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Pham Hoang Ha (Hanoi Nat. Univ. Education, Vietnam),
Si Duc Quang (Hanoi Nat. Univ. Education; Thang Long Inst. Math. and Appl. Sci., Hanoi, Vietnam)

UNICITY THEOREMS WITH TRUNCATED MULTIPLICITIES OF MEROMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES FOR FEW FIXED TARGETS*

ТЕОРЕМИ ЄДИНОСТІ З ОБРІЗАНИМИ КРАТНОСТЯМИ ДЛЯ МЕРОМОРФНИХ ВІДОБРАЖЕНЬ ЗА КІЛЬКОМА ЗМІННИМИ ДЛЯ НЕВЕЛИКОЇ КІЛЬКОСТІ ОБ'ЄКТІВ

The purpose of our paper is twofold. Our first aim is to prove a uniqueness theorem for meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ sharing $2N+2$ hyperplanes in the general position with truncated multiplicities, where all common zeros with multiplicities more than a certain number do not need to be counted. Second, we consider the case of mappings sharing less than $2N+2$ hyperplanes. These results are improvements of some recent results.

Робота має дві основні мети. По-перше, доведено теорему єдиності для мероморфних відображень з $\mathbb{C}^n$ в $\mathbb{P}^N(\mathbb{C})$, що поділяють $2N+2$ гіперплощини загального положення з обрізаними кратностями, де всі спільні нулі з кратностями, що перевищують деяке число, можна не враховувати. По-друге, розглянуто випадок, коли відображення поділяють менше, ніж $2N+2$ гіперплощини. Отримані результаті покращують деякі відомі нові результати.

1. Introduction. In 1926, R. Nevanlinna showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ cannot have the same inverse images for five distinct values. This result is usually called the Nevanlinna's five values theorem, which is the first theorem on the uniqueness problem of meromorphic mappings. After that, the uniqueness problems with truncated multiplicities of meromorphic mappings of $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$ sharing a finite set of hyperplanes in $\mathbb{P}^N(\mathbb{C})$ has been studied intensively by many authors such as H. Fujimoto [5], L. Smiley [11], S. Ji [7], Z.-H. Tu [16], G. Dethloff, T. V. Tan [3], D. D. Thai, S. D. Quang [10, 14], Z. Chen, Q. Yan [2] and others.

To state some of them, first of all we recall the following.

(a) Let f be a nonconstant meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with a reduced representation $f=(f_0:\ldots:f_N)$. Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$ given by $H=\{a_0\omega_0+\ldots+a_N\omega_N\}$. We set $(f,H)=\sum_{i=0}^{N}a_if_i$. Then we can define the corresponding divisor $\nu_{(f,H)}$ which is rephrased as the intersection multiplicity of the image of f and H at $f(z)$. Let k be a positive integer or $k=\infty$. For every $z\in\mathbb{C}^n$, we set

$$\nu_{(f,H),\le k}(z)=\begin{cases}0 & \text{if}\quad \nu_{(f,H)}(z)>k,\\ \nu_{(f,H)}(z) & \text{if}\quad \nu_{(f,H)}(z)\le k,\end{cases}$$

and

$$\nu_{(f,H),\ge k}(z)=\begin{cases}0 & \text{if}\quad \nu_{(f,H)}(z)<k,\\ \nu_{(f,H)}(z) & \text{if}\quad \nu_{(f,H)}(z)\ge k.\end{cases}$$

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Take a meromorphic mapping f of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, a positive integer d, a positive integer k or $k=\infty$ and q hyperplanes $H_1,\ldots,H_q$ in $\mathbb{P}^N(\mathbb{C})$ located in general position with

$$\dim\big\{z\in\mathbb{C}^n:\nu_{(f,H_i),\le k}(z)>0 \text{ and } \nu_{(f,H_j),\le k}(z)>0\big\}\le n-2,\quad 1\le i<j\le q,$$

and consider the set $\mathcal{F}(f,\{H_j\}_{j=1}^q,k,d)$ of all linearly nondegenerate meromorphic maps $g\colon\mathbb{C}^n\to\to\mathbb{P}^N(\mathbb{C})$ satisfying the conditions

(a) $\min\,(\nu_{(f,H_j),\le k},d)=\min\,(\nu_{(g,H_j),\le k},d),\ 1\le j\le q,$

(b) $f(z)=g(z)$ on $\bigcup_{j=1}^q\big\{z\in\mathbb{C}^n:\nu_{(f,H_j),\le k}(z)>0\big\}$.

Denote by $\sharp\, S$ the cardinality of the set S.

In 1983, L. Smiley [11] proved the following, which is usually called the unicity theorem for meromorphic mapping sharing few hyperplanes regardless of multiplicity.

Theorem A [11]. *If* $q\ge 3N+2$, *then* $\sharp\mathcal{F}\big(f,\{H_i\}_{i=1}^q,\infty,1\big)=1$.

In 2006, D. D. Thai and S. D. Quang [14] improved slightly the result of Smiley for the case of $N\ge 2$ to the following.

Theorem B [14]. *If* $N\ge 2$, *then* $\sharp\mathcal{F}\big(f,\{H_i\}_{i=1}^{3N+1},\infty,1\big)=1$.

In 2009, Z. Chen and Q. Yan [2] showed that the above unicity theorems are still valid for the case of meromorphic mapping sharing $2N+3$ hyperplanes. They proved the following theorem.

Theorem C [2]. $\sharp\mathcal{F}\big(f,\{H_i\}_{i=1}^{2N+3},\infty,1\big)=1$.

Recently, S. D. Quang [10] improved the above result of Chen – Yan by omitting all zeros with multiplicities larger than a certain number. He proved the following theorem.

Theorem D [10]. *If* $k>\dfrac{N(4N^2+11N+4)}{3N+2}-1$, *then* $\sharp\ \mathcal{F}\big(f,\{H_i\}_{i=1}^{2N+3},k,1\big)=1$.

Then a natural question arise here: *Are there any unicity theorems with truncated multiplicities in the case where* $q\le 2N+2$?

This question is first considered in [10] by S. D. Quang. He gave the following theorem.

Theorem E [10]. *Let* f_1 *and* f_2 *be two linearly nondegenerate meromorphic mappings of* $\mathbb{C}^n$ *into* $\mathbb{P}^N(\mathbb{C})$, $N\ge 2$, *and let* $H_1,\ldots,H_{2N+2}$ *be hyperplanes in* $\mathbb{P}^N(\mathbb{C})$ *located in general position such that*

$$\dim\big\{z\in\mathbb{C}^n:\nu_{(f_1,H_i)}(z)>0 \textit{ and } \nu_{(f_1,H_j)}(z)>0\big\}\le n-2$$

for every $1\le i<j\le 2N+2$. *Assume that the following conditions are satisfied*:

(a) $\min\{\nu_{(f_1,H_j),\le N},1\}=\min\{\nu_{(f_2,H_j),\le N},1\}$ *and* $\min\{\nu_{(f_1,H_j),\ge N},1\}=\min\{\nu_{(f_2,H_j),\ge N},1\}$, $1\le j\le 2N+2$,

(b) $f_1(z)=f_2(z)$ *on* $\bigcup_{j=1}^{2N+2}\big\{z\in\mathbb{C}^n:\nu_{(f_1,H_j)}(z)>0\big\}$.

Then $f_1\equiv f_2$.

However, we see that in the above theorem all zeros of the functions (f,H_i)'s are counted.

In the first part of this paper, we will improve Theorem E by omitting the zeros with multiplicity larger than a certain number k. Namely, we will prove the following theorem.

Theorem 1.1. *Let f_1 and f_2 be two linearly nondegenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, $N \geq 2$, and let $H_1, \ldots, H_{2N+2}$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in general position such that*

$$\dim\left\{z \in \mathbb{C}^n : \nu_{(f_1,H_i),\leq k}(z) > 0 \ \ \textit{and} \ \ \nu_{(f_1,H_j),\leq k}(z) > 0\right\} \leq n-2$$

for every $1 \leq i < j \leq 2N+2$, where k be a positive integer such that

$$k > 6(m-2)m(N^2-1) + 2N - 1 \quad \textit{with} \quad m = \binom{2N+2}{N+1}.$$

Assume that the following conditions are satisfied:

(a) $\min\{\nu_{(f_1,H_j)(z),\leq N}, 1\} = \min\{\nu_{(f_2,H_j)(z),\leq N}, 1\}$ *and* $\min\{\nu_{(f_1,H_j),\geq N}(z), 1\} = \min\{\nu_{(f_2,H_j),\geq N}(z), 1\}$ *for all* $z \in \operatorname{Supp}(\nu_{(f_1,H_i),\leq k}) \cup \operatorname{Supp}(\nu_{(f_1,H_j),\leq k})$, $1 \leq j \leq 2N+2$,

(b) $f_1(z) = f_2(z)$ *on* $\bigcup_{j=1}^{2N+2}\{z \in \mathbb{C}^n : \nu_{(f_1,H_j),\leq k}(z) > 0\}$.

Then $f_1 \equiv f_2$.

We would like to emphasize that our paper is a apart of the doctoral thesis of the first author at Hanoi National University of Education with some slight improvements. Recently, motivated by our method, H. Z. Cao and T. B. Cao [6] (Theorem 1.9) have proved a result similar to Theorem 1.1, where the condition "$z \in \operatorname{Supp}(\nu_{(f_1,H_i),\leq k}) \cup \operatorname{Supp}(\nu_{(f_1,H_j),\leq k})$" is replaced by "$\nu_{(f_1,H_j)}(z) \equiv \nu_{(f_2,H_j)}(z) (\operatorname{mod} T)$" with a large enough positive integer T. The proof of Theorem 1.1 is presented in Section 3.

(b) In the last part of this paper, we will consider the uniqueness problem for the case meromorphic mappings sharing less than $2N+2$ hyperplanes. In fact, we will prove a uniqueness theorem for the case where the mappings share q hyperplanes with $N+3 \leq q < 2N+2$. To state our results, we give the following.

Take a meromorphic mapping f of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, a positive integer d, a positive integer k or $k = +\infty$ and q hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^N(\mathbb{C})$ located in general position with

$$\dim\left\{z \in \mathbb{C}^n : \nu_{(f,H_i),\leq k}(z) > 0 \ \text{ and } \ \nu_{(f,H_j),\leq k}(z) > 0\right\} \leq n-2, \quad 1 \leq i < j \leq q.$$

With the above notations, we have the following definition.

Definition 1.1. *We denote by $\mathcal{G}(f, \{H_j\}_{j=1}^q, k, d)$ the set of all linearly nondegenerate meromorphic maps $g \colon \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions*:

(a) $\min\{\nu_{(f,H_j),\leq k}, d\} = \min\{\nu_{(g,H_j),\leq k}, d\}$, $1 \leq j \leq q$.

(b) *Let $f = (f_0 : \ldots : f_N)$ and $g = (g_0 : \ldots : g_N)$ be reduced representations of f and g, respectively. Then, for each $0 \leq j \leq N$ and for each $\omega \in \bigcup_{i=1}^q\{z \in \mathbb{C}^n : \nu_{(f,H_i),\leq k}(z) > 0\}$, the following two conditions are satisfied*:

(i) *if $f_j(\omega) = 0$, then $g_j(\omega) = 0$,*

(ii) *if $f_j(\omega)g_j(\omega) \neq 0$, then $\mathcal{D}^\alpha\left(\dfrac{f_i}{f_j}\right)(\omega) = \mathcal{D}^\alpha\left(\dfrac{g_i}{g_j}\right)(\omega)$ for each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers with $|\alpha| = \alpha_1 + \ldots + \alpha_n < d$ and for each $i \neq j$, where $\mathcal{D}^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 \ldots \partial^{\alpha_n} z_n}$.*

Remark that the condition (b) does not depend on the choice of reduced representations.

For each real number x, by $[x]$ we denote the largest integer which does not exceed x. We will prove the following, which is the last purpose of this paper.

Theorem 1.2. *Let* f_1 *and* f_2 *be two meromorphic mappings in* $\mathcal{G}\big(f, \{H_i\}_{i=1}^q, k, d\big)$, *where* $d, k, q,\ q \geq N+2$, *are positive integers.*

(a) *If* $q > \dfrac{N+3}{2} + \sqrt{\dfrac{2(N+1)(N-d)}{d} + \left(\dfrac{N+3}{2}\right)^2}$ *and*

$$k > \frac{2d(Nq - 2N - 1) + 2q + 2N^2 + 2N}{dq(q-N-3) - 2(N+1)(N-d)},$$

then there exist $[q/2]+1$ *indices* $i_1, \ldots, i_{[q/2]+1}$ *such that*

$$\frac{(f_1, H_{i_1})}{(f_2, H_{i_1})} = \ldots = \frac{(f_1, H_{[q/2]})}{(f_2, H_{[q/2]+1})}.$$

In particular, if $q \geq 2N$, *then* $f_1 = f_2$.

(b) *If* $1 \leq d \leq N,\ q > \dfrac{(N+1)(d+1)}{2d} + \sqrt{\dfrac{(N+1)^2(d+1)^2}{4d^2} + \dfrac{N^2-1}{d}}$ *and*

$$k > \frac{q(N-1)(dq-N-1) + (dq+N-1)(N+1)}{dq^2 - (N+1)(d+1)q - N^2 + 1},$$

then $f_1 = f_2$.

(c) *If* $d \geq N+1,\ q > N+1+\dfrac{2N}{d}$ *and* $k > \dfrac{d(Nq-q+N+1) - 2N^2 + 2N}{d(q-N-1) - 2N}$, *then* $f_1 = f_2$.

We note that, in [15] we together with Do Duc Thai also proved a similar result for the case where the meromorphic mappings sharing $N+2$ moving hyperplanes. Our this result deals with the general case where the number q of fixed hyperplanes belongs to $[N+2; 2N+2]$.

We distinguish here some cases where the assumptions of Theorem 1.2 are satisfied.

1. The assumptions of the assertion (a) satisfies in the following cases:

$d = 1, q = 2N+3$ and $k > \dfrac{6N^2+4N+4}{3N+2}$ (therefore, in this case Theorem 1.2 is improvement of Theorems C and D);

$2 \leq d \leq N, q = 2N+4-d$ and $k > \dfrac{d(4N^2+4N-2) + 2N^2 + 2N}{(N-d)(d(2N+4-d) - 2(N+1)) + 2N + 4 - d}$;

$d \geq N+1, q = N+3$ and $k > \dfrac{2d(N^2+N-1) + 2N^2 + 4N + 6}{2(d-N)(N+1)}$.

2. The assumptions of the assertion (b) satisfies in the following cases:

$d = N, q = N+3, N \geq 2$ and $k > \dfrac{N^4 + 5N^3 + 4N^2 - 7N + 3}{2(N-1)}$;

$N = d = 1, q = 5$ and $k > 2$ (therefore, in this case we have an improvement of the Nevanlinna's five values theorem).

3. The assumptions of the assertion (c) satisfies in the following cases:

$N+1 \leq d \leq 2N, q = N+2$ and $k > \dfrac{d(N^2+3N-2) - 2n^2 + 2N}{2(d-N)}$;

$d \geq 2N+1, q = N+2$ and $k > \dfrac{d(N^2+2N-1) - 2n^2 + 2N}{d-2N}$.

2. Basic notions in Nevanlinna theory. ***2.1. Divisors on*** $\mathbb{C}^n$**.** We set $\|z\| = \big(|z_1|^2 + \ldots$ $\ldots + |z_n|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define $B(r) := \{z \in \mathbb{C}^n : \|z\| < r\}$, $S(r) := \{z \in$ $\in \mathbb{C}^n : \|z\| = r\}$, $0 < r < \infty$.

Define $d = \partial + \bar{\partial}$, $d^c = \dfrac{i}{4\pi}(\bar{\partial} - \partial)$ and

$$v_{n-1}(z) := \big(dd^c\|z\|^2\big)^{n-1},$$

$$\sigma_n(z) := d^c \log\|z\|^2 \wedge \big(dd^c \log\|z\|^2\big)^{n-1} \quad \text{on} \quad \mathbb{C}^n \setminus \{0\}.$$

Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^n$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, we set $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\mathcal{D}^\alpha F = \dfrac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 \ldots \partial^{\alpha_n} z_n}$. We define the mapping $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\big\{m : \mathcal{D}^\alpha F(z) = 0 \ \text{ for all } \ \alpha \ \text{ with } \ |\alpha| < m\big\}, \quad z \in \Omega.$$

We mean by a divisor on a domain Ω in $\mathbb{C}^n$ a mapping $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood U of a $(\subset \Omega)$ such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq n - 2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq n - 2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n - 2$, and we define the divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G. Hence, they are globally well-defined on Ω.

2.2. Counting functions. For a divisor ν on $\mathbb{C}^n$ and for positive integers k, M (or $M = \infty$), we define the counting functions of ν as follows. Set

$$\nu^{(M)}(z) = \min\{M, \nu(z)\},$$

$$\nu^{(M)}_{\leq k}(z) = \begin{cases} 0 & \text{if} \quad \nu(z) > k, \\ \nu^{(M)}(z) & \text{if} \quad \nu(z) \leq k, \end{cases}$$

$$\nu^{(M)}_{> k}(z) = \begin{cases} \nu^{(M)}(z) & \text{if} \quad \nu(z) > k, \\ 0 & \text{if} \quad \nu(z) \leq k. \end{cases}$$

We define $n(t)$ by

$$n(t) = \begin{cases} \displaystyle\int_{|\nu| \cap B(t)} \nu(z) v_{n-1} & \text{if} \quad n \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if} \quad n = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t)$, $n^{(M)}_{\leq k}(t)$, $n^{(M)}_{>k}(t)$.

Define

$$N(r,\nu) = \int_1^r \frac{n(t)}{t^{2n-1}}\,dt, \quad 1 < r < \infty.$$

Similarly, we define $N\big(r,\nu^{(M)}\big)$, $N\big(r,\nu^{(M)}_{\le k}\big)$, $N\big(r,\nu^{(M)}_{>k}\big)$ and denote them by $N^{(M)}(r,\nu)$, $N^{(M)}_{\le k}(r,\nu)$, $N^{(M)}_{>k}(r,\nu)$, respectively.

Let $\varphi\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r,\nu_\varphi), \qquad N^{(M)}_\varphi(r) = N^{(M)}(r,\nu_\varphi),$$

$$N^{(M)}_{\varphi,\le k}(r) = N^{(M)}_{\le k}(r,\nu_\varphi), \qquad N^{(M)}_{\varphi,>k}(r) = N^{(M)}_{>k}(r,\nu_\varphi).$$

For brevity we will omit the superscript $^{(M)}$ if $M = \infty$.

2.3. Characteristic and proximity functions. Let $f : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \ldots : w_N)$ on $\mathbb{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \ldots : f_N)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and $f(z) = \big(f_0(z) : \ldots : f_N(z)\big)$ outside the analytic set $\{f_0 = \ldots = f_N = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(|f_0|^2 + \ldots + |f_N|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T(r,f) = \int\limits_{S(r)} \log\|f\|\sigma_n - \int\limits_{S(1)} \log\|f\|\sigma_n.$$

Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$ given by $H = \{a_0\omega_0 + \ldots + a_N\omega_N\}$. We define the proximity function of H by

$$m_{f,H}(r) = \int\limits_{S(r)} \log\frac{\|f\|\|H\|}{|(f,H)|}\sigma_n - \int\limits_{S(1)} \log\frac{\|f\|\|H\|}{|(f,H)|}\sigma_n,$$

where $\|H\| = \left(\sum_{i=0}^N |a_i|^2\right)^{1/2}$.

Let φ be a nonzero meromorphic function on $\mathbb{C}^n$, which are occasionally regarded as a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$. The proximity function of φ is defined by

$$m(r,\varphi) := \int\limits_{S(r)} \log\max\,(|\varphi|,1)\sigma_n.$$

As usual, by the notation "$\|\, P$" we mean the assertion P holds for all $r \in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\displaystyle\int_E dr < \infty$.

2.4. Some lemmas. The following results play essential roles in Nevanlinna theory (see [9, 12, 13]).

Theorem 2.1 (first main theorem). *Let $f\colon \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T(r,f), \quad r > 1.$$

Theorem 2.2 (second main theorem)**.** *Let $f: \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, \ldots, H_q$ be hyperplanes in general position in $\mathbb{P}^N(\mathbb{C})$. Then*

$$\Big\| \, (q - N - 1)T(r, f) \leq \sum_{i=1}^{q} N^{(N)}_{(f,H_i)}(r) + o(T(r, f)).$$

Lemma 2.1 (see [14])**.** *Let $f: \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping. Let $H_1, H_2, \ldots, H_q$ be q hyperplanes in $\mathbb{P}^N(\mathbb{C})$ located in general position. Assume that $k \geq N - 1$. Then*

$$\Big\| \left(q - N - 1 - \frac{Nq}{k+1}\right)T(r, f) \leq \sum_{j=1}^{q} \left(1 - \frac{N}{k+1}\right) N^{(N)}_{(f,H_j),\leq k}(r) + o(T(r, f)) \, .$$

Lemma 2.2 (lemma on logarithmic derivative)**.** *Let f be a nonzero meromorphic function on $\mathbb{C}^n$. Then*

$$\Big\| \, m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O\big(\log^+ T(r, f)\big), \quad \alpha \in \mathbb{Z}^n_+.$$

Denote by ${\mathcal{M}^*}_n$ the Abelian multiplicative group of all nonzero meromorphic functions on $\mathbb{C}^n$. Then the multiplicative group ${\mathcal{M}^*}_n/\mathbb{C}^*$ is a torsion free Abelian group.

Definition 2.1. *Let G be a torsion free Abelian group and $A = (a_1, a_2, \ldots, a_q)$ be a q-tuple of elements a_i in G. Let $q \geq r > s > 1$. We say that the q-tuple A has the property $(P_{r,s})$ if any r elements $a_{l(1)}, \ldots, a_{l(r)}$ in A satisfy the condition that for any given $i_1, \ldots, i_s$, $1 \leq i_1 < \ldots \ldots < i_s \leq r$, there exist $j_1, \ldots, j_s$, $1 \leq j_1 < \ldots < j_s \leq r$, with $\{i_1, \ldots, i_s\} \neq \{j_1, \ldots, j_s\}$ such that $a_{l(i_1)} \ldots a_{l(i_s)} = a_{l(j_1)} \ldots a_{l(j_s)}$.*

Proposition 2.1 (see [4])**.** *Let G be a torsion free Abelian group and $A = (a_1, \ldots, a_q)$ be a q-tuple of elements a_i in G. If A has the property $(P_{r,s})$ for some r, s with $q \geq r > s > 1$, then there exist $i_1, \ldots, i_{q-r+2}$ with $1 \leq i_1 < \ldots < i_{q-r+2} \leq q$ such that $a_{i_1} = a_{i_2} = \ldots = a_{i_{q-r+2}}$.*

Lemma 2.3. *Let f be a linearly nondegenerate meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let H be a hyperplanes of $\mathbb{P}^N(\mathbb{C})$, d be a positive integer and k is a positive integer or $+\infty$ with $k \geq d$, then*

$$N^{(d)}(r, \nu^0_{(f,H),\leq k}) \geq \frac{k+1}{k+1-d} N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1-d} T(r, f).$$

Proof. We have

$$N^{(d)}(r, \nu_{(f,H),\leq k}) = N^{(d)}(r, \nu_{(f,H)}) - N^{(d)}(r, \nu_{(f,H),>k}) \geq$$

$$\geq N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1} N(r, \nu^0_{(f,H),>k}) =$$

$$= N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1}\big(N(r, \nu_{(f,H)}) - N(r, \nu_{(f,H),\leq k})\big) \geq$$

$$\geq N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1} T(r, f) + \frac{d}{k+1} N^{(d)}(r, \nu_{(f,H),\leq k}).$$

This implies that

$$N^{(d)}(r, \nu_{(f,H),\leq k}) \geq \frac{k+1}{k+1-d} N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1-d} T(r, f).$$

The lemma is proved.

3. Proof of Theorem 1.1. Suppose that $f_1 \not\equiv f_2$. We set $T(r) = T(r, f_1) + T(r, f_2)$.
For two index $i, j \in \{1, \ldots, 2N+2\}$, we set

$$P_{ij} = (f_1, H_i)(f_2, H_j) - (f_1, H_j)(f_2, H_i).$$

By Jensen's formula, we easily see that

$$N_{P_{ij}}(r) \le \int\limits_{S(r)} \log\left|(f_1, H_i)(f_2, H_j) - (f_1, H_j)(f_2, H_i)\right|\sigma_n \le$$

$$\le \int\limits_{S(r)} \log\|f_1\|\sigma_n + \int\limits_{S(r)} \log\|f_2\|\sigma_n + O(1) = T(r) + O(1). \tag{3.1}$$

Claim 3.1. Assume that $P_{ij} \not\equiv 0$. Then we have

$$\sum_{v=i,j} N\big(r, \min\{\nu_{(f_1,H_v),\le k}, \nu_{(f_2,H_v),\le k}\}\big) + \sum_{\substack{v=1\\ v\ne i,j}}^{2N+2} N^{(1)}_{(f_s,H_v),\le k}(r) \le T(r) + O(1), \quad s = 1, 2.$$

In fact, we will prove the above inequality for $s = 1$. We see that for each $z \in \operatorname{Supp}\nu_{(f_1,H_v),\le k}$, $v = i, j$, one has

$$\nu_{P_{ij}}(z) \ge \min\{\nu_{(f_1,H_v),\le k}(z), \nu_{(f_2,H_v),\le k}(z)\}.$$

Also, for each $z \in \bigcup_{\substack{v\ne i,j\\ 1\le v\le 2N+2}} \operatorname{Supp}\nu_{(f_1,H_v),\le k}$, since $f(z) = g(z)$ we get $P_{ij}(z) = 0$ and then

$$\nu_{P_{ij}}(z) \ge \sum_{\substack{v\ne i,j\\ v=1}}^{2N+2} N^{(1)}_{(f_1,H_v),\le k}(z).$$

Therefore, we obtain

$$\nu_{P_{ij}}(z) \ge \min\{\nu_{(f_1,H_v),\le k}(z), \nu_{(f_2,H_v),\le k}(z)\} + \sum_{\substack{v\ne i,j\\ v=1}}^{2N+2} N^{(1)}_{(f_1,H_v),\le k}(z),$$

for all $z \in \mathbb{C}^n$ outside an analytic set of codimension two. By integrating both sides of the above inequality, we have

$$\sum_{v=i,j} N\big(r, \min\{\nu_{(f_1,H_v),\le k}, \nu_{(f_2,H_v),\le k}\}\big) + \sum_{\substack{v=1\\ v\ne i,j}}^{2N+2} N^{(1)}_{(f_1,H_v)}(r) \le N_{P_{ij}}(r).$$

Combining this inequality and (3.1), we obtain

$$\sum_{v=i,j} N\big(r, \min\{\nu_{(f_1,H_v),\le k}, \nu_{(f_2,H_v),\le k}\}\big) + \sum_{\substack{v=1\\ v\ne i,j}}^{2N+2} N^{(1)}_{(f_1,H_v)}(r) \le T(r) + O(1).$$

This proves the claim.

For each $i \in \{1, \ldots, 2N+2\}$, we define the divisor μ_i as follows:

$$\mu_i(z) = \begin{cases} 1 & \text{if} \quad \nu_{(f_1,H_i)}(z) \neq \nu_{(f_2,H_i)}(z), \\ 0 & \text{for otherwise.} \end{cases}$$

For each $z \in \operatorname{Supp}(\nu_{(f_1,H_i)}) \cup \operatorname{Supp}(\nu_{(f_1,H_j)})$, we easily see that

if $z \in \operatorname{Supp}(\nu_{(f_1,H_i),\le k}) \cup \operatorname{Supp}(\nu_{(f_1,H_j),\le k})$, then by the assumption (a) of the theorem, for each $s = 1, 2$, we have

$$\mu_i(z) \le \min\{\nu_{(f_1,H_i),\le k}(z), \nu_{(f_2,H_i),\le k}(z)\} -$$
$$-\nu^{(N)}_{(f_1,H_i),\le k}(z) - \nu^{(N)}_{(f_2,H_i),\le k}(z) + N\nu^{(1)}_{(f_s,H_i),\le k}(z);$$

otherwise $z \in \operatorname{Supp}(\nu_{(f_1,H_i),>k}) \cup \operatorname{Supp}(\nu_{(f_1,H_j),>k})$, then

$$\mu_i(z) \le \nu^{(1)}_{(f_1,H_i),>k}(z) + \nu^{(1)}_{(f_2,H_i),>k}(z).$$

Therefore, we have

$$\mu_i \le \min\left\{\nu_{(f_1,H_i),\le k}, \nu_{(f_2,H_i),\le k}\right\} - \nu^{(N)}_{(f_1,H_i),\le k} -$$
$$-\nu^{(N)}_{(f_2,H_i),\le k} + N\nu^{(1)}_{(f_s,H_i),\le k} + \nu^{(1)}_{(f_1,H_i),>k} + \nu^{(1)}_{(f_2,H_i),>k}$$

outside an analytic set of codimension two. This yields that

$$N(r,\mu_i) \le N\big(r, \min\left\{\nu_{(f_1,H_i),\le k}, \nu_{(f_2,H_i),\le k}\right\}\big) - N^{(N)}_{(f_1,H_i),\le k}(r) - N^{(N)}_{(f_2,H_i),\le k}(r) +$$
$$+NN^{(1)}_{(f_s,H_i),\le k}(r) + N^{(1)}_{(f_1,H_i),>k}(r) + N^{(1)}_{(f_2,H_i),>k}(r). \tag{3.2}$$

Claim 3.2:
$$\sum_{i=1}^{2N+2} N(r,\mu_i) \le \frac{(N+1)(N+8)}{2(k+1-N)}.$$

Indeed, by changing indices if necessary, we may assume that

$$\underbrace{\frac{(f_1,H_1)}{(f_2,H_1)} \equiv \frac{(f_1,H_2)}{(f_2,H_2)} \equiv \ldots \equiv \frac{(f_1,H_{k_1})}{(f_2,H_{k_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f_1,H_{k_1+1})}{(f_2,H_{k_1+1})} \equiv \ldots \equiv \frac{(f_1,H_{k_2})}{(f_2,H_{k_2})}}_{\text{group 2}} \not\equiv$$
$$\not\equiv \underbrace{\frac{(f_1,H_{k_2+1})}{(f_2,H_{k_2+1})} \equiv \ldots \equiv \frac{(f_1,H_{k_3})}{(f_2,H_{k_3})}}_{\text{group 3}} \not\equiv \ldots \not\equiv \underbrace{\frac{(f_1,H_{k_{s-1}+1})}{(f_2,H_{k_{s-1}+1})} \equiv \ldots \equiv \frac{(f_1,H_{k_s})}{(f_2,H_{k_s})}}_{\text{group } s},$$

where $k_s = 2N+2$.

For each $1 \le i \le 2N+2$, we set

$$\tau(i) = \begin{cases} i+N+1 & \text{if} \quad i \le N+1, \\ i-N-1 & \text{if} \quad i > N+1. \end{cases}$$

Since $f_1 \not\equiv f_2$, the number of elements of every group is at most N. Hence, $\dfrac{(f_1, H_i)}{(f_2, H_i)}$ and $\dfrac{(f_1, H_{\tau(i)})}{(f_2, H_{\tau(i)})}$ belong to distinct groups. This means that $P_{\tau(i)i} \not\equiv 0,\ 1 \le i \le 2N+2$. From Claim 3.1, we have

$$T(r) \ge \sum_{v=i,\tau(i)} N\big(r, \min\{\nu_{(f_1,H_v),\le k}, \nu_{(f_2,H_v),\le k}\}\big) + \sum_{\substack{v=1\\ v\neq i,\tau(i)}}^{2N+2} N^{(1)}_{(f_s,H_v),\le k}(r) + O(1), \quad s = 1,2.$$

Summing-up of both sides of the above inequality over all pairs $(i, \tau(i))$, by (3.2) and Lemma 2.3, for each $s = 1,2$ we get

$$(N+1)T(r) \ge \sum_{v=1}^{2N+2} N\big(r, \min\{\nu_{(f_1,H_v),\le k}, \nu_{(f_2,H_v),\le k}\}\big) +$$

$$+ N \sum_{v=1}^{2N+2} N^{(1)}_{(f_s,H_v),\le k}(r) + O(1) \ge$$

$$\ge \sum_{v=1}^{2N+2} \left(N(r,\mu_i) + \sum_{t=1,2} \Big(N^{(N)}_{(f_t,H_v),\le k}(r) - N^{(1)}_{(f_1,H_v),>k}(r) \Big) \right) =$$

$$= \sum_{v=1}^{2N+2} \left(N(r,\mu_i) + \sum_{t=1,2} \Big(N^{(N)}_{(f_t,H_v),\le k}(r) - \frac{1}{N} N^{(N)}_{(f_1,H_v),>k}(r) \Big) \right) =$$

$$= \sum_{v=1}^{2N+2} \left(N(r,\mu_i) + \sum_{t=1,2} \Big(\frac{N+1}{N} N^{(N)}_{(f_t,H_v),\le k}(r) - \frac{1}{N} N^{(N)}_{(f_1,H_v)}(r) \Big) \right) =$$

$$= \sum_{v=1}^{2N+2} \left(N(r,\mu_i) + \sum_{t=1,2} \Big(\frac{N+1}{N} \Big(\frac{k+1}{k+1-N} N^{(N)}_{(f_t,H_v)}(r) - \frac{N}{k+1-N} T(r,f_t) \Big) - \right.$$

$$\left. - \frac{1}{N} N^{(N)}_{(f_1,H_v)}(r) \Big) \right) \ge$$

$$\ge \sum_{v=1}^{2N+2} \left(N(r,\mu_i) + \sum_{t=1,2} \frac{1}{N} \Big(\frac{(N+1)(k+1)}{k+1-N} - 1 \Big) N^{(N)}_{(f_t,H_v)}(r) - \frac{N}{k+1-N} T(r) \right) \ge$$

$$\ge (N+1) \Big(\frac{1}{N} \Big(\frac{(N+1)(k+1)}{k+1-N} - 1 \Big) - \frac{2N}{k+1-N} \Big) T(r) + \sum_{v=1}^{2N+2} N(r,\mu_i).$$

Thus,

$$\sum_{i=1}^{2N+2} N(r,\mu_i) \le \frac{N^2-1}{k+1-N} T(r).$$

This proves the claim.

Claim 3.3. For $i,j\in\{1,\ldots,2N+2\}$ such that $P_{ij}\not\equiv 0$, we have

$$\Big\| T(r)\overset{(1)}{\geq} N_{P_{ij}}(r)\overset{(2)}{\geq}\sum_{v=i,j}N\big(r,\min\{\nu_{(f_1,H_v),\leq k},\nu_{(f_2,H_v),\leq k}\}\big)+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f_s,H_v),\leq k}(r)\overset{(3)}{\geq}$$

$$\overset{(3)}{\geq}\sum_{v=i,j}\left(N^{(N)}_{(f_1,H_v),\leq k}(r)+N^{(N)}_{(f_2,H_v),\leq k}(r)-NN^{(1)}_{(f_s,H_v),\leq k}(r)\right)+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f_s,H_v),\leq k}(r)\overset{(4)}{\geq}$$

$$\overset{(4)}{\geq}\left(1-\frac{N(N+1)}{k+1-N}\right)T(r)+O(1),\quad s=1,2.$$

Indeed, inequalities (1) and (2) are clear. The third inequality follows from the inequality $\min\{a,b\}\geq\min\{a,n\}-\min\{b,n\}-1$ for two integers a and b. We will prove the last inequality. Without loss of generality, we may assume that $i=1$ and $j=\tau(1)$. Then we obtain

$$\sum_{v=1,\tau(1)}\left(N^{(N)}_{(f_1,H_v),\leq k}(r)+N^{(N)}_{(f_2,H_v),\leq k}(r)-NN^{(1)}_{(f_s,H_v),\leq k}(r)\right)+$$

$$+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f_s,H_v),\leq k}(r)-T(r)\geq$$

$$\geq\sum_{t=1}^{N+1}\left(\sum_{v=t,\tau(t)}\left(N^{(N)}_{(f_1,H_v),\leq k}(r)+N^{(N)}_{(f_2,H_v),\leq k}(r)-NN^{(1)}_{(f_s,H_v),\leq k}(r)\right)+\right.$$

$$\left.+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f_s,H_v),\leq k}(r)-T(r)\right)\geq$$

$$\geq\sum_{t=1}^{2N+2}\left(N^{(N)}_{(f_1,H_t),\leq k}(r)+N^{(N)}_{(f_2,H_t),\leq k}(r)\right)-(N+1)T(r)\geq$$

$$\geq\left(\frac{(N+1)(k+1)-N(2N+2)}{k+1-N}\right)T(r)-(N+1)T(r)=\frac{-N(N+1)}{k+1-N}T(r).$$

This proves the last inequality of the claim.

Assume that $H_i=\{a_{i0}\omega_0+\ldots+a_{iN}\omega_N=0\}$. We set $h_i=\dfrac{(f_1,H_i)}{(f_2,H_i)}$, $1\leq i\leq 2N+2$. Then $\dfrac{h_i}{h_j}=\dfrac{(f_1,H_i)(f_2,H_j)}{(f_1,H_j)(f_2,H_i)}$ does not depend on representations of f_1 and f_2, respectively. Since $\sum_{k=0}^N a_{ik}f_{1k}-h_i\sum_{k=0}^N a_{ik}f_{2k}=0$, $1\leq i\leq 2N+2$, it implies that $\det(a_{i0},\ldots,a_{iN},a_{i0}h_i,\ldots$ $\ldots,a_{iN}h_i;1\leq i\leq 2N+2)=0$.

For each subset $I\subset\{1,2,\ldots,2N+2\}$, put $h_I=\prod_{i\in I}h_i$. Denote by $\mathcal{I}$ the set of all combinations $I=(i_1,\ldots,i_{N+1})$ with $1\leq i_1<\ldots<i_{N+1}\leq 2N+2$.

For each $I=(i_1,\ldots,i_{N+1})\in\mathcal{I}$, define

$$A_I = (-1)^{\frac{(N+1)(N+2)}{2}+i_1+\ldots+i_{N+1}} \det(a_{i_r l};\ 1 \le r \le N+1,\ 0 \le l \le N)\times$$
$$\times \det(a_{j_s l};\ 1 \le s \le N+1,\ 0 \le l \le N),$$

where $J = (j_1, \ldots, j_{N+1}) \in \mathcal{I}$ such that $I \cup J = \{1, 2, \ldots, 2N+2\}$.

Then $\sum_{I\in\mathcal{I}} A_I h_I = 0$.

Take $I_0 \in \mathcal{I}$. Then

$$A_{I_0} h_{I_0} = -\sum_{I\in\mathcal{I}, I\neq I_0} A_I h_I, \quad \text{i.e.,} \quad h_{I_0} = -\sum_{I\in\mathcal{I}, I\neq I_0} \frac{A_I}{A_{I_0}} h_I.$$

Remark that for each $I \in \mathcal{I}$, then $\dfrac{A_I}{A_{I_0}} \not\equiv 0$.

Denote by t the minimal number satisfying the following: There exist t elements $I_1, \ldots, I_t \in \in \mathcal{I} \setminus \{I_0\}$ and t nonzero constants $b_i \in \mathbb{C}$ such that $h_{I_0} = \sum_{i=1}^{t} b_i h_{I_i}$. It is easy to see that $t \le \binom{2N+2}{N+1} - 1$.

Since $h_{I_0} \not\equiv 0$ and by the minimality of t, it follows that the family $\{h_{I_1}, \ldots, h_{I_t}\}$ is linearly independent over $\mathbb{C}$.

Assume that $t \ge 2$. Consider the meromorphic mapping $h\colon \mathbb{C}^n \to \mathbb{P}^{t-1}(\mathbb{C})$ with a reduced representation $h = (dh_{I_1} : \ldots : dh_{I_t})$, where d is meromorphic on $\mathbb{C}^n$. We see that if z is a zero or pole of some dh_{I_j}, then it must be zero or pole of some h_i, i.e., $\mu_i(z) = 1$. Then by the second main theorem, we have

$$\Big\| \, T(r,h) \le \sum_{i=1}^{t} N^{(t-1)}_{dh_{I_i}}(r) + N^{(t-1)}_{dh_{I_0}}(r) + o(T(r,h)) \le$$
$$\le (t-1)\left(\sum_{i=1}^{t} N^{(1)}_{dh_{I_i}}(r) + N^{(1)}_{dh_{I_0}}(r)\right) + o(T(r,h)) \le$$
$$\le (t-1)(t+1)\sum_{i=1}^{2N+2} N(r,\mu_i) + o(T(r,h)) + o(T(r)) \le$$
$$\le \frac{(t-1)(t+1)(N^2-1)}{k+1-N} T(r) + o(T(r,h)) + o(T(r)).$$

This yields that $\| \, T(r,h) = o(T(r))$.

On the other hand, we get

$$3T(r,h) \ge N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r) + N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r) + O(1).$$

Since $\dfrac{h_I}{h_J} = 1$ on the set $\bigcup_{j\in((I\cup J)\setminus(I\cap J))^c} E_j$, where $E_j = \{z \in \mathbb{C}^n : \nu_{(f_s,H_j),\le k}(z) > 0\}$ and

$$\big((I_1 \cup I_2) \setminus (I_1 \cap I_2)\big)^c \cup \big((I_2 \cup I_0) \setminus (I_2 \cap I_0)\big)^c \cup \big((I_0 \cup I_1) \setminus (I_0 \cap I_1)\big)^c =$$
$$= \{1, \ldots, 2N+2\}, \quad s = 1, 2,$$

it implies that

$$N^{(1)}_{\frac{h_{I_1}}{h_{I_2}}-1}(r)+N^{(1)}_{\frac{h_{I_2}}{h_{I_0}}-1}(r)+N^{(1)}_{\frac{h_{I_0}}{h_{I_1}}-1}(r)\geq\sum_{i=1}^{2N+2}N^{(1)}_{(f_s,H_i),\leq k}(r)\geq\frac{1}{N}\sum_{i=1}^{2N+2}N^{(N)}_{(f_s,H_i),\leq k}(r).$$

Hence, for each $s=1,2$, we obtain

$$\Big\|\, 3T(r,h)\geq\frac{1}{N}\sum_{i=1}^{2N+2}N^{(N)}_{(f_s,H_i),\leq k}(r)\geq$$

$$\geq\frac{1}{N(k+1-N)}\Big((N+1)(k+1)-N(2N+2)\Big)T(r,f_s)+$$

$$+o(T(r))=\frac{(N+1)(k-2N+1)}{N(k+1-N)}T(r,f_s)+o(T(r)).$$

Therefore, we have

$$\frac{6(t-1)(t+1)(N^2-1)}{k+1-N}T(r)\geq\frac{(N+1)(k-2N+1)}{N(k+1-N)}T(r)+o(T(r)).$$

This implies that

$$k\leq 6(t-1)(t+1)(N^2-N)+2N-1\leq 6(m-2)m(N^2-N)+2N-1.$$

This is a contradiction.

Thus, $t=1$. Then $\frac{h_{I_0}}{h_{I_1}}=\text{constant}\neq 0$. Hence, for each $I\in\mathcal{I}$, there is $J\in\mathcal{I}\setminus\{I\}$ such that $\frac{h_I}{h_J}=\text{constant}\neq 0$. Consider the free Abelian subgroup generated by the family $\{[h_1],\ldots,[h_{2N+2}]\}$ of the torsion free Abelian group $\mathcal{M}^*{}_n/\mathbb{C}^*$. Then the family $\{[h_1],\ldots,[h_{2N+2}]\}$ has the property $P_{2N+2,N+1}$. It implies that there exist $2N+2-2N=2$ elements, without loss of generality we may assume that they are $[h_1],[h_2]$, such that $[h_1]=[h_2]$. Then $\frac{h_1}{h_2}=\tau\in\mathbb{C}^*$.

Suppose that $\tau\neq 1$. Since for each $z\in\bigcup_{i=3}^{2N+2}\operatorname{Supp}\nu_{(f_1,H_i),\leq k}\setminus\bigcup_{i=1,2}(f_i^{-1}(H_1)\cup f_i^{-1}(H_2))$ we have $\frac{h_1(z)}{h_2(z)}=1$, the set $\bigcup_{i=3}^{2N+2}\operatorname{Supp}\nu_{(f_1,H_i),\leq k}$ is a subset of $\bigcup_{i=1,2}(f_i^{-1}(H_1)\cup f_i^{-1}(H_2))$, and, hence, it is a subset of $\bigcup_{i=1,2}\big(\operatorname{Supp}\nu_{(f_i,H_1),>k}\cup\operatorname{Supp}\nu_{(f_i,H_2),>k}\big)$. By Lemma 2.1, we have

$$\Big\|\,\frac{(N-1)(k+1)-2N^2}{k+1-N}(T(r,f_1)+T(r,f_2))\leq$$

$$\leq\sum_{s=1,2}\sum_{i=3}^{2N+2}N^{(N)}_{(f_s,H_i),\leq k}(r)+o(T(r))\leq$$

$$\leq\sum_{s=1,2}N\sum_{i=1,2}N^{(1)}_{(f_s,H_i),>k}+o(T(r))\leq$$

$$\leq\sum_{s=1,2}\frac{N}{k+1}\sum_{i=1,2}N_{(f_s,H_i),>k}+o(T(r))\leq$$

$$\leq \frac{2N}{k+1}T(r) + o(T(r)).$$

Letting $r \longrightarrow +\infty$, we get $\dfrac{(N-1)(k+1) - 2N^2}{k+1-N} \leq \dfrac{2N}{k+1}$. This is a contradiction. Thus, $\tau = 1$, i.e., $h_1 = h_2$.

Now we consider

$$P_{1\tau(1)} = P_{1(N+2)} = (f_1, H_1)(f_2, H_{N+2}) - (f_2, H_1)(f_1, H_{N+2}) =$$

$$= \frac{(f_1, H_1)}{(f_1, H_2)}\left(\frac{(f_1, H_2)}{(f_1, H_{N+2})} - \frac{(f_2, H_2)}{(f_2, H_{N+2})}\right) \not\equiv 0.$$

For a point $z \notin \cup \operatorname{Supp}\nu_{(f_1,H_2),>k} \cup \operatorname{Supp}\nu_{(f_1,H_{N+2}),>k} \cup \operatorname{Supp}\nu_{(f_2,H_{N+2}),>k}$, we see that if $z \in \operatorname{Supp}\nu_{(f_1,H_v),\leq k}$ for some $v \neq 1, N+1$, then $P_{1(N+2)}(z) = 0$, i.e.,

$$\nu_{P_{1(N+2)}}(z) \geq \sum_{v\neq 1} \min\{1, \nu_{(f_1,H_v),\leq k}\},$$

if $z \in \operatorname{Supp}\nu_{(f_s,H_1),\leq k}$, then z is a zero of $P_{1(N+2)}$ with multiplicity at least $\nu_{(f_1,H_1),\leq k}(z) + \min\{1, \nu_{(f_1,H_1),\leq k}\}$, hence,

$$\nu_{P_{1(N+2)}}(z) \geq \min\{N, \nu_{(f_1,H_1),\leq k}\}(z) + \min\{N, \nu_{(f_2,H_1),\leq k}(z)\} -$$

$$-(N-1)\min\{1, \nu_{(f_1,H_1),\leq k}(z)\},$$

if $z \in \operatorname{Supp}\nu_{(f_s,H_{N+2}),\leq k}$, then z is a zero of $P_{1(N+2)}$ with multiplicity at least $\nu_{(f_1,H_{N+2}),\leq k}(z) + \min\{1, \nu_{(f_1,H_{N+2}),\leq k}\}$, hence,

$$\nu_{P_{1(N+2)}}(z) \geq \min\{N, \nu_{(f_1,H_{N+2}),\leq k}(z)\} +$$

$$+ \min\{N, \nu_{(f_2,H_{N+2}),\leq k}(z)\} - N\min\{1, \nu_{(f_1,H_1),\leq k}(z)\}.$$

Therefore, this implies that

$$\nu_{P_{1(N+2)}} \geq \sum_{v=1,N+2} \big(\min\{N, \nu_{(f_1,H_v),\leq k}\} + \min\{N, \nu_{(f_1,H_v),\leq k}\} - N\min\{1, \nu_{(f_1,H_v),\leq k}\}\big) +$$

$$+ \sum_{\substack{v\neq 1,N+2\\ v=1}}^{2N+2} \min\{1, \nu_{(f_1,H_v),\leq k}\} + \min\{1, \nu_{(f_1,H_1),\leq k}\} -$$

$$-(N+1)\big(\min\{1, \nu_{(f_1,H_2),>k}\} + \min\{1, \nu_{(f_1,H_{N+2}),>k}\} + \min\{1, \nu_{(f_2,H_{N+2}),>k}\}\big).$$

By integrating both sides of the above inequality, we obtain

$$T(r) \geq N_{P_{1(N+2)}}(r) \geq \sum_{v=1,N+2} \Big(N^{(N)}_{(f_1,H_v),\leq k}(z) + N^{(N)}_{(f_2,H_v),\leq k}(z) - NN^{(1)}_{(f_1,H_v),\leq k}(z)\Big) +$$

$$+ \sum_{\substack{v\neq 1,N+2\\ v=1}}^{2N+2} N^{(1)}_{(f_1,H_v),\leq k}(r) + N^{(1)}_{(f_1,H_1),\leq k}(r) -$$

$$-(N+1)\Big(N^{(1)}_{(f_1,H_2),>k}(r)+N^{(1)}_{(f_1,H_{N+2}),>k}(r)+N^{(1)}_{(f_2,H_{N+2}),>k}(r)\Big)\overset{\text{(by Claim 3.3(4))}}{\geq}$$

$$\overset{\text{(by Claim 3.3(4))}}{\geq}\left(1-\frac{N(N+1)}{k+1-N}\right)T(r)+N^{(1)}_{(f_1,H_1),\leq k}(r)-$$

$$-\frac{N+1}{k+1}\big(T(r,f_1)+T(r)\big)+o(T(r)).$$

Similarly, we have

$$T(r)\geq\left(1-\frac{N(N+1)}{k+1-N}\right)T(r)+N^{(1)}_{(f_1,H_1),\leq k}(r)-\frac{N+1}{k+1}\big(T(r,f_2)+T(r)\big)+o(T(r)).$$

Therefore,

$$T(r)\geq\left(1-\frac{N(N+1)}{k+1-N}\right)T(r)+N^{(1)}_{(f_1,H_1),\leq k}(r)-\frac{3(N+1)}{2(k+1)}T(r)+o(T(r)).$$

Thus,

$$\Big\|\ N^{(1)}_{(f_1,H_1),\leq k}(r)\leq\left(\frac{N(N+1)}{k+1-N}+\frac{3(N+1)}{2(k+1)}\right)T(r)+o(T(r)). \qquad (3.3)$$

On the other hand, since $f_1\not\equiv f_2$, for each $i\neq 1$ there exists an index j such that $P_{1j}\not\equiv 0$ and $P_{ij}\not\equiv 0$. Therefore, by Claim 3.3, we easily see that

$$\frac{N(N+1)}{k+1-N}T(r)\geq\sum_{v=i,j}N\big(r,\min\{\nu_{(f_1,H_v),\leq k},\nu_{(f_2,H_v),\leq k}\}\big)+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f_1,H_v),\leq k}(r)-$$

$$-\left(\sum_{v=1,j}\Big(N^{(N)}_{(f_1,H_v),\leq k}(r)+N^{(N)}_{(f_2,H_v),\leq k}(r)-NN^{(1)}_{(f_1,H_v),\leq k}(r)\Big)+\sum_{\substack{v=1\\ v\neq i,j}}^{2N+2}N^{(1)}_{(f_1,H_v),\leq k}(r)\right)\geq$$

$$\geq N\big(r,\min\{\nu_{(f_1,H_i),\leq k},\nu_{(f_2,H_i),\leq k}\}\big)+N^{(1)}_{(f_1,H_i),\leq k}(r)-$$

$$-\Big(N^{(N)}_{(f_1,H_1),\leq k}(r)+N^{(N)}_{(f_2,H_1),\leq k}(r)-(N+1)N^{(1)}_{(f_1,H_1),\leq k}(r)\Big)\geq$$

$$\geq 2N^{(1)}_{(f_1,H_i),\leq k}(r)-(N-1)N^{(1)}_{(f_1,H_1),\leq k}(r).$$

This implies that

$$\Big\|\ N^{(1)}_{(f_1,H_i),\leq k}(r)\leq\frac{N(N+1)}{2(k+1-N)}T(r)+\frac{N-1}{2}N^{(1)}_{(f_1,H_1),\leq k}(r)\leq$$

$$\leq\left(\frac{N^2(N+1)}{2(k+1-N)}+\frac{3(N^2-1)}{4(k+1)}\right)T(r)+o(T(r)). \qquad (3.4)$$

Now applying Lemma 2.1 and using (3.3) and (3.4), we have

$$\frac{(N+1)(k+1)-N(2N+2)}{k+1-N}T(r,f_1)\leq\sum_{i=1}^{2N+1}N^{(N)}_{(f,H_i),\leq k}(r)+o(T(r))\leq$$

$$\leq \sum_{i=1}^{2N+1} NN^{(1)}_{(f,H_i),\leq k}(r) + o(T(r)) \leq$$

$$\leq \left(N(2N+1)\left(\frac{N^2(N+1)}{2(k+1-N)} + \frac{3(N^2-1)}{4(k+1)}\right)+\right.$$

$$\left.+\left(\frac{N(N+1)}{k+1-N} + \frac{3(N+1)}{2(k+1)}\right)\right)T(r) + o(T(r)) \leq$$

$$\leq N(2N+1)\frac{N^2(N+1)}{k+1-N}T(r) + o(T(r)).$$

Similarly, we obtain

$$\frac{(N+1)(k+1) - N(2N+2)}{k+1-N}T(r,f_2) \leq N(2N+1)\frac{N^2(N+1)}{k+1-N}T(r) + o(T(r)).$$

Then

$$\frac{(N+1)(k+1) - N(2N+2)}{k+1-N}T(r) \leq 2N(2N+1)\frac{N^2(N+1)}{k+1-N}T(r) + o(T(r)).$$

Letting $r \longrightarrow +\infty$, we get

$$\frac{(N+1)(k+1) - N(2N+2)}{k+1-N} \leq 2N(2N+1)\frac{N^2(N+1)}{k+1-N}.$$

This implies that

$$k \leq 4N^4 + 2N^3 + 2N - 1 \leq 6N^4 + 2N - 1 \leq$$

$$\leq 6(2^{N+1}-2)2^{N+1} + 2N - 1 \leq 6(m-2)m + 2N - 1.$$

This is a contradiction. Hence, $f_1 \equiv f_2$.

Theorem 1.1 is proved.

4. Proof of Theorem 1.2. Assume that f_1, f_2 have reduce representation $f_i = (f_{i0} : \ldots : f_{iN})$, $i = 1, 2$. We will use the same notations which are introduced in the proof of Theorem 1.1. Define

$$I = I(f_1) \cup I(f_2) \cup_{1 \leq t < s \leq q} \left\{z \in \mathbb{C}^n \mid \nu_{(f_1,H_t),\leq k}(z)\nu_{(f_1,H_s),\leq k}(z) > 0\right\}.$$

Then I is an analytic set of codimension 2 or empty set. For each $i \in \{1, \ldots, q\}$, we set

$$N_i(r) = N^{(N)}_{(f_1,H_i),\leq k}(r) + N^{(N)}_{(f_1,H_i),\leq k}(r) - (N+d)N^{(1)}_{(f_1,H_i),\leq k}(r).$$

For each permutation $I = (i_1, \ldots, i_q)$ of $\{1, \ldots, q\}$, we define T_I the set of all $r \in [1, +\infty)$ such that

$$N_{i_1}(r) \geq N_{i_2}(r) \geq \ldots \geq N_{i_q}(r).$$

Then we see that $\bigcup_I T_I = [1, +\infty)$. Therefore, there exists a permutation, for instance it is $I = (1, \ldots, q)$ such that $\int_{T_I} dr = +\infty$.

We also remark that with the assumptions of (a) or (b) or (c), one always has $k \geq N$.

We first prove the assertion (a).

Claim 4.1. If $P_{ij} \not\equiv 0$, then the following assertion holds:

$$T(r) \geq \sum_{v=i,j} \Big(N^{(N)}_{(f,H_v),\leq k}(r)+N^{(N)}_{(f,H_v),\leq k}(r)-(N+d)N^{(1)}_{(f,H_v),\leq k}(r)\Big)+\sum_{v=1}^{q} dN^{(1)}_{(f,H_v),\leq k}(r)+o(T(r)).$$

Indeed, we fix a point $z \notin I$ satisfying $\nu_{(f_1,H_t),\leq k}(z) > 0,\ t \neq i,j$. Suppose that $f_{1l}(z)f_{2l}(z) = 0,\ 0 \leq l \leq N$. Then $f_{2l}(z) = 0,\ 0 \leq l \leq N$. This means that $z \in I(f_2)$. This is impossible. Hence, there exists an index l such that $f_{1l}(z)f_{2l}(z) \neq 0$. This implies that

$$\mathcal{D}^{\alpha}\left(\frac{P_{ij}}{f_{1l}f_{2l}}\right)(z) = 0 \quad \forall |\alpha| < d.$$

Hence, $\nu_{P_{ij}}(z) \geq d$.

Now we fix a point $z \notin I$ satisfying $\nu_{(f_1,H_t),\leq k}(z) > 0,\ t = i,j$, for instance $t = i$. Then we easily have the following:

$$\nu_{P_{ij}}(z) \geq \min\left\{\nu_{(f_1,H_t),\leq k}(z), \nu_{(f_2,H_t),\leq k}(z)\right\} \geq$$
$$\geq \min\left\{N, \nu_{(f_1,H_t),\leq k}(z)\right\} + \min\left\{N, \nu_{(f_2,H_t),\leq k}(z)\right\} - N\min\left\{1, \nu_{(f_1,H_t),\leq k}(z)\right\}.$$

Therefore, we get

$$\nu_{P_{ij}} \sum_{v=i,j} \big(\min\{N, \nu_{(f_1,H_t),\leq k}\} + \min\{N, \nu_{(f_2,H_t),\leq k}\} -$$
$$-(N+d)\min\{1, \nu_{(f_1,H_t),\leq k}\}\big) + d\sum_{v=1}^{q} \min\{1, \nu_{(f_1,H_v),\leq k}\}$$

outside an analytic set of codimension two. This implies that

$$N_{P_{ij}} \geq \sum_{v=i,j} \Big(N^{(N)}_{(f_1,H_v),\leq k}(r) + N^{(N)}_{(f_2,H_v),\leq k}(r) - (N+d)N^{(1)}_{(f_1,H_v),\leq k}(r)\Big) + d\sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\leq k}(r).$$

By Jensen's formula, it is also clear that

$$T(r) \geq N_{P_{ij}}(r) + o(T(r)).$$

This proves the claim.

Now we suppose that there exists an index $i \in \left\{1, \ldots, \left[\frac{q}{2}\right] + 1\right\}$ such that $P_{1i} \not\equiv 0$. For $r \in T_I$, we obtain

$$\Big\| \ T(r) \geq \sum_{v=1,i} \Big(N^{(N)}_{(f_1,H_v),\leq k}(r) + N^{(N)}_{(f_2,H_v),\leq k}(r) -$$
$$-(N+d)N^{(1)}_{(f_1,H_v),\leq k}(r)\Big) + d\sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\leq k}(r) \geq$$
$$\geq \frac{2}{[q/2]} \sum_{v=1}^{2[q/2]} \Big(N^{(N)}_{(f_1,H_v),\leq k}(r) + N^{(N)}_{(f_2,H_v),\leq k}(r) -$$

$$-(N+d)N^{(1)}_{(f_1,H_v),\le k}(r)\Big)+d\sum_{v=1}^{q}N^{(1)}_{(f_1,H_v),\le k}(r)\ge$$

$$\ge\frac{2}{q}\sum_{v=1}^{q}\Big(N^{(N)}_{(f_1,H_v),\le k}(r)+N^{(N)}_{(f_2,H_v),\le k}(r)-$$

$$-(N+d)N^{(1)}_{(f_1,H_v),\le k}(r)\Big)+d\sum_{v=1}^{q}N^{(1)}_{(f_1,H_v),\le k}(r)=$$

$$=\frac{2}{q}\sum_{v=1}^{q}\Big(N^{(N)}_{(f_1,H_v),\le k}(r)+N^{(N)}_{(f_2,H_v),\le k}(r)\Big)+$$

$$+\left(d-\frac{2(N+d)}{q}\right)\sum_{v=1}^{q}N^{(1)}_{(f_1,H_v),\le k}(r)\ge$$

$$\ge\left(\frac{2}{q}+\frac{d}{2N}-\frac{N+d}{Nq}\right)\sum_{v=1}^{q}(N^{(N)}_{(f_1,H_v),\le k}(r)+N^{(N)}_{(f_2,H_v),\le k}(r))\ge$$

$$\ge\left(\frac{2}{q}+\frac{d}{2N}-\frac{N+d}{Nq}\right)\left(\frac{(q-N-1)(k+1)-Nq}{k+1-N}\right)T(r)+o(T(r))=$$

$$=\frac{dq+2N-2d}{2Nq}\frac{(q-N-1)k-Nq+q-N-1}{k+1-N}T(r)+o(T(r)).$$

Letting $r\longrightarrow+\infty$ $(r\in T_I)$, we get

$$1\ge\frac{dq+2N-2d}{2Nq}\frac{(q-N-1)k-Nq+q-N-1}{k+1-N}.$$

Thus,

$$k(dq(q-N-3)-2(N+1)(N-d))\le d(N-1)(q-2)q+(qd+2N-2d)(N+1).$$

By the assumption of the theorem we see that $dq(q-N-3)-2(N+1)(N-d)>0$, and then the above inequality yields that

$$k\le\frac{d(N-1)(q-2)q+(qd+2N-2d)(N+1)}{dq(q-N-3)-2(N+1)(N-d)}.$$

This is a contradiction. Thus, there does not exist the index i such that $P_{1i}\not\equiv 0$ with $i\le[q/2]+1$.

Therefore, we have

$$\frac{(f_1,H_1)}{(f_2,H_1)}=\ldots=\frac{(f_1,H_{[q/2]+1})}{(f_2,h_{[q/2]+1})}.$$

The assertion (a) is proved.

We prove the assertion (b). As the first part, we use the same notations.

Claim 4.2. If $P_{ij}\not\equiv 0$, then the following assertion holds:

$$T(r)\ge\Big(N^{(N)}_{(f_1,H_i),\le k}(r)+N^{(N)}_{(f_2,H_i),\le k}(r)-(N+d)N^{(1)}_{(f_1,H_i),\le k}(r)\Big)+$$

$$+\sum_{v=1}^{q} dN^{(1)}_{(f,H_v),\le k}(r) + o(T(r)).$$

As in the proof of the first assertion, we fix $z \notin I$ satisfying $\nu_{(f_1,H_t),\le k}(z) > 0,\ t \neq i$. Then there exists an index l such that $f_{1l}(z)f_{2l}(z) \neq 0$. This implies that

$$\mathcal{D}^{\alpha}\left(\frac{P_{ij}}{f_{1l}f_{2l}}\right)(z) = 0 \quad \forall|\alpha| < d.$$

Hence, $\nu_{P_{ij}}(z) \ge d$.

For a point $z \notin I$ satisfying $\nu_{(f_1,H_i),\le k}(z) > 0$, we also get

$$\nu_{P_{ij}}(z) \ge \min\{N, \nu_{(f_1,H_i),\le k}(z)\} + \min\{N, \nu_{(f_2,H_i),\le k}(z)\} - N\min\{1, \nu_{(f_1,H_i),\le k}(z)\}.$$

Therefore, we have

$$\nu_{P_{ij}} \ge \min\{N, \nu_{(f_1,H_i),\le k}\} + \min\{N, \nu_{(f_2,H_i),\le k}\} -$$
$$-(N+d)\min\{1, \nu_{(f_1,H_i),\le k}\} + d\sum_{v=1}^{q}\min\{1, \nu_{(f_1,H_v),\le k}\}$$

outside an analytic set of codimension two. This implies that

$$N_{P_{ij}} \ge N^{(N)}_{(f_1,H_1),\le k}(r) + N^{(N)}_{(f_2,H_1),\le k}(r) - (N+d)N^{(1)}_{(f_1,H_1),\le k}(r) + d\sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\le k}(r).$$

By Jensen's formula, we get

$$T(r) \ge N_{P_{ij}}(r) + o(T(r)).$$

This proves the claim.

Now we suppose that $f_1 \neq f_2$, then there exists an index $i \neq 1$ such that $P_{1i} \not\equiv 0$. For $r \in T_I$, we obtain

$$\Big\| T(r) \ge \left(N^{(N)}_{(f_1,H_1),\le k}(r) + N^{(N)}_{(f_2,H_1),\le k}(r) - (N+d)N^{(1)}_{(f_1,H_1),\le k}(r)\right) + d\sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\le k}(r) \ge$$

$$\ge \frac{1}{q}\sum_{v=1}^{q}\left(N^{(N)}_{(f_1,H_v),\le k}(r) + N^{(N)}_{(f_2,H_v),\le k}(r) - (N+d)N^{(1)}_{(f_1,H_v),\le k}(r)\right) + d\sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\le k}(r) =$$

$$= \frac{1}{q}\sum_{v=1}^{q}\left(N^{(N)}_{(f_1,H_v),\le k}(r) + N^{(N)}_{(f_2,H_v),\le k}(r)\right) + \left(d - \frac{N+d}{q}\right)\sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\le k}(r) \ge$$

$$\ge \left(\frac{1}{q} + \frac{d}{2N} - \frac{N+d}{2Nq}\right)\sum_{v=1}^{q}(N^{(N)}_{(f_1,H_v),\le k}(r) + N^{(N)}_{(f_1,H_v),\le k}(r)) \ge$$

$$\ge \left(\frac{1}{q} + \frac{d}{2N} - \frac{N+d}{2Nq}\right)\left(\frac{(q-N-1)(k+1) - Nq}{k+1-N}\right)T(r) + o(T(r)) =$$

$$= \frac{dq+N-d}{2Nq}\,\frac{(q-N-1)k - Nq + q - N - 1}{k+1-N}T(r) + o(T(r)).$$

Letting $r \longrightarrow +\infty$ $(r \in T_I)$, we have

$$1 \geq \frac{dq + N - d}{2Nq} \frac{(q - N - 1)k - Nq + q - N - 1}{k + 1 - N}.$$

Thus,

$$k(dq^2 - (N+1)(d+1)q - N^2 + 1) \leq q(N-1)(dq - N - 1) + (dq + N - 1)(N+1).$$

By the assumption of the theorem, we see that $dq^2 - (N+1)(d+1)q - N^2 + 1 > 0$, and then the above inequality yields that

$$k \leq \frac{q(N-1)(dq - N - 1) + (dq + N - 1)(N+1)}{dq^2 - (N+1)(d+1)q - N^2 + 1}.$$

This is a contradiction. Thus, $f_1 = f_2$. The assertion (b) is proved.

We prove the assertion (c). Suppose that $f_1 \not\equiv f_2$, then there exist two index $i, j \in \{0, \ldots, N\}$ such that $P = f_{1i}f_{2j} - f_{1j}f_{2i} \not\equiv 0$.

We fix a point $z \notin I$ with $\nu_{(f_1,H_v)(z),\leq k} > 0$. Then there exists an index l such that $f_{1l}(z)f_{2l}(z) \neq \neq 0$. This implies that

$$\mathcal{D}^\alpha\left(\frac{P}{f_{1l}f_{2l}}\right)(z) = 0 \quad \forall |\alpha| < N.$$

Hence, $\nu_P(z) \geq d$. Therefore, we have

$$\nu_P \geq d \sum_{v=1}^{q} \min\{1, \nu_{(f_1,H_v),\leq k}\}$$

outside an analytic set of codimension two. By Lemma 2.3, this implies that

$$N_{P_{ij}} \geq d \sum_{v=1}^{q} N^{(1)}_{(f_1,H_v),\leq k}(r) \geq \frac{d}{N} \sum_{v=1}^{q} N^{(N)}_{(f_1,H_v),\leq k}(r) \geq \frac{d}{N} \frac{(q - N - 1)(k+1) - Nq}{k + 1 - N} T(r, f_1).$$

By Jensen's formula and the above inequality, we have

$$T(r) \geq N_{P_{ij}}(r) + o(T(r)) \geq \frac{d}{N} \frac{(q - N - 1)(k+1) - Nq}{k + 1 - N} T(r, f_1) + o(T(r)).$$

Similarly, we get

$$T(r) \geq \frac{d}{N} \frac{(q - N - 1)(k+1) - Nq}{k + 1 - N} T(r, f_2) + o(T(r)).$$

Therefore,

$$T(r) \geq \frac{d}{2N} \frac{(q - N - 1)(k+1) - Nq}{k + 1 - N} T(r) + o(T(r)).$$

Letting $r \longrightarrow +\infty$ $(r \in T_I)$, we get

$$1 \geq \frac{d}{2N} \frac{(q - N - 1)(k+1) - Nq}{k + 1 - N}.$$

Thus,

$$k(d(q-N-1)-2N) \leq d(Nq-q+N+1)-2N^2+2N.$$

By the assumption of the theorem, we see that $d(q-N-1)-2N>0$, and then the above inequality yields that

$$k \leq \frac{d(Nq-q+N+1)-2N^2+2N}{d(q-N-1)-2N}.$$

This is a contradiction. Thus, $f_1 = f_2$. The assertion (c) is proved.

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