UDC 517.9

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EXISTENCE RESULTS FOR A CLASS OF KIRCHHOFF-TYPE SYSTEMS WITH COMBINED NONLINEAR EFFECTS

РЕЗУЛЬТАТИ ПРО ІСНУВАННЯ РОЗВ'ЯЗКІВ ДЛЯ ОДНОГО КЛАСУ СИСТЕМ ТИПУ КІРХГОФА З КОМБІНОВАНИМИ НЕЛІНІЙНИМИ ЕФЕКТАМИ

We study the existence of positive solutions for a nonlinear system

$$-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=\lambda|x|^{-(a+1)p+c_1}f(u,v),\quad x\in\Omega,$$

$$-M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v)=\lambda|x|^{-(b+1)q+c_2}g(u,v),\quad x\in\Omega,$$

$$u=v=0,\quad x\in\partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $0\in\Omega,\ 1<p,q<N, 0\le a<\dfrac{N-p}{p}, 0\le b<\dfrac{N-q}{q}$, and c_1, c_2, and λ are positive parameters. Here, M_1, M_2, f, and g satisfy certain conditions. We use the method of sub- and supersolutions to establish our results.

Розглянуто проблему існування додатних розв'язків нелінійної системи

$$-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=\lambda|x|^{-(a+1)p+c_1}f(u,v),\quad x\in\Omega,$$

$$-M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v)=\lambda|x|^{-(b+1)q+c_2}g(u,v),\quad x\in\Omega,$$

$$u=v=0,\quad x\in\partial\Omega,$$

де Ω — обмежена гладка область в $\mathbb{R}^N$ з $0\in\Omega,\ 1<p,q<N, 0\le a<\dfrac{N-p}{p}, 0\le b<\dfrac{N-q}{q}$, а c_1, c_2, λ — додатні параметри. Величини M_1, M_2, f та g задовольняють деякі умови. Наші результати отримано за допомогою методу суб- та суперрозв'язків.

1. Introduction. In this paper we study the existence of positive solution for the nonlinear system

$$-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)=\lambda|x|^{-(a+1)p+c_1}f(u,v),\quad x\in\Omega,$$

$$-M_2\left(\int\limits_{\Omega} |\nabla v|^q\,dx\right)\mathrm{div}\left(|x|^{-bq}|\nabla v|^{q-2}\nabla v\right) = \lambda|x|^{-(b+1)q+c_2}g(u,v), \quad x\in\Omega, \tag{1.1}$$

$$u = v = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0\in\Omega,\ 1<p,\ q<N,\ 0\le a<\dfrac{N-p}{p}$, $0\le b<\dfrac{N-q}{q}$ and $c_1,\ c_2$, and λ are positive parameters. Here $M_1,\ M_2$ satisfy the following condition:

(H$_1$) $M_i\colon \mathbb{R}_0^+\to\mathbb{R}^+,\ i=1,2$, are two continuous and increasing functions and $0<m_i\le$ $\le M_i(t)\le m_{i,\infty}$ for all $t\in\mathbb{R}_0^+$, where $\mathbb{R}_0^+:\ =[0,+\infty)$.

Moreover $f,g\colon[0,\infty)\times[0,\infty)\to[0,\infty)$ are nondecreasing continuous functions. System (1.1) is related to the stationary problem of a model introduced by Kirchhoff [16]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int\limits_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where $\rho,\ P_0,\ h$, and E are all constants. This equation extends the classical D'Alembert wave equation. A distinguishing feature of equation (1.2) is that the equations a nonlocal coefficient $\dfrac{P_0}{h}+$ $+\dfrac{E}{2L}\int_0^L\left|\dfrac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\dfrac{1}{2L}\int_0^L\left|\dfrac{\partial u}{\partial x}\right|^2 dx$; hence, the equation is no longer a pointwise identity. Nonlocal problems can be used for modeling, for example, physical and biological systems for which u describes a process which depends on the average of itself, such as the population density. The elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-\,\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)$, were motivated by the following Caaffarelli, Kohn and Nirenberg's inequality (see [6, 24]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in Newtonian fluids, in flow through porous media and in glaciology (see [4, 8]). On the other hand, quasilinear elliptic systems has an extensive practical background. It can be used to describe the multiplicate chemical reaction catalyzed by the catalyst grains under constant or variant temperature, it can be used in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [13, 23]), and can be a simple model of tubular chemical reaction, more naturally, it can be a correspondence of the stable station of dynamical system determined by the reaction-diffusion system, see Ladde and Lakshmikantham et al. [17]. More naturally, it can be the populations of two competing species [10]. So, the study of positive solutions of elliptic systems has more practical meanings. We refer to [5, 12, 14, 22] for additional results on elliptic problems. We are inspired by the ideas in the interesting paper [21], in which the authors considered (1.1) in the case $M_1(t)=M_2(t)\equiv 1$. Using the sub-supersolution method combining a comparison principle introduced in [3], the authors established the existence of a positive solution for (1.1) when the parameter λ is large. The concepts of sub- and supersolution were introduced by Nagumo [19] in 1937 who proved, using also the shooting method, the existence of at least one solution for a class of nonlinear Sturm–Liouville problems. In fact, the premises of the sub- and supersolution method can be traced back to Picard. He applied, in the early 1880s, the method of successive approximations to argue the existence of solutions for nonlinear elliptic equations that are suitable perturbations of uniquely solvable linear problems. This is the starting point of the use of sub- and supersolutions in connection with monotone methods. Picard's techniques were applied later by Poincaré [20] in connection with problems arising in astrophysics.

2. Preliminary results. In this paper, we denote $W_0^{1,r}\big(\Omega, |x|^{-ar}\big)$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = \left(\int_\Omega |x|^{-ar}|\nabla u|^r dx\right)^{1/r}$ with $r = p, q$. To precisely state our existence result we consider the eigenvalue problem

$$-\operatorname{div}\big(|x|^{-sr}|\nabla\phi|^{r-2}\nabla\phi\big) = \lambda|x|^{-(s+1)p+t}|\phi|^{r-2}\phi, \qquad x \in \Omega, \quad \phi = 0, \quad x \in \partial\Omega. \tag{2.1}$$

For $r = p, s = a$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2.1) such that $\phi_{1,p}(x) > 0$ in Ω and $\|\phi_{1,p}\|_\infty = 1$, and for $r = q,\ s = b$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1;q}$ of (2.1) such that $\phi_{1,q}(x) > 0$ in Ω and $\|\phi_{1,q}\|_\infty = 1$ (see [18, 25]). It can be shown that $\dfrac{\partial\phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$. Here n is the outward normal. This result is well known and hence, depending on Ω there exist positive constants $\epsilon,\ \delta,\ \sigma_r$ such that

$$\lambda_{1,r}|x|^{-(s+1)r+t}\phi_{1,r}^r - |x|^{-sr}|\nabla\phi_{1,r}|^r \le -\epsilon, \quad x \in \overline{\Omega}_\delta, \tag{2.2}$$

$$\phi_{1,r} \ge \sigma_r, \quad x \in \Omega \setminus \overline{\Omega}_\delta, \tag{2.3}$$

with $r = p, q;\ s = a,\ b;\ t = c_1,\ c_2$ and $\overline{\Omega}_\delta = \big\{x \in \Omega \mid d(x, \partial\Omega) \le \delta\big\}$ (see [18]). We will also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0^{1,p}(\Omega, \|x\|^{-ap}) \times W_0^{1,q}(\Omega, \|x\|^{-bq})$ for the system

$$-\operatorname{div}\big(|x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p\big) = |x|^{-(a+1)p+c_1}, \quad x \in \Omega,$$

$$-\operatorname{div}\big(|x|^{-bq}|\nabla\zeta_q|^{q-2}\nabla\zeta_q\big) = |x|^{-(b+1)q+c_2}, \quad x \in \Omega,$$

$$u = v = 0, \quad x \in \partial\Omega,$$

to discuss our existence result. It is known that $\zeta_r(x) > 0$ in Ω and $\dfrac{\partial\zeta_r(x)}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$ (see [18]).

3. Existence results. In this section, we shall establish our existence result via the method of sub- and supersolutions. A pair of nonnegative functions $(\psi_1, \psi_2),\ (z_1, z_2)$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$M_1\left(\int_\Omega |\nabla\psi_1|^p\, dx\right)\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w\, dx \le \lambda\int_\Omega |x|^{-(a+1)p+c_1} f(\psi_1, \psi_2) w\, dx,$$

$$M_2\left(\int_\Omega |\nabla\psi_2|^q\, dx\right)\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w\, dx \le \lambda\int_\Omega |x|^{-(b+1)q+c_2} g(\psi_1, \psi_2) w\, dx,$$

and

$$M_1\left(\int\limits_\Omega |\nabla z_1|^p\,dx\right)\int\limits_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx \ge \lambda\int\limits_\Omega |x|^{-(a+1)p+c_1}f(z_1,z_2)w\,dx,$$

$$M_2\left(\int\limits_\Omega |\nabla z_2|^q\,dx\right)\int\limits_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx \ge \lambda\int\limits_\Omega |x|^{-(b+1)q+c_2}g(z_1,z_2)w\,dx,$$

for all $w\in W=\big\{w\in C_0^\infty(\Omega)|w\ge 0\in\Omega\big\}$.

We make the following assumptions:

(H_2) $f,g\colon[0,\infty)\times[0,\infty)\to[0,\infty)$ are C^1 functions such that $f_u,\ f_v,\ g_u,\ g_v\ge 0$ and

$$\lim_{u,v\to\infty} f(u,v)=\lim_{u,v\to\infty} g(u,v)=\infty;$$

(H_3) for every $A>0$,

$$\lim_{x\to\infty}\frac{f\big(x,A[g(x,x)]^{1/q-1}\big)}{x^{p-1}}=0;$$

(H_4) $\lim_{x\to\infty}\dfrac{g(x,x)}{x^{q-1}}=0$.

A key role in our arguments will be played by the following auxiliary result. Its proof is similar to those presented in [9], the reader can consult further the papers [1, 2, 15].

Lemma 3.1. *Assume that $M:\mathbb{R}_0^+\to\mathbb{R}^+$ is continuous and increasing, and there exists $m_0>0$ such that $M(t)\ge m_0$ for all $t\in\mathbb{R}_0^+$. If the functions $u,v\in W_0^{1,p}(\Omega,|x|^{-ap})$ satisfy*

$$M\left(\int\limits_\Omega |\nabla u|^p\,dx\right)\int\limits_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx\le$$

$$\le M\left(\int\limits_\Omega |\nabla v|^p\,dx\right)\int\limits_\Omega |x|^{-ap}|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx \tag{3.1}$$

for all $\varphi\in W_0^{1,p}(\Omega,|x|^{-ap})$, $\varphi\ge 0$, then $u\le v$ in Ω.

From Lemma 3.1 we can establish the basic principle of the sub- and supersolutions method for nonlocal systems. Indeed, we consider the nonlocal system

$$-M_1\left(\int\limits_\Omega |\nabla u|^p\,dx\right)\operatorname{div}\big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\big)=|x|^{-(a+1)p+c_1}h(x,u,v)\quad\text{in}\quad\Omega,$$

$$-M_2\left(\int\limits_\Omega |\nabla v|^q\,dx\right)\operatorname{div}\big(|x|^{-bq}|\nabla v|^{q-2}\nabla v\big)=|x|^{-(b+1)q+c_2}k(x,u,v)\quad\text{in}\quad\Omega, \tag{3.2}$$

$$u=v=0\quad\text{on}\quad x\in\partial\Omega,$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h,k\colon \overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:

(HK$_1$) $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s, t belong to bounded sets;

(HK$_2$) there exists a function $g\colon\mathbb{R}\to\mathbb{R}$ being continuous, nondecreasing, with $g(0)=0$, $0\le g(s)\le C\big(1+|s|^{\min\{p,q\}-1}\big)$ for some $C>0$, and applications $s\mapsto h(x,s,t)+g(s)$ and $t\mapsto k(x,s,t)+g(t)$ are nondecreasing, for a.e. $x\in\Omega$.

If $u,v\in L^\infty(\Omega)$, with $u(x)\le v(x)$ for a.e. $x\in\Omega$, we denote by $[u,v]$ the set $\{w\in L^\infty(\Omega)\colon u(x)\le w(x)\le v(x)$ for a.e. $x\in\Omega\}$. Using Lemma 3.1 and the method as in the proof of Theorem 2.4 of [18] (see also Section 4 of [7]), we can establish a version of the abstract lower and upper solution method for our class of the operators as follows.

Proposition 3.1. *Let $M_1,M_2\colon\mathbb{R}_0^+\to\mathbb{R}^+$ be two functions satisfying the condition* (H$_1$). *Assume that the functions h,k satisfy the conditions* (HK$_1$) *and* (HK$_2$). *Assume that $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$, are respectively, a weak subsolution and a weak supersolution of system* (3.2) *with $\underline{u}(x)\le\overline{u}(x)$ and $\underline{v}(x)\le\overline{v}(x)$ for a.e. $x\in\Omega$. Then there exists a minimal (u_*,v_*) $\big($and, respectively, a maximal $(u^*,v^*)\big)$ weak solution for system* (3.2) *in the set $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. In particular, every weak solution $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ of system* (3.2) *satisfies $u_*(x)\le u(x)\le u^*(x)$ and $v_*(x)\le v(x)\le v^*(x)$ for a.e. $x\in\Omega$.*

Now we are ready to state our existence result.

Theorem 3.1. *Assume* (H$_1$) – (H$_4$) *hold. Then the system* (1.1) *admits a positive solution when λ is large enough.*

Proof. Since f,g are continuous and nondecreasing, we have $f(x,y),\ g(x,y)\ge -a_0$ for all $x,y\ge 0$ and for some $a_0>0$. Choose $\eta>0$ such that

$$\eta\le \operatorname{Min}\left\{|x|^{-(a+1)p+c_1},|x|^{-(b+1)q+c_2}\right\}$$

in $\overline{\Omega}_\delta$. We shall verify that

$$(\psi_{1,\lambda},\psi_{2,\lambda})=\left(\left[\frac{\lambda a_0\eta}{\epsilon m_{1,\infty}}\right]^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\phi_{1,p}^{\frac{p}{p-1}},\left[\frac{\lambda a_0\eta}{\epsilon m_{2,\infty}}\right]^{\frac{1}{q-1}}\left(\frac{q-1}{q}\right)\phi_{1,q}^{\frac{q}{q-1}}\right)$$

is a subsolution of (1.1). Let $w\in W$. Then a calculation shows that

$$M_1\left(\int\limits_\Omega|\nabla\psi_{1,\lambda}|^p\,dx\right)\int\limits_\Omega|x|^{-ap}|\nabla\psi_{1,\lambda}|^{p-2}\nabla\psi_{1,\lambda}\cdot\nabla w\,dx=$$

$$=M_1\left(\int\limits_\Omega|\nabla\psi_{1,\lambda}|^p\,dx\right)\left(\frac{\lambda a_0\eta}{\epsilon m_{1,\infty}}\right)\int\limits_\Omega|x|^{-ap}\phi_{1,p}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\cdot\nabla w\,dx=$$

$$=M_1\left(\int\limits_\Omega|\nabla\psi_{1,\lambda}|^p\,dx\right)\left(\frac{\lambda a_0\eta}{\epsilon m_{1,\infty}}\right)\int\limits_\Omega|x|^{-ap}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\big[\nabla(\phi_{1,p}w)-|\nabla\phi_{1,p}|^pw\big]dx\le$$

$$\le\left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_\Omega\big[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\big]w\,dx.$$

Similarly

$$M_2\left(\int\limits_\Omega |\nabla\psi_{2,\lambda}|^q\,dx\right)\int\limits_\Omega |x|^{-bq}|\nabla\psi_{2,\lambda}|^{q-2}\nabla\psi_{2,\lambda}\cdot\nabla w\,dx \le$$

$$\le \left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_\Omega \left[\lambda_{1,q}|x|^{-(b+1)q+c_2}\phi_{1,q}^q - |x|^{-bq}|\nabla\phi_{1,q}|^q\right]w\,dx.$$

First we consider the case $x\in\overline{\Omega}_\delta$. We have

$$\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p \le -\epsilon$$

on $\overline{\Omega}_\delta$. Since $\psi_{1,\lambda}(x),\psi_{2,\lambda}(x)\ge 0$ in Ω it follows that

$$-a_0\eta \le \mathrm{Min}\left\{|x|^{-(a+1)p+c_1}f(\psi_{1,\lambda},\psi_{2,\lambda}), |x|^{-(b+1)q+c_2}g(\psi_{1,\lambda},\psi_{2,\lambda})\right\}$$

in $\overline{\Omega}_\delta$. Hence, we have

$$\left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_{\overline{\Omega}_\delta}\left[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right]w\,dx \le$$

$$\le -\lambda a_0\eta\int\limits_{\overline{\Omega}_\delta} w\,dx \le \lambda\int\limits_{\overline{\Omega}_\delta}|x|^{-(a+1)p+c_1}f(\psi_{1,\lambda},\psi_{2,\lambda})w\,dx.$$

A similar argument shows that

$$\left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_{\overline{\Omega}_\delta}\left[\lambda_{1,p}|x|^{-(b+1)q+c_2}\phi_{1,q}^q - |x|^{-bq}|\nabla\phi_{1,q}|^q\right]w\,dx \le$$

$$\le \lambda\int\limits_{\overline{\Omega}_\delta}|x|^{-(b+1)q+c_2}g(\psi_{1,\lambda},\psi_{2,\lambda})w\,dx.$$

On the other hand, on $\Omega\setminus\overline{\Omega}_\delta$ we have $\phi_{1,p}\ge\sigma_p$ and $\phi_{1,q}\ge\sigma_q$ for some $0<\sigma_p,\ \sigma_q<1$. Therefore,

$$\psi_{1,\lambda}\ge\left(\frac{\lambda a_0\eta}{m_{1,\infty}\epsilon}\right)^{\frac{1}{p-1}}\left(\frac{p-1}{p}\right)\sigma_p^{\frac{p}{p-1}}\to\infty, \tag{3.3}$$

$$\psi_{2,\lambda}\ge\left(\frac{\lambda a_0\eta}{m_{2,\infty}\epsilon}\right)^{\frac{1}{q-1}}\left(\frac{q-1}{q}\right)\sigma_q^{\frac{q}{q-1}}\to\infty \tag{3.4}$$

as $\lambda\to\infty$, uniformly in $\Omega\setminus\overline{\Omega}_\delta$. By (3.3), (3.4) and ($H_2$) we can find λ_* sufficiently large such that

$$f(\psi_{1,\lambda},\psi_{2,\lambda}), g(\psi_{1,\lambda},\psi_{2,\lambda})\ge\frac{a_0\eta}{\epsilon}\,\mathrm{Max}\left\{\lambda_{1,p},\lambda_{1,q}\right\}$$

for all $x\in\Omega\setminus\overline{\Omega}_\delta$ and for all $\lambda\ge\lambda_*$. Hence,

$$\left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_{\Omega\backslash\overline{\Omega}_\delta}\left[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\right]w\,dx\le$$

$$\le\left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_{\Omega\backslash\overline{\Omega}_\delta}|x|^{-(a+1)p+c_1}\lambda_{1,p}w\,dx\le$$

$$\le\lambda\int\limits_{\Omega\backslash\overline{\Omega}_\delta}|x|^{-(a+1)p+c_1}f(\psi_{1,\lambda},\psi_{2,\lambda})w\,dx.$$

Similarly

$$\left(\frac{\lambda a_0\eta}{\epsilon}\right)\int\limits_{\Omega\backslash\overline{\Omega}_\delta}\left[\lambda_{1,p}|x|^{-(b+1)q+c_2}\phi_{1,q}^q-|x|^{-bq}|\nabla\phi_{1,q}|^q\right]w\,dx\le$$

$$\le\int\limits_{\Omega\backslash\overline{\Omega}_\delta}|x|^{-(b+1)q+c_2}g(\psi_{1,\lambda},\psi_{2,\lambda})w\,dx.$$

Hence,

$$M_1\left(\int\limits_\Omega|\nabla\psi_{1,\lambda}|^p\,dx\right)\int\limits_\Omega|x|^{-ap}|\nabla\psi_{1,\lambda}|^{p-2}|\nabla\psi_{1,\lambda}|\cdot\nabla w\,dx\le$$

$$\le\int\limits_\Omega|x|^{-(a+1)p+c_1}f(\psi_{1,\lambda},\psi_{2,\lambda})w\,dx,$$

$$M_2\left(\int\limits_\Omega|\nabla\psi_{2,\lambda}|^q\,dx\right)\int\limits_\Omega|x|^{-bq}|\nabla\psi_{2,\lambda}|^{q-2}|\nabla\psi_{2,\lambda}|\cdot\nabla w\,dx\le$$

$$\le\int\limits_\Omega|x|^{-(b+1)q+c_2}g(\psi_{1,\lambda},\psi_{2,\lambda})w\,dx,$$

i.e., $(\psi_{1,\lambda},\psi_{2,\lambda})$ is a subsolution of (1.1).

Now, we will prove there exists a M large enough so that

$$(z_1,z_2)=$$

$$=\left(M\theta_p^{-1}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\zeta_p(x),\left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}},M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]^{\frac{1}{q-1}}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{q-1}}\zeta_q(x)\right)$$

is a supersolution of (1.1); where $\theta_r=\|\zeta_r\|_\infty;\ r=p,\ q,\lambda\ge\lambda_*$ and $m_0=\mathrm{Min}\{m_1,m_2\}$. A calculation shows that

$$M_1\left(\int\limits_\Omega|\nabla z_1|^p\,dx\right)\int\limits_\Omega|x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx=$$

$$= M_1\left(\int\limits_\Omega |\nabla z_1|^p\,dx\right)\left(\frac{\lambda}{m_0}\right)\left(M\theta_p^{-1}\right)^{p-1}\int\limits_\Omega |x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p\nabla w\,dx =$$

$$= M_1\left(\int\limits_\Omega |\nabla z_1|^p\,dx\right)\theta_p^{1-p}\left[M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right]^{p-1}\int\limits_\Omega |x|^{-(a+1)p+c_1}w\,dx \geq$$

$$\geq m_1\theta_p^{1-p}\left[M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right]^{p-1}\int\limits_\Omega |x|^{-(a+1)p+c_1}w\,dx.$$

By monotonicity condition on f and (H$_3$) we can choose M large enough so that

$$m_1\theta_p^{1-p}\left[M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right]^{p-1} \geq$$

$$\geq \lambda f\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, \left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]^{\frac{1}{q-1}}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{q-1}}\theta_q\right) \geq$$

$$\geq \lambda f\left(M\theta_p^{-1}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\zeta_p(x), \left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]^{\frac{1}{q-1}}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{q-1}}\zeta_q(x)\right) =$$

$$= \lambda f(z_1, z_2).$$

Hence,

$$M_1\left(\int\limits_\Omega |\nabla z_1|^p\,dx\right)\int\limits_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w\,dx \geq \lambda\int\limits_\Omega |x|^{-(a+1)p+c_1}f(z_1, z_2)w\,dx.$$

Next, by (H$_4$) for M large enough we have

$$\frac{\left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]^{\frac{1}{q-1}}}{M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}} \leq \left(\frac{\lambda}{m_0}\right)^{\frac{1}{1-q}}\theta_q^{-1}.$$

Hence,

$$M_2\left(\int\limits_\Omega |\nabla z_2|^q\,dx\right)\int\limits_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w\,dx =$$

$$= M_2\left(\int\limits_\Omega |\nabla z_2|^q\,dx\right)\left(\frac{\lambda}{m_0}\right)\left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]\times$$

$$\times \int_\Omega |x|^{-bq}|\nabla \zeta_p|^{q-2}\nabla\zeta_p\cdot\nabla w\,dx =$$

$$= M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\left(\frac{\lambda}{m_0}\right)\left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]\int_\Omega |x|^{-(b+1)q+c_2}w\,dx \geq$$

$$\geq \lambda\int_\Omega |x|^{-(b+1)q+c_2} g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}},\right.$$

$$\left.\left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]^{\frac{1}{q-1}}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{q-1}}\theta_q\right) w\,dx \geq$$

$$\geq \lambda\int_\Omega |x|^{-(b+1)q+c_2} g\left(M\theta_p^{-1}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\zeta_p(x),\right.$$

$$\left.\left[g\left(M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}, M\left(\frac{\lambda}{m_0}\right)^{\frac{1}{p-1}}\right)\right]^{\frac{1}{q-1}}\left(\frac{\lambda}{m_0}\right)^{\frac{1}{q-1}}\zeta_q(x)\right) w\,dx =$$

$$= \lambda\int_\Omega |x|^{-(b+1)q+c_2} g(z_1,z_2)w\,dx,$$

i.e., (z_1, z_2) is a supersolution of (1.1) with $z_i \geq \psi_{i,\lambda}$, $i = 1, 2$, for a M large enough. Thus, by Proposition 3.1 there exists a positive solution (u, v) of (1.1) such that $(\psi_{1,\lambda}, \psi_{2,\lambda}) \leq (u, v) \leq \leq (z_1, z_2)$.

Theorem 3.1 is proved.

***Example* 3.1.** Consider the problem

$$-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = \lambda|x|^{-(a+1)p+c_1}(v^\alpha + (uv)^\beta - 1) \quad \text{in} \quad \Omega,$$

$$-M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\operatorname{div}\left(|x|^{-bq}|\nabla v|^{q-2}\nabla v\right) = \lambda|x|^{-(b+1)q+c_2}(u^\sigma + (uv)^{\frac{\gamma}{2}} - 1) \quad \text{in} \quad \Omega, \tag{3.5}$$

$$u = v = 0 \quad \text{in} \quad \partial\Omega,$$

where α, β, σ, and γ are positive parameters. Then it is easy to see that (3.5) satisfies the hypotheses of Theorem 3.1 if $\max\{\sigma,\gamma\}\dfrac{\alpha}{q-1} < p-1$, $\Big(\max\{\sigma,\gamma\}\dfrac{1}{q-1}+1\Big)\beta < p-1$ and $\max\{\sigma,\gamma\} < q-1$.

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Received 03.06.16