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RESONANT EQUATIONS WITH CLASSICAL ORTHOGONAL POLYNOMIALS. II

РЕЗОНАНСНІ РІВНЯННЯ З КЛАСИЧНИМИ ОРТОГОНАЛЬНИМИ ПОЛІНОМАМИ. II

We study some resonant equations related to the classical orthogonal polynomials on infinite intervals, i.e., the Hermite and the Laguerre orthogonal polynomials, and propose an algorithm of finding their particular and general solutions in the closed form. The algorithm is especially suitable for the computer-algebra tools, such as Maple. The resonant equations form an essential part of various applications, e.g., of the efficient functional-discrete method for the solution of operator equations and eigenvalue problems. These equations also appear in the context of supersymmetric Casimir operators for the di-spin algebra, as well as of the square operator equations $A^2u = f$, e.g., of the biharmonic equation.

Вивчаються резонансні рівняння, що пов'язані з класичними ортогональними многочленами, заданими на нескінченних інтервалах, тобто з ортогональними многочленами Ерміта і Лагерра. Запропоновано алгоритм знаходження їхніх частинних розв'язків і загального розв'язку в замкненому вигляді. Цей алгоритм є особливо зручним в імплементації засобами комп'ютерної алгебри, наприклад, Maple. Резонансні рівняння є вагомою складовою різних застосувань, наприклад ефективного функціонально-дискретного методу розв'язування операторних рівнянь і задач на власні значення. Такі рівняння виникають також у контексті суперсиметричних операторів Казиміра для ді-спінової алгебри, а також при розв'язуванні операторних рівнянь з квадратом деякого оператора, наприклад бігармонічного рівняння.

1. Introduction. This paper represents the second part of the eponymous paper from the previous issue of this journal. Here we study the resonant equations with the differential operators defining the classical orthogonal polynomials on infinity intervals, namely the Hermite and the Laguerre orthogonal polynomials. We use the Algorithm 3.1 from part I (see [4]) to obtain the particular solutions of the corresponding resonant equations of the first and of the second kind. We obtain explicit formulas for the general solutions of the corresponding inhomogeneous resonant differential equations.

2. Resonant equation of the Hermite type. ***2.1. The Hermite resonant equation of the first kind.*** In this section we consider the following resonant gather of the Hermite type:

$$\exp\left(x^2\right)\frac{d}{dx}\left[\exp\left(-x^2\right)\frac{du(x)}{dx}\right]+2nu(x)=H_n(x), \tag{2.1}$$

where $H_n(x)$ is the Hermite polynomial, satisfying the homogeneous differential equation. The Hermite polynomial $H_n(x)$ can be represented through the hypergeometric function

$$H_\nu(x)=\frac{2^\nu\sqrt{\pi}\left(\dfrac{1-\nu}{2}\right)_{\left[\frac{n}{2}\right]+1}}{\Gamma\left(\left[\dfrac{n}{2}\right]+1+\dfrac{1-\nu}{2}\right)}\,{}_1F_1\left[-\frac{\nu}{2};\frac{1}{2};x^2\right]-\frac{2^{\nu+1}\sqrt{\pi}\left(\dfrac{-\nu}{2}\right)_{\left[\frac{n}{2}\right]+1}}{\Gamma\left(\left[\dfrac{n}{2}\right]+1-\dfrac{\nu}{2}\right)}\,x\,{}_1F_1\left[\frac{1-\nu}{2};\frac{3}{2};x^2\right] \tag{2.2}$$

for $\nu = n$ (see [6, p. 147]). The general solution of the homogeneous equation (2.1) is given by

$$u(x) = c_1 H_n(x) + c_2 h_n(x),$$

where

$$h_n(x) = -\int\limits_{\infty}^{\infty} \frac{\exp\left(-t^2\right) H_n(t)}{t-z}\, dt, \quad n = 0, 1, 2, \ldots, \quad x \in \mathbb{C} \setminus (-\infty, \infty),$$

is the Hermite functions of the second kind, which satisfies the recurrence equation for the Hermite polynomials. This function can be expressed also through the confluent hypergeometric function in the following way [5]:

$$h_{2n}(x) = (-1)^n 2^{n+1} (2n)!! x {}_1F_1\left(-\frac{2n-1}{2}; \frac{3}{2}; x^2\right) =$$

$$= \left[p_{2n}(x) \exp\left(x^2\right) + \sqrt{\pi} H_{2n}(x) \mathrm{erfi}(x)\right], \quad n = 1, 2, \ldots,$$

$$h_{2n+1}(x) = (-1)^{n+1} (2n)!! 2^{n+1}_1 F_1\left(-\frac{2n+1}{2}; \frac{1}{2}; x^2\right) =$$

$$= \left[p_{2n+1}(x) \exp\left(x^2\right) + \sqrt{\pi} H_{2n+1}(x) \mathrm{erfi}(x)\right], \quad n = 0, 1, \ldots.$$

These formulas were obtained by Maple solving the Hermite differential equation and they satisfy the difference equation

$$p_{n+1}(x) = 2x p_n(x) - 2n p_{n-1}(x), \quad n = 1, 2, \ldots,$$
$$p_0(x) = 0, \quad p_1(x) = -2. \tag{2.3}$$

The formulas for the odd and the even indexes can be unified in the following formula:

$$h_n(x) = (-1)^{\left[\frac{n+1}{2}\right]} 2^{\left[\frac{n}{2}\right]+1} \left(2\left[\frac{n}{2}\right]\right)!!\, x^{2\left\{\frac{n+1}{2}\right\}} \times$$

$$\times {}_1F_1\left(-\frac{n}{2} + \left\{\frac{n+1}{2}\right\}; \frac{1}{2} + 2\left\{\frac{n+1}{2}\right\}; x^2\right), \quad n = 0, 1, \ldots, \tag{2.4}$$

where $[x]$ and $\{x\}$ denote the integer and the fractional parts of the real number x.

The last expression can be transformed to

$$h_n(x) = H_n(x)\sqrt{\pi}\mathrm{erfi}(x) + p_n(x) \exp\left(x^2\right),$$

where the polynomials $p_n(x)$ satisfy the recurrence equation (2.3).

We use Theorem 3.1 of [4] to find a particular solution of the inhomogeneous equation. We begin with the case $n = 0$, i.e., we differentiate representation (2.2) by ν, i.e.,

$$\tilde{u}_0(x) = -\frac{1}{2} \frac{d}{d\nu}\, {}_1F_1\left(\frac{-\nu}{2}; \frac{1}{2}; x^2\right)\bigg|_{\nu=0},$$

set thereafter $\nu = 0$ and omit some summands, which satisfy the homogeneous equation

$$u_0(x) = \frac{1}{4}\sum_{p=1}^{\infty} \frac{x^{2p}}{p\left(\frac{1}{2}\right)_p} = \frac{\sqrt{\pi}}{2}\int\limits_0^x \exp\left(t^2\right)[1-\operatorname{erfc}(t)]\,dt. \tag{2.5}$$

Analogously in order to obtain $u_1(x)$ we set $n = 1$ in (2.2), differentiate by ν, substitute $\nu = 1$ and omit some summands satisfying the homogeneous differential equation. Then we obtain with the assistance of Maple

$$u_1(x) = -x\frac{d}{d\nu}\,{}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};x^2\right)\Bigg|_{\nu=1} = \frac{1}{2}\,x\sum_{p=1}^{\infty}\frac{x^{2p}}{p\left(\frac{3}{2}\right)_p} =$$

$$= \frac{x}{2}\int\limits_0^x \frac{1}{t^2}\left\{\sqrt{\pi}\exp\left(t^2\right)[1-\operatorname{erfc}(t)] - 2t\right\}\,dt, \tag{2.6}$$

where $\operatorname{erfc}(x)$ is the imaginary error function [2].

One can observe that this way to obtain particular solutions is very cumbersome. Below one can see that Algorithm 3.1 from part I [4] provides a more comfortable way.

Actually, let us differentiate the recurrence relation for the Hermite polynomials

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

by n, then using Theorem 3.1 of [4] we obtain the following recursion:

$$u_{n+1}(x) = 2xu_n(x) - 2nu_{n-1}(x) + H_{n-1}(x), \quad n = 1, 2, \ldots. \tag{2.7}$$

Using (2.5), (2.6) we have the following expressions as particular solutions:

$$\chi_0(x) = E(x) = \frac{\sqrt{\pi}}{2}\int\limits_0^x \operatorname{erf}(t)\exp\left(t^2\right)\,dt,$$

$$\chi_1(x) = \sum_{p=0}^{1}(-1)^p C_1^p H_{1-p}(x)\frac{d^p}{dx^p}E(x) + x.$$

Further we use the ansatzes

$$u_0(x) = \chi_0(x) + c_0, \quad u_1(x) = \chi_1(x) + c_1 x \tag{2.8}$$

with undefined coefficients c_0, c_1 for the initial values of Algorithm 3.1 from part I (see [4]). Substituting these into the recurrence equation (2.7) with $n = 1$ and choosing these coefficients so that $u_2(x)$ satisfies the resonant equation, we obtain that c_0 can be arbitrary and c_1 should satisfy the equation

$$4 + 4c_1 = 0,$$

i.e., $c_1 = -1$. Note that if we choose $c_0 = 0$, then we arrive at the representation

$$u_n(x) = \sum_{p=0}^{n} (-1)^p C_n^p H_{n-p}(x) \frac{d^p}{dx^p} E(x),$$
$$E(x) = \frac{\sqrt{\pi}}{2} \int_0^x \operatorname{erf}(t) \exp\left(t^2\right) dt \tag{2.9}$$

which for $n = 0, 1, 2$ was obtained in [3].

We have constructed $u_k(x),\ k = 0, 1, 2,$ so, that these functions are particular solutions of the Hermite resonant equation of the first kind. The next theorem shows that it is the case for all $n = 0, 1, 2, \ldots$.

Theorem 2.1. *The functions* $u_k(x),\ k = 3, 4, \ldots,$ *obtained by the recursion* (2.7) *with the initial conditions* $u_k(x),\ k = 0, 1,$ *given by* (2.8) *and* $c_0 = 0,\ c_1 = -1,$ *satisfy the resonant Hermite differential equation of the first kind.*

Proof. We prove this assertion by induction.

Let us assume that $u_p(x),\ p = 0, 1, \ldots, n,$ all satisfy the resonant Hermite differential equation of the first kind (2.1). Applying to the recurrence equation (2.7) the Hermite differential operator

$$\mathcal{A}_{n+1} = \frac{d^2}{dx^2} - 2x \frac{d}{dx} + 2(n+1),$$

and using the induction assumption, we obtain

$$\mathcal{A}_{n+1} u_{n+1}(x) = H_{n+1}(x) + \left[4 \frac{du_n(x)}{dx} - 8n\, u_{n-1}(x) + 4\, H_{n-1}(x)\right]. \tag{2.10}$$

Further we use the classical relation (see, e.g., [3], § 10.13)

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x).$$

Differentiating this equality by n and using Theorem 3.1 of [4] we get

$$-2 \frac{du_n(x)}{dx} = -4nu_{n-1}(x) + 2H_{n-1}(x),$$

which shows that the square bracket in (2.10) is equal to zero.

Theorem 2.1 is proved.

***Remark* 2.1.** Despite their beauty the formulas (2.9) are uncomfortable for the practical calculations because it requires differentiation. From this point of view our recurrent algorithm is more comfortable and can be easily performed using a computer algebra tool like Maple.

Now, the general solution of the resonant equation (2.1) is given by

$$u(x) = c_1 H_n(x) + c_2 h_n(x) + u_n(x), \tag{2.11}$$

where $c_1,\ c_2$ are arbitrary constants.

2.2. The Hermite resonant equation of the second kind. Let us consider the resonant equation

$$\exp\left(x^2\right) \frac{d}{dx} \left[\exp\left(-x^2\right) \frac{du_n(x)}{dx}\right] + 2nu_n(x) = h_n(x), \tag{2.12}$$

where $h_n(x)$ are the Hermite functions of the second kind defined by (2.4).

Due to Theorem 3.1 of [4] we have a particular solution of the Hermite resonant equation (2.12) of the second kind in the form

$$u_n(x) = (-1)^{\left[\frac{n+1}{2}\right]} 2^{\left[\frac{n}{2}\right]+1} \left(2\left[\frac{n}{2}\right]\right)!! \left[\frac{\left(-\frac{n-1}{2}\right)_n}{\left(\frac{1}{2}\right)^2_{\left[\frac{n}{2}\right]}} x \frac{\partial}{\partial \nu} {}_1F_1\left(-\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}; x^2\right) - \right.$$

$$\left. - \frac{\left(-\frac{n}{2}+1\right)_{n-1}}{\left(\frac{1}{2}\right)^2_{\left[\frac{n}{2}\right]}} \frac{\partial}{\partial \nu} {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; x^2\right) \right]_{\nu=n}, \tag{2.13}$$

where $(a)_{-1} = 0$. The general solution of (2.12) possesses the form (2.11).

To obtain a recursive algorithm for particular solutions, we differentiate the recurrence equation for the Hermite functions of the second kind by n and obtain

$$u_{n+1}(x) = 2xu_n(x) - 2nu_{n-1}(x) + h_{n-1}(x), \quad n = 1, 2, \ldots. \tag{2.14}$$

From (2.13) we extract the following particular solutions for $n = 0, 1$:

$$\chi_0(x) = \sqrt{\pi} \int_0^x \int_0^t \operatorname{erfi}(\xi) \exp\left(-\xi^2\right) d\xi \exp\left(t^2\right) dt,$$

$$\chi_1(x) = \left(-2 \int_0^x \left(\sqrt{\pi}\operatorname{erfi}(\xi)\xi - \exp\left(\xi^2\right)\right)^2 \exp\left(-\xi^2\right) d\xi\, x + \right.$$

$$\left. + 2\int_0^x \left(\sqrt{\pi}\operatorname{erfi}(\xi)\xi \exp\left(-\xi^2\right) - 1\right)\xi\, d\xi \right) \left(\sqrt{\pi}\operatorname{erfi}(x)x - \exp\left(x^2\right)\right) \tag{2.15}$$

which we modify to the initial values for the recursion (2.14) so that $u_2(x)$ satisfies the differential equation. With this aim we use the ansatzes

$$u_0(x) = \chi_0(x) + c_0 h_0(x), \quad u_1(x) = \chi_1(x) + c_1 h_1(x) \tag{2.16}$$

with undefined coefficients c_1, c_2. Substituting these into (2.14) with $n = 1$ we demand that $u_2(x)$ satisfies the resonant differential equation and obtain for the arbitrary constants and for the particular solution $u_2(x)$ the following formulas:

$$c_0 = \frac{3}{8}, \quad c_1 = \frac{1}{4},$$

$$u_2(x) = 2xu_1(x) - 2u_0(x) + h_0(x) =$$

$$= -2\chi_0(x) + 2x\chi_1(x) + \left(x^2 + \frac{1}{4}\right)\sqrt{\pi}\,\mathrm{erfi}(x) - x\exp\left(x^2\right).$$

Thus, we have a particular solution in the form

$$u_n(x) = -2H_{n-2}(x)\chi_0(x) + p_n(x)\chi_1(x) + q_n(x)\sqrt{\pi}\mathrm{erfi}(x) + v_n(x)\exp\left(x^2\right), \qquad (2.17)$$

where the polynomials $p_n(x)$ satisfy the recurrence equation for the Hermite polynomials with the initial conditions $p_0(x) = 0,\ p_1(x) = 1,$ the polynomials $q_n(x)$ solve the initial value problem

$$q_{n+1}(x) = 2xq_n(x) - 2nq_{n-1}(x) + H_{n-1}(x), \quad n = 1, 2, \ldots,$$

$$q_0(x) = \frac{3}{8}, \quad q_1(x) = \frac{x}{2},$$

and the polynomials $v_n(x)$ solve the discrete problem

$$v_{n+1}(x) = 2xv_n(x) - 2nv_{n-1}(x) + p_{n-1}(x), \quad n = 1, 2, \ldots,$$

$$v_0(x) = 0, \quad v_1(x) = -\frac{1}{2}.$$

The particular solutions $u_n(x)$ of the Hermite resonant equation of the second kind satisfy the resonant differential equation by construction for $n = 0, 1, 2$. The next theorem shows that it is the case for all $n = 0, 1, 2, \ldots.$

Theorem 2.2. *The functions $u_k(x)$ obtained by the recursion* (2.17) *with the initial conditions $u_k(x),\ k = 0, 1,$ given by* (2.16) *satisfy the resonant Hermite differential equation of the second kind for all $k = 3, 4, \ldots.$*

The proof is completely analogous to the one of Theorem 2.1 if we take into account that the Hermite functions of the second kind (which are not polynomials!) satisfy the same recurrence equation as the Hermite polynomials and the same differentiation formula.

3. Resonant equation of the Laguerre type. ***3.1. The Laguerre resonant equation of the first kind.*** In this section we consider the following equation of the Laguerre type:

$$x\frac{d^2u(x)}{dx^2} + (1 + \alpha - x)\frac{du(x)}{dx} + n\,u(x) = L_n^{\alpha}(x), \qquad (3.1)$$

where

$$L_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!}\,\Phi(-n, \alpha+1, x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$$

is the Laguerre polynomial satisfying the homogeneous differential equation corresponding to (3.1). This polynomial can be represented through the confluent hypergeometric function (i.e., through the solution of a confluent hypergeometric equation, which is a degenerate form of the hypergeometric differential equation when two of the three regular singularities merge into an irregular singularity) [1, p. 189] (formula (14)). Since the Laguerre polynomial solves the homogeneous equation, the inhomogeneous equation (3.1) is resonant.

The second linear independent solution of the homogeneous differential equation is the Laguerre function of the second kind [7, p. 16, 20]. Solving the corresponding differential equation by Maple we obtain the following representation of the Laguerre function of the second kind for non-integer α:

$$l_n^\alpha(x) = x^{-\alpha}\, {}_1F_1(-n-\alpha; -\alpha+1; x) = \Gamma(1-\alpha, -x) L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x) \exp(x),$$

$$p_{n+1}^\alpha(x) = \frac{1}{n+1}\left[(2n+\alpha+1-x)p_n^\alpha(x) - (n+\alpha)p_{n-1}^\alpha(x)\right], \quad n = 1, 2, \ldots, \tag{3.2}$$

$$p_0^\alpha(x) = 1, \quad p_1^\alpha(x) = 1 - x,$$

where

$$\Gamma(a, z) = \int_z^\infty e^{-t} t^{a-1} dt$$

is the incomplete Gamma function. For non-negative integer α we have

$$l_n^\alpha(x) = \mathrm{Ei}_1(-x) L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x) \exp(x),$$

$$p_1^\alpha(x) = -x^\alpha + \sum_{p=1}^{\alpha} (p-1)!(\alpha - p + 1)_+ x^{\alpha - p}, \quad (y)_+ = \begin{cases} y, & y > 0, \\ 0, & y \le 0, \end{cases}$$

$$p_0^\alpha(x) = x^{\alpha-1} + x^\alpha \left[U(2, 2, -x) + (-1)^\alpha \alpha! U(1+\alpha, 1+\alpha, -x)\right] = \sum_{p=1}^{\alpha} x^{\alpha-p}(p-1)!,$$

where

$$\mathrm{Ei}_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt, \quad |\mathrm{Arg}(z)| < \pi,$$

is the exponential integral, and $U(a, b, z)$ is the Kummer's function of the second kind. The last one is a solution of the Kummer's differential equation

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0.$$

The other linear independent solution is the Kummer's function of the first kind M defined, e.g., by a generalized hypergeometric series:

$$M(a, b, z) = \sum_{n=0}^\infty \frac{a_{(n)} z^n}{b_{(n)} n!} = {}_1F_1(a; b; z),$$

where $a_{(0)} = 1,\ a_{(n)} = a(a+1)(a+2)\ldots(a+n-1)$ is the Pochhammer symbol. The Kummer's function of the second kind can be represented as

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)}\, M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)}\, z^{1-b} M(a+1-b, 2-b, z).$$

Let $f^\alpha(z,x) = \sum_{n=0}^{\infty} z^n p_n^\alpha(x)$ be the generating function for the polynomials $p_n^\alpha(x)$ for both non-integer and integer non-negative α, then multiplying the the second equation (3.2) by z^n and summing up over n we obtain the Cauchy problem

$$(1-z)^2 \frac{\partial}{\partial z} f^\alpha(z,x) = [\alpha + 1 - x - z(1+a)]\, f^\alpha(z,x) +$$

$$+ \left(1 + 2z^2\right) p_1^\alpha(x) - (\alpha + 1 - x) p_0^\alpha(x), \quad f^\alpha(0,x) = p_0^\alpha(x),$$

with the solution

$$f^\alpha(z,x) = \sum_{n=0}^{\infty} z^n p_n^\alpha(x) = (1-z)^{-\alpha-1} \exp\left(\frac{x}{z-1}\right) \times$$

$$\times \left\{ -\int\limits_0^z \left(-\left(2t^2+1\right) p_1^\alpha(x) + (\alpha+1-x) p_0^\alpha(x) \right) (t-1)^{\alpha-1} \exp\left(-\frac{x}{t-1}\right) dt + \right.$$

$$\left. + (-1)^{-\alpha-1} p_0^\alpha(x) \exp(x) \right\}.$$

In particulary, for $\alpha = 0$, we have

$$f^0(z,x) = (1-z)^{-1}(3-x) \exp\left(\frac{xz}{z-1}\right) + (1-z)^{-1}(x-1)(x-3) \exp\left(\frac{x}{z-1}\right) \times$$

$$\times \left\{ -\mathrm{Ei}_1(-x) + \mathrm{Ei}_1\left(-\frac{x}{z-1}\right) \right\} + z - x + 3,$$

for $\alpha = 1$, we obtain

$$f^1(z,x) = (1-z)^{-2} \left(x^2 - 5x + 2\right)(1-x) \frac{1}{3} \exp\left(\frac{xz}{z-1}\right) +$$

$$+ (1-z)^{-2} \left(x^3 - 7x^2 + 12x - 3\right) \frac{x}{3} \exp\left(\frac{x}{z-1}\right) \left\{ -\mathrm{Ei}_1(-x) + \mathrm{Ei}_1\left(-\frac{x}{z-1}\right) \right\} -$$

$$-\frac{1}{3(z-1)} \left[x^3 - (z+6)x^2 + \left(2z^2 + 3z + 7\right) x - 2z^2 - 2z + 1 \right].$$

For $\alpha = 1/2$, it holds

$$f^{1/2}(z,x) = -(z-1)^{-3/2} \exp\left(\frac{xz}{z-1}\right) \times$$

$$\times \left\{ \frac{\exp(-x)}{2} \int\limits_0^z \frac{\left[1 + 4(x-1)t^2\right]}{\sqrt{t-1}} \exp\left(\frac{x}{t-1}\right) dt + 1 \right\}.$$

The general solution of the homogeneous Laguerre differential equation is given by

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x)$$

with arbitrary constants $c_1,\ c_2$. Due to Theorem 3.1 of [4] a particular solution of the Laguerre resonant differential equation of the first kind

$$x\,\frac{d^2u_\nu(x)}{dx^2} + (1-x)\,\frac{du_\nu(x)}{dx} + \nu u_\nu(x) = L_\nu(x) \equiv \Phi(-\nu,1,x)$$

is given by

$$u_\nu(x) = \frac{d}{d\nu}\,\Phi(-\nu,1,x) = -\sum_{k=1}^{\infty}\frac{x^k}{(k!)^2}\,(-\nu)_k\sum_{i=0}^{p-1}\frac{1}{-\nu+i},$$

where $\Phi(a,c;x)$ is the confluent hypergeometric function [1] (Ch. 6). Changing here $\nu\in\mathbb{R}$ to $n\in\mathbb{N}$ we obtain the resonant Laguerre equation and the corresponding particular solution

$$u_n(x) = \sum_{k=1}^{n}\frac{x^k}{(k!)^2}\,\frac{d(-\nu)_k}{d\nu}\bigg|_{\nu=n} - (-1)^n n!\sum_{k=n+1}^{\infty}\frac{x^k}{k!}\,\frac{1}{\prod_{i=0}^{n}(k-i)} = u_{n,1}(x) + u_{n,2}(x). \qquad (3.3)$$

Using the relation

$$\frac{1}{\prod_{i=0}^{n}(k-i)} = \sum_{i=0}^{n}\frac{a_i^{(n)}}{k-n+i}, \qquad a_i^{(n)} = \frac{(-1)^i}{i!(n-i)!}, \qquad (3.4)$$

we transform the sums $u_{n,1}(x),\ u_{n,2}(x)$ to a shapes, which can be computed in closed form, e.g., by Maple and we get

$$u_0(x) = -\left[\mathrm{Ei}_1(-x) + \ln(-x) + \gamma\right],$$

$$u_1(x) = -L_1(x)u_0(x) - \exp(x) + 1 + x,$$

$$u_2(x) = -L_2(x)u_0(x) + \frac{1}{2}\,(x-3)\exp(x) - \frac{3}{4}\,x^2 + x + \frac{3}{2},$$

$$u_3(x) = -L_3(x)u_0(x) + \frac{1}{6}\,\left(-x^2+8x-11\right)\exp(x) + \frac{11}{36}\,x^3 - \frac{7}{4}\,x^2 + \frac{1}{2}\,x + \frac{11}{6},$$

$$u_4(x) = -L_4(x)u_0(x) + \frac{1}{24}\,\left(x^3-15x^2+58x-50\right)\exp(x) -$$

$$-\frac{25}{288}\,x^4 + \frac{19}{18}\,x^3 - \frac{7}{4}\,x^2 - \frac{1}{3}\,x + \frac{25}{12},$$

where $\gamma = 0.5772156649\ldots$ is the Euler's constant. Having in mind to obtain a closed form of the sum $u_{n,2}(x)$, we note that

$$v_i^{(n)}(x) = \sum_{p=n+1}^{\infty}\frac{x^p}{p!(p-i)} = -\frac{x^i}{i!}\,w(x) - \frac{\exp(x)}{i!}\sum_{p=0}^{i-1}p!x^{i-1-p} -$$

$$-\sum_{p=0}^{n-i-1}\frac{x^{n-p}}{(n-p)!(n-p-i)}+\frac{x^i}{i!}\sum_{p=0}^{i-1}\frac{1}{p+1}+\sum_{p=0}^{i-1}\frac{x^p}{p!(i-p)}, \tag{3.5}$$

$$w(x)=\mathrm{Ei}_1(-x)+\ln(-x)+\gamma.$$

Then from (3.3)–(3.5) we have

$$u_{n,2}(x)=-(-1)^n n!\sum_{i=0}^{n}a_i^{(n)}v_{n-i}^{(n)}(x)=\sum_{i=0}^{n}(-1)^{n+1-i}C_n^i v_{n-i}^{(n)}(x)=$$

$$=L_n(x)w(x)-\exp(x)\sum_{p=0}^{n-1}x^p\sum_{i=0}^{n-p-1}\frac{(-1)^{n+i}n!(n-i-1-p)!}{i![(n-i)!]^2}+$$

$$+\sum_{p=0}^{n}x^p\sum_{i=0}^{n}\frac{(-1)^{n+1-i}n!}{i!(n-i)!}\,b_{p,i},$$

where

$$b_{p,i}=\begin{cases}\dfrac{1}{p!(i-p)}, & p\neq i,\\[2mm] \dfrac{1}{i!}\sum_{t=0}^{i-1}\dfrac{1}{t+1}, & p=i.\end{cases}$$

The technique presented above for $\alpha=0$ is even more cumbersome in the case $\alpha\neq 0$. This is why below we use our recursive algorithm for the particular solutions in order to be able to write down the general solution of the Laguerre resonant equation (3.1) in the form

$$u(x)=c_1L_n^{\alpha}(x)+c_2l_n^{\alpha}(x)+u_n(x),$$

with arbitrary constants $c_1,\ c_2$.

Differentiating the recurrence equation for the Laguerre polynomials by n and using Theorem 3.1 of [4] we obtain for the particular solutions the recurrence formula

$$u_{n+1}^{\alpha}(x)=\frac{2n+\alpha+1-x}{n+1}\,u_n^{\alpha}(x)-\frac{n+\alpha}{n+1}\,u_{n-1}^{\alpha}(x)+$$

$$+\frac{\alpha-1-x}{(n+1)^2}\,L_n^{\alpha}(x)-\frac{\alpha-1}{(n+1)^2}\,L_{n-1}^{\alpha}(x),\quad n=1,2,\ldots, \tag{3.6}$$

with the corresponding initial conditions. For example, in the case $\alpha=1$ we have

$$u_0^1(x)=\frac{1}{x}-\ln(x),\quad u_1^1(x)=(2-x)u_0^1(x)-x-\frac{1}{x}$$

and the following representation of the particular solution of the resonant equation

$$u_n(x)=L_n^1(x)u_0(x)+q_n(x)\left(x+\frac{1}{x}\right)+v_n(x).$$

Here the polynomial $q_n(x)$ satisfies the recurrence equation for the Laguerre polynomials with the initial conditions

$$v_0(x) = 0, \quad v_1(x) = -1.$$

The polynomial $v_n(x)$ solves the difference problem

$$v^1_{n+1}(x) = \frac{2n+2-x}{n+1}\, v^1_n(x) - v^1_{n-1}(x) - \frac{x}{(n+1)^2}\, L^1_n(x), \quad n = 1, 2, \ldots,$$
$$v^1_0(x) = 0, \quad v^1_1(x) = 0. \tag{3.7}$$

For an arbitrary α due to Theorem 3.1 of [4] we have a particular solution

$$u^\alpha_n(x) = -\frac{d}{d\nu}\, L^\alpha_\nu(x)\Big|_{\nu=n} = -\Phi(-n, \alpha+1; x)\frac{d}{d\nu}\, \frac{\Gamma(\alpha+1+\nu)}{\Gamma(\alpha+1)\Gamma(\nu+1)}\Big|_{\nu=n} -$$
$$-\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)\Gamma(n+1)}\, \frac{d}{d\nu}\, \Phi(-\nu, \alpha+1; x)\Big|_{\nu=n},$$

from where we obtain the following particular solutions for $n = 0, 1$:

$$\chi_0(x) = \frac{x}{\alpha+1}\; {}_2F_2(1, 1; 2, 2+\alpha; x),$$

$$\chi_1(x) = x\; {}_2F_2(1, 1; 2, 2+\alpha; x) - \frac{x^2}{\alpha+2}\; {}_2F_2(1, 1; 2, 3+\alpha; x).$$

With the aim to obtain from the recurrence formula solutions of the resonant differential equation we use the ansatzes

$$u^\alpha_0(x) = \chi_0(x) + c_0, \quad u^\alpha_1(x) = \chi_1(x) + c_1 L^\alpha_1(x)$$

with undefined coefficients $c_0,\ c_1$. Substituting these into (3.7) and demanding, that the particular solution $u^\alpha_1(x)$ satisfies the resonant differential equation we get

$$c_0 = -\frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)}, \quad c_1 = -\frac{\alpha}{2(\alpha+2)}.$$

Now, the initial values for the recursive algorithm for the particular solutions become to

$$u^\alpha_0(x) = \frac{x}{\alpha+1}\; {}_2F_2(1, 1; 2, 2+\alpha; x) - \frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)},$$
$$u^\alpha_1(x) = x\; {}_2F_2(1, 1; 2, 2+\alpha; x) - \frac{x^2}{\alpha+2}\; {}_2F_2(1, 1; 2, 3+\alpha; x) - \frac{\alpha}{2(\alpha+2)}\, L^\alpha_1(x). \tag{3.8}$$

The next assertion shows that the functions $u^\alpha_n(x)$ generated by recursion (3.6) with the initial values (3.8) satisfy the Laguerre resonant differential equation of the first kind for all $n = 0, 1, 2, \ldots$.

Theorem 3.1. *The functions $u^\alpha_n(x)$ generated by the recursive algorithm* (3.6) *with the initial values* (3.8) *are particular solutions of the Laguerre resonant differential equation of the first kind for all $n = 0, 1, 2, \ldots$.*

Proof. We prove the assertion by the mathematical induction. First of all we note that the functions $u_n^\alpha(x)$ for $n = 0, 1, 2$ are the particular solution due to their construction. We assume that all functions $u_p^\alpha(x),\ p = 0, 1, \ldots, n,$ are particular solutions and prove that then $u_{n+1}^\alpha(x)$ is a solution too.

Actually, the application to the both sides of (3.6) of the Laguerre differential operator

$$\mathcal{A}_{n+1}^\alpha = x\,\frac{d^2}{dx^2} + (\alpha + 1 - x)\,\frac{d}{dx} + n + 1$$

and the induction assumption provide

$$\mathcal{A}_{n+1}^\alpha u_{n+1}^\alpha(x) = L_{n+1}^\alpha(x) + \frac{2}{n+1}\left[n u_n^\alpha(x) - x\,\frac{du_n^\alpha(x)}{dx} - (n+\alpha)u_{n-1}^\alpha(x)\right] -$$

$$-\frac{2}{n+1}\left[L_n^\alpha(x) - L_{n-1}^\alpha(x)\right]. \tag{3.9}$$

Further we use the relation (see, e.g., [8], § 10.12)

$$x\,\frac{d\,L_n^\alpha(x)}{dx} = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x).$$

Differentiating this relation by n and using Theorem 3.1 of [4] we see that the both square brackets in (3.9) are equal to zero and herewith the assertion is proven.

The general representation of the particular solutions is

$$u_n^\alpha(x) = p_n^\alpha(x)\ {}_2F_2(1,1;2,2+\alpha;x) + q_n^\alpha(x)\ {}_2F_2(1,1;2,3+\alpha;x) + v_n^\alpha(x), \quad n = 2, 3, \ldots,$$

where the polynomials $p_n^\alpha(x),\ q_n^\alpha(x)$ satisfy the classical Laguerre recurrence equation with the initial conditions

$$p_0^\alpha(x) = \frac{x}{\alpha+1}, \quad p_1^\alpha(x) = x,$$

$$q_0^\alpha(x) = 0, \quad q_1^\alpha(x) = -\frac{x^2}{\alpha+2},$$

respectively. The polynomials $v_n^\alpha(x)$ satisfies the inhomogeneous recurrence equation

$$v_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1}\,v_n^\alpha(x) - \frac{n+\alpha}{n+1}\,v_{n-1}^\alpha(x) +$$

$$+\frac{\alpha-1-x}{(n+1)^2}\,L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2}\,L_{n-1}^\alpha(x), \quad n = 1, 2, \ldots,$$

with the initial conditions

$$v_0^\alpha(x) = -\frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)}, \quad v_1^\alpha(x) = -\frac{\alpha(\alpha+1-x)}{2(\alpha+2)}.$$

3.2. The Laguerre resonant equation of the first kind (revisited). In this section we consider again the resonant Laguerre differential equation of the first type (3.1) and show that the particular solutions can be represented by elementary functions only.

We know that one of the linear independent solutions of the homogeneous differential equation is the Laguerre function of the second kind [7, p. 16, 20]. Solving the corresponding differential equation by Maple we obtain the following representation of the Laguerre function of the second kind for non-integer α:

$$l_n^\alpha(x) = x^{-\alpha}\ {}_1F_1(-n-\alpha, -\alpha+1; x) = \Gamma(1-\alpha, -x) L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x)\exp(x),$$

$$p_{n+1}^\alpha(x) = \frac{1}{n+1}\left[(2n+\alpha+1-x)p_n^\alpha(x) - (n+\alpha)p_{n-1}^\alpha(x)\right], \quad n = 1, 2, \ldots,$$

$$p_0^\alpha(x) = 0, \quad p_1^\alpha(x) = 1 - x.$$

For non-negative natural $\alpha \in \mathbb{N}$ we have

$$l_n^\alpha(x) = \mathrm{Ei}_1(-x) L_n^\alpha(x) - (-x)^{-\alpha} p_n^\alpha(x)\exp(x),$$

$$p_{-1}^\alpha(x) = (\alpha-1)!, \tag{3.10}$$

$$p_0^\alpha(x) = x^{\alpha-1} + x^\alpha\left[U(2, 2, -x) + (-1)^\alpha \alpha! U(1+\alpha, 1+\alpha, -x)\right].$$

Note that the function at the second initial condition in (3.10) solves the following difference initial value problem:

$$p_0^\alpha(x) = x p_0^{\alpha-1}(x) + (\alpha-1)!, \quad \alpha = 1, 2, \ldots, \quad p_0^0(x) = 0.$$

Using Theorem 3.1 of [4] we can represent the particular solutions of the Laguerre resonant equation of the first kind by

$$u_n(x) = \frac{(-1)^{n+1}}{n!}\frac{\partial}{\partial \nu} U(-\nu, 1+\alpha, -x)\Big|_{n=\nu}, \quad n = 0, 1 \ldots.$$

This representation provides the particular solutions

$$\chi_0^\alpha(x) = u_0(x) = -\ln(x) + \sum_{p=0}^{\alpha-1}\frac{(\alpha-p)_{p+1}}{(p+1)x^{p+1}},$$

$$\chi_1^\alpha(x) = u_1(x) = -L_1^\alpha(x)\ln(x) + \sum_{p=0}^{\alpha}\frac{k_p(\alpha)}{x^p},$$

where

$$k_{p+1}(\alpha) = p\sum_{i=1}^{\alpha-1} k_p(i), \quad p = 1, 2, \ldots, \alpha-1,$$

$$k_1(\alpha) = \frac{\alpha(\alpha+1)}{2}, \quad k_0(\alpha) = -\alpha-2, \quad \alpha = 2, 3, \ldots.$$

At the first step of Algorithm 3.1 of [4] we use the ansatzes

$$u_0^\alpha(x) = \chi_0^\alpha(x) + c_0 L_0^\alpha(x) + d_0 L_1^\alpha(x),$$

$$u_1^\alpha(x) = \chi_1^\alpha(x) + c_1 L_0^\alpha(x) + d_1 L_1^\alpha(x)$$

with undefined coefficients $c_0,\ d_0,\ c_1,\ d_1$, substitute them into (3.6) with $n = 1$, obtain $u_2^\alpha(x)$ and choose $c_0,\ d_0,\ c_1,\ d_1$ so that $u_2^\alpha(x)$ satisfies the resonant differential equation. We get $d_0 = 0$, $d_1 = 0$ and $c_1 = 1 + c_0$.

Now one can prove that

$$u_n^\alpha(x) = -L_n^\alpha(x)\ln(x) + \frac{p_n^\alpha(x)}{x^\alpha},$$

where the polynomials $p_n^\alpha(x)$ satisfy the recurrence equation

$$p_{n+1}^\alpha(x) = \frac{2n+\alpha+1-x}{n+1}\,p_n^\alpha(x) - \frac{n+\alpha}{n+1}\,p_{n-1}^\alpha(x) +$$

$$+\frac{\alpha-1-x}{(n+1)^2}\,L_n^\alpha(x) - \frac{\alpha-1}{(n+1)^2}\,L_{n-1}^\alpha(x), \quad n = 1, 2, \ldots,$$

with the initial conditions

$$p_0^\alpha(x) = \sum_{p=0}^{\alpha-1} \frac{x^{\alpha-p-1}(\alpha-p)_{p+1}}{p+1} + c_0 x^\alpha, \qquad p_1^\alpha(x) = \sum_{p=0}^{\alpha} x^{\alpha-p} k_p(\alpha) + (1+c_0)x^\alpha L_1^\alpha(x).$$

3.3. The Laguerre resonant equation of the second kind. In this subsection we consider the resonant equation

$$x\,\frac{d^2u(x)}{dx^2} + (1+\alpha-x)\,\frac{du(x)}{dx} + nu(x) = l_n^\alpha(x), \tag{3.11}$$

where $l_n^\alpha(x)$ is the Laguerre function of the second kind given by (3.2).

Due to Theorem 3.1 of [4] the formula

$$u_n(x) = -\frac{d}{d\nu}\,l_\nu^\alpha(x)|_{\nu=n} \tag{3.12}$$

defines a particular solution of (3.11), so that its general solution is given by

$$u(x) = c_1 L_n^\alpha(x) + c_2 l_n^\alpha(x) + u_n(x).$$

The use of formula (3.12) for arbitrary n is rather burdensome, therefore we use Algorithm 3.1 of [4], where we for the sake of simplicity set $\alpha = 0$. Solving differential equation (3.11) with Maple for $n = 0,\ n = 1$ we get

$$\chi_0(x) = -\int_1^x \frac{\exp(t)}{t} \int_1^t \mathrm{Ei}_1(-\xi)\exp(-\xi)d\xi\,dt,$$

$$\tag{3.13}$$

$$\chi_1(x) = [(1-x)\mathrm{Ei}_1(-x) - \exp(x)] \int_1^x [1 + \mathrm{Ei}_1(-\xi)(-1+\xi)\exp(-\xi)]\,(-1+\xi)d\xi +$$

$$+\int\limits_1^x \exp(-\xi)\left[\mathrm{Ei}_1(-\xi)(-1+\xi)+\exp(-\xi)\right]^2 d\xi(-1+x).$$

As the ansatzes for initial values of our algorithm we use

$$u_0^0(x)=\chi_0(x)+c_0\mathrm{Ei}_1(-x)+d_0, \quad u_1^0(x)=\chi_1(x)+c_1 l_1^0(x)+d_1 L_1^0(x) \tag{3.14}$$

with undefined constants $c_0,\ d_0,\ c_1,\ d_1$. Differentiating the recurrence equation for the Laguerre functions of the second kind by n in regard of (3.12) we obtain the following recurrence relation for particular solutions:

$$u_{n+1}^0(x)=\frac{2n+1-x}{n+1}\,u_n^0(x)-\frac{n}{n+1}\,u_{n-1}^0(x)-\frac{1+x}{(n+1)^2}\,l_n^0(x)+\frac{1}{(n+1)^2}\,l_{n-1}^0(x). \tag{3.15}$$

We substitute (3.14) into this equation with $n=1$ and demand that the obtained function $u_2^0(x)$ satisfies the resonant differential equation (3.11) with $n=2$, then we obtain

$$\begin{gathered} c_0=-\mathrm{Ei}_1(-1)\exp(-1)-1, \quad d_0=-[\mathrm{Ei}_1(-1)\exp(-1/2)+\exp(1/2)]^2, \\ c_1=0, \quad d_1=0. \end{gathered} \tag{3.16}$$

Analogously to Theorem 3.1 the following assertion can be proven.

Theorem 3.2. *The functions* $u_n^0(x)$ *generated by the recursive algorithm* (3.15) *with the initial values* (3.14) *with the constants given by* (3.16) *are particular solutions of the Laguerre resonant differential equation of the second kind for all* $n=0,1,2,\ldots$.

It can be proven by substitution into (3.15) that the following representation holds true:

$$u_n^0(x)=p_n^0(x)\chi_1(x)+q_n^0(x)\chi_0(x)+v_n^0(x)\mathrm{Ei}_1(-x)+w_n^0(x)\exp(x)+q_n^0(x)d_0, \tag{3.17}$$

where the polynomials $p_n^0(x),\ q_n^0(x)$ satisfy the recurrence relation for the Laguerre polynomials with the initial conditions

$$p_0^0(x)=0, \quad p_1^0(x)=1, \quad q_0^0(x)=1, \quad q_1^0(x)=0.$$

The polynomials $w_n^0(x)$ satisfy the inhomogeneous recurrence relation for the Laguerre polynomials

$$w_{n+1}^0(x)=\frac{2n+1-x}{n+1}\,w_n^0(x)-\frac{n}{n+1}\,w_{n-1}^0(x)-$$

$$-\frac{1+x}{(n+1)^2}\,p_n^0(x)+\frac{1}{(n+1)^2}\,p_{n-1}^0(x), \quad n=1,2,\ldots,$$

with the initial conditions

$$w_1^0(x)=0, \quad w_2^0(x)=\frac{x+1}{4}.$$

Here $p_n^0(x)$ are the same polynomials as in (3.17).

The polynomials $v_n^0(x)$ solve the following discrete initial value problem:

$$v_{n+1}^0(x)=\frac{2n+1-x}{n+1}\,v_n^0(x)-\frac{n}{n+1}\,v_{n-1}^0(x)-\frac{1+x}{(n+1)^2}\,L_n^0(x)+$$

$$+\frac{1}{(n+1)^2}\,L^0_{n-1}(x), \quad n=1,2,\ldots,$$

$$v^0_1(x)=0, \quad v^0_2(x)=\frac{x^2-2c_0}{4}.$$

Below we give some particular solutions of the Laguerre resonant equation of the second kind obtained by our algorithm:

$$u^0_0(x)=\chi_0(x)+c_0\mathrm{Ei}_1(-x)+d_0, \quad u^0_1(x)=\chi_1(x),$$

$$u^0_2(x)=-\frac{x-3}{2}\,\chi_1(x)-\frac{1}{2}\,\chi_0(x)+\frac{x^2-2c_0}{4}\,\mathrm{Ei}_1(-x)-\frac{x^2-1}{8}\,\exp(x)-\frac{1}{2}\,d_0,$$

$$u^0_3(x)=\left(\frac{1}{6}\,x^2-\frac{4}{3}\,x+\frac{11}{6}\right)\chi_1(x)+\left(\frac{1}{6}\,x-\frac{5}{6}\right)\chi_0(x)+$$

$$+\left(-\frac{5}{36}\,x^3+\frac{7}{12}\,x^2+\frac{c_0}{6}\,x-\frac{5c_0}{6}\right)\mathrm{Ei}_1(-x)+$$

$$+\left(\frac{1}{24}\,x^3-\frac{11}{72}\,x^2-\frac{23}{72}\,x-\frac{1}{72}\right)\exp(x)+\left(\frac{1}{6}\,x-\frac{5}{6}\right)d_0,$$

where $c_0,\ d_0$ are given by (3.16) and $\chi_0,\ \chi_1$ — by (3.3).

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