\infty}\left\|\frac{1}{nm}\sum_{l=1}^{n}\sum_{r=1}^{m}\left(e^{A\left|S_{l,r}(f)-f\right|^{1/2}}-1\right)\right\|_C=0. \tag{14}$$

The validity of equality of (14) immediately follows from Theorem 1.

(b) First of all we prove that if $\psi\in\Psi$ and

$$\overline{\lim_{u\to\infty}}\frac{\psi(u)}{u}=\infty$$

then there exists a function $f\in C(G_m)$ and sequence of positive integers $\{A_k: k\geq 1\}$ such that

$$\psi\left(\left|S_{N_{A_k}}(f,0)\right|\right)>5(A_k-1)\ln a, \tag{15}$$

where $a:=\sup\limits_j m_j$ and $N_{A_j}:=\sum_{k=A_{j-1}}^{A_j-1}\left[\frac{m_{2k}}{2}\right]M_{2k}$.

Let $\{B_k: k\geq 1\}$ be an increasing sequence of positive integers such that

$$B_1>c', \tag{16}$$

$$B_j>2B_{j-1}, \tag{17}$$

$$\frac{\psi\left(B_j\right)}{B_j}>\frac{5j\ln a}{c'}, \tag{18}$$

where the constant $c'$ would be definen below.

Set

$$A_k := \left[\frac{kB_k}{c'}\right] + 1,$$

$$f_j(x) := \frac{1}{j+1} \sum_{s=A_{j-1}}^{A_j-1} \sum_{x_{2s+1}=0}^{m_{2s+1}-1} \cdots \sum_{x_{2A_j-1}=0}^{m_{2A_j}-1} \exp\left(-i\arg\left(\overline{D}_{N_{A_j}}(x)\right)\right) \times$$

$$\times \mathbb{I}_{I_{2A_j}\left(0,\ldots,0,x_{2s}=m_{2s}-1,x_{2s+1},\ldots,x_{2A_j-1}\right)}(x),$$

$$f(x) := \sum_{j=1}^{\infty} f_j(x), \qquad f(0) = 0,$$

where $\mathbb{I}_E$ is characteristic function of the set $E \subset G_m$.

Since

$$I_{2A_j}\left(0,\ldots,0,x_{2s}=m_{2s}-1,x_{2s+1},\ldots,x_{2A_j-1}\right) \cap$$

$$\cap I_{2A_j}\left(0,\ldots,0,x_{2l}=m_{2l}-1,x_{2l+1},\ldots,x_{2A_j-1}\right) = \varnothing, \quad l \neq s,$$

and $1/(j+1) \to 0$ as $j \to \infty$ we conclude that $f \in C(G_m)$.

We can write

$$\left|S_{N_{A_k}}(f;0) - f(0)\right| = \left|S_{N_{A_k}}(f;0)\right| =$$

$$= \left|\int_{G_m} f(t)\overline{D}_{N_{A_k}}(t)d\mu(t)\right| \geq$$

$$\geq \left|\int_{G_m} f_k(t)\overline{D}_{N_{A_k}}(t)d\mu(t)\right| - \sum_{j=k+1}^{\infty} \left|\int_{G_m} f_j(t)\,\overline{D}_{N_{A_k}}(t)d\mu(t)\right| -$$

$$- \sum_{j=0}^{k-1} \left|\int_{G_m} f_j(t)\overline{D}_{N_{A_k}}(t)d\mu(t)\right| =$$

$$= J_1 - J_2 - J_3. \tag{19}$$

From the definition of the function $f$ we have

$$J_1 = \frac{1}{k+1} \left| \sum_{s=A_{k-1}}^{A_k-1} \sum_{t_{2s+1}=0}^{m_{2s+1}-1} \cdots \right.$$

$$\left. \cdots \sum_{t_{2A_k-1}=0}^{m_{2A_k}-1} \int_{I_{2A_k}\left(0,\ldots,0,t_{2s}=m_{2s}-1,t_{2s+1},\ldots,t_{2A_k-1}\right)} \exp\left(-i\arg\left(\overline{D}_{N_{A_k}}(t)\right)\right) \overline{D}_{N_{A_k}}(t)d\mu(t) \right| =$$

$$=\frac{1}{k+1}\sum_{s=A_{k-1}}^{A_k-1}\sum_{t_{2s+1}=0}^{m_{2s+1}-1}\ldots$$

$$\ldots\sum_{t_{2A_k-1}=0}^{m_{2A_k}-1}\int\limits_{I_{2A_k}\left(0,\ldots,0,t_{2s}=m_{2s}-1,t_{2s+1},\ldots,t_{2A_k-1}\right)}\left|D_{N_{A_k}}(t)\right|d\mu(t).$$

Since (see [10])

$$\left|D_{N_{A_k}}(t)\right|\geq cM_{2s+1}$$

for

$$t\in I_{2s+1}\left(0,\ldots,0,t_{2s}=m_{2s}-1\right),\qquad s=A_{k-1},\ldots,A_k-1,$$

from (17) we can write

$$J_1\geq\frac{c}{k+1}\sum_{s=A_{k-1}}^{A_k-1}M_{2s+1}\sum_{t_{2s+1}=0}^{m_{2s+1}-1}\ldots$$

$$\ldots\sum_{t_{2A_k-1}=0}^{m_{2A_k-1}-1}\mu\left(I_{2A_k}\left(0,\ldots,0,t_{2s}=m_{2s}-1,t_{2s+1},\ldots,t_{2A_k-1}\right)\right)=$$

$$=\frac{c}{k+1}\sum_{s=A_{k-1}}^{A_k-1}\frac{M_{2s+1}m_{2s+1}\ldots m_{2A_k-1}}{M_{2A_k}}=$$

$$=\frac{c}{k+1}\left(A_k-A_{k-1}\right).$$

Since (see (16))

$$A_k-A_{k-1}=\left[\frac{kB_k}{c'}\right]-\left[\frac{(k-1)B_{k-1}}{c'}\right]\geq$$

$$\geq\frac{kB_k}{c'}-\frac{(k-1)\,B_{k-1}}{c'}-1=$$

$$=\frac{k\left(B_k-B_{k-1}\right)}{c'}+\frac{B_{k-1}}{c'}-1>$$

$$>\frac{k\left(B_k-B_{k-1}\right)}{c'}>\frac{kB_k}{2c'}\geq\frac{A_k}{2}-\frac{1}{2}>\frac{A_k}{4},$$

for $J_1$ we have

$$J_1\geq\frac{c}{k+1}\frac{A_k}{4}.$$

For $J_2$ we get

$$J_2\leq\sum_{j=k+1}^{\infty}\frac{1}{j+1}\sum_{s=A_{j-1}}^{A_j-1}\frac{1}{M_{2s}}N_{A_k}\leq$$

$$\leq \frac{1}{k}\sum_{s=A_k}^{\infty}\frac{1}{M_{2s}}N_{A_k} \leq \frac{c}{k}. \tag{20}$$

By (2) and from the construction of the function $f_j$ we can write

$$\operatorname{supp}(f_j) \cap \operatorname{supp}\left(D_{N_{A_k}}\right) = \varnothing, \quad j = 1, 2, \ldots, k-1,$$

consequently,

$$J_3 = 0. \tag{21}$$

Combining (18) – (21) we conclude that

$$\left|S_{N_{A_k}}(f;0)\right| = \left|S_{N_{A_k}}(f;0) - f(0)\right| \geq \frac{c' A_k}{k} \geq B_k,$$

$$\psi\left(\left|S_{N_{A_k}}(f;0)\right|\right) \geq \psi(B_k) \geq \frac{5k\ln a}{c'}B_k \geq 5(A_k-1)\ln a.$$

Hence, (15) is proved.

Write $\varphi(u) = \lambda(u)\sqrt{u}$ and define $\psi(u) := \lambda\left(u^2\right)u$. Then

$$\overline{\lim_{u\to\infty}}\frac{\psi(u)}{u} = +\infty.$$

Therefore there exist a function $f \in C(G_m)$ and sequence of positive integers $\{A_k : k \geq 1\}$ for which

$$\psi\left(\left|S_{N_{A_k}}(f,0)\right|\right) > 5(A_k-1)\ln a. \tag{22}$$

Set

$$F(x,y) := f(x)f(y).$$

It is easy to show that

$$\varphi\left(\left|S_{N_{A_k},N_{A_k}}(F;0,0)\right|\right) = \varphi\left(\left|S_{N_{A_k}}(f;0)\right|^2\right) =$$

$$= \lambda\left(\left|S_{N_{A_k}}(f;0)\right|^2\right)\left|S_{N_{A_k}}(f;0)\right| = \psi\left(\left|S_{N_{A_k}}(f;0)\right|\right).$$

Since $N_{A_k} \leq a^{2A_k}$, from (22) we have

$$\frac{1}{N_{A_k}^2}\sum_{i=1}^{N_{A_k}}\sum_{j=1}^{N_{A_k}} e^{\varphi(|S_{i,j}(F;0,0)|)} \geq \frac{1}{N_{A_k}^2}e^{\varphi\left(\left|S_{N_{A_k},N_{A_k}}(F;0,0)\right|\right)} =$$

$$= \frac{1}{N_{A_k}^2}e^{\psi\left(\left|S_{N_{A_k}}(f;0)\right|\right)} \geq \frac{e^{5(A_k-1)\ln a}}{a^{4A_k}} \to \infty \quad \text{as} \quad k \to \infty.$$

Theorem 2 is proved.

## References


1. *Alexits A., Králik D.* Über den Annäherungagred der Approximation im starken Sinne von stetigen Funktionen // Magyar Tud. Acad. Mat. Kut. Int. Kozl. – 1963. – **8**. – P. 317 – 327.
2. *Fejér L.* Untersuchungen über Fouriersche Reihen // Math. Ann. – 1904. – **58**. – P. 501 – 569.
3. *Fridli S.* On integrability and strong summability of Walsh – Kaczmarz series // Anal. Math. – 2014. – **40**, № 3. – P. 197 – 214.
4. *Fridli S., Schipp F.* Strong summability and Sidon type inequality // Acta Sci. Math. (Szeged). – 1985. – **60**. – P. 277 – 289.
5. *Fridli S., Schipp F.* Strong approximation via Sidon type inequalities // J. Approxim. Theory. – 1998. – **94**. – P. 263 – 284.
6. *Gát G., Goginava U., Karagulyan G.* Almost everywhere strong summability of Marcinkiewicz means of double Walsh – Fourier series // Anal. Math. – 2014. – **40**, № 4. – P. 243 – 266.
7. *Gát G., Goginava U., Karagulyan G.* On everywhere divergence of the strong $\Phi$-means of Walsh – Fourier series // J. Math. Anal. and Appl. – 2015. – **421**, № 1. – P. 206 – 214.
8. *Glukhov V. A.* Summation of multiple Fourier series in multiplicative systems // Mat. Zametki. – 1986. – **39**, № 5. – P. 665 – 673 (in Russian).
9. *Gogoladze L.* On the exponential uniform strong summability of multiple trigonometric Fourier series // Georg. Math. J. – 2009. – **16**, № 3. – P. 517 – 532.
10. *Goginava U.* Convergence in measure of partial sums of double Vilenkin – Fourier series // Georg. Math. J. – 2009. – **16**, № 3. – P. 507 – 516.
11. *Goginava U., Gogoladze L.* Strong approximation by Marcinkiewicz means of two-dimensional Walsh – Fourier series // Constr. Approxim. – 2012. – **35**, № 1. – P. 1 – 19.
12. *Goginava U., Gogoladze L.* Strong approximation of two-dimensional Walsh – Fourier series // Stud. Sci. Math. Hung. – 2012. – **49**, № 2. – P. 170 – 188.
13. *Goginava U., Gogoladze L., Karagulyan G.* BMO-estimation and almost everywhere exponential summability of quadratic partial sums of double Fourier series // Constr. Approxim. – 2014. – **40**, № 1. – P. 105 – 120.
14. *Goginava U.* Almost everywhere convergence of $(C, \alpha)$-means of cubical partial sums of $d$-dimensional Walsh – Fourier series // J. Approxim. Theory. – 2006. – **141**, № 1. – P. 8 – 28.
15. *Goginava U.* The weak type inequality for the Walsh system // Stud. Math. – 2008. – **185**, № 1. – P. 35 – 48.
16. *Hardy G. H., Littlewood J. E.* Sur la serie de Fourier d'une fonction a carre sommable // C. R. Acad. Sci. Paris. – 1913. – **156**. – P. 1307 – 1309.
17. *Leindler L.* Über die Approximation im starken Sinne // Acta Math. Acad. Hung. – 1965. – **16**. – P. 255 – 262.
18. *Leindler L.* On the strong approximation of Fourier series // Acta Sci. Math. (Szeged). – 1976. – **38**. – P. 317 – 324.
19. *Leindler L.* Strong approximation and classes of functions // Mitt. Math. Sem. Giessen. – 1978. – **132**. – S. 29 – 38.
20. *Leindler L.* Strong approximation by Fourier series. – Budapest: Akademiai Kiado, 1985.
21. *Rodin V. A.* BMO-strong means of Fourier series // Funct. Anal. and Appl. – 1989. – **23**. – P. 73 – 74 (in Russian).
22. *Schipp F.* Über die starke Summation von Walsh – Fourier Reihen // Acta Sci. Math. (Szeged). – 1969. – **30**. – P. 77 – 87.
23. *Schipp F.* On strong approximation of Walsh – Fourier series // MTA III. Oszt. Kozl. – 1969. – **19**. – P. 101 – 111 (in Hungarian).
24. *Schipp F., Ky N. X.* On strong summability of polynomial expansions // Anal. Math. – 1986. – **12**. – P. 115 – 128.
25. *Schipp F., Wade W. R., Simon P., Pál J.* Walsh series: an introduction to dyadic harmonic analysis. – Bristol: New York: Adam Hilger, 1990.
26. *Totik V.* On the strong approximation of Fourier series // Acta Math. Sci. Hung. – 1980. – **35**. – P. 151 – 172.
27. *Totik V.* On the generalization of Fejér's summation theorem // Functions, Series, Operators: Coll. Math. Soc. J. Bolyai (Budapest) Hungary. – Amsterdam etc.: North-Holland, 1980. – **35**. – P. 1195 – 1199.
28. *Totik V.* Notes on Fourier series: s trong approximation // J. Approxim. Theory. – 1985. – **43**. – P. 105 – 111.
29. *Weisz F.* Strong summability of Ciesielski – Fourier series // Stud. Math. – 2004. – **161**, № 3. – P. 269 – 302.
30. *Weisz F.* Strong summability of more-dimensional Ciesielski – Fourier series // East J. Approxim. – 2004. – **10**, № 3. – P. 333 – 354.
31. *Weisz F.* Summability of multi-dimensional Fourier series and Hardy space. – Dordrecht: Kluwer Acad., 2002.


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