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DIVIDEND PAYMENTS IN A PERTURBED COMPOUND POISSON MODEL WITH STOCHASTIC INVESTMENT AND DEBIT INTEREST *

ДИВІДЕНДНІ ВИПЛАТИ У ЗБУРЕНІЙ СКЛАДНІЙ ПУАССОНІВСЬКІЙ МОДЕЛІ ЗІ СТОХАСТИЧНИМИ ІНВЕСТИЦІЯМИ ТА ДЕБЕТОВИМИ ВІДСОТКАМИ

We consider a compound Poisson insurance risk model perturbed by diffusion with stochastic return on investment and debit interest. If the initial surplus is nonnegative, then the insurance company can invest its surplus in a risky asset and risk-free asset based on a fixed proportion. Otherwise, the insurance company can get the business loan when the surplus is negative. The integrodifferential equations for the moment generating function of the cumulative dividends value are obtained under the barrier and threshold dividend strategies, respectively. The closed-form of the expected dividend value is obtained when the claim amount is exponentially distributed.

Розглядається складна пуассонівська модель страхових ризиків, збурена дифузією, зі стохастичним доходом по інвестиціях та дебетовим відсотком. Якщо початковий надлишок є невід'ємним, то страхова компанія може вкладати цей надлишок у ризикові або безризикові активи в фіксованій пропорції. В протилежному випадку, коли надлишок є від'ємним, страхова компанія може отримувати бізнесові кредити. Інтегро-диференціальні рівняння для функції, що породжує моменти значень кумулятивних дивідендів, отримано для бар'єрних та порогових дивідендних стратегій відповідно. Очікувану величину дивідендів отримано в замкненій формі у випадку експоненціального розподілу суми позову.

1. Introduction. The compound Poisson insurance risk model perturbed by diffusion was first introduced by Gerber [8] and further studied by many authors, such as Dufresne and Gerber [7], Yuen et al. [23], Asmussen [1], Chiu and Yin [5], Lu et al. [15]. If investment income is introduced in the perturbed compound Poisson model, the security loading will be a variable. In the field of actuarial mathematics, researchers paid lots of attention to the issue of stochastic investment and debit, for example, Gerber [9], Zhu and Yang [24] considered the absolute ruin problem in the compound Poisson model when the debt and credit interest rates were the same. Cai [4] studied the Gerber – Shiu function in the classical insurance risk model with debit interest. Paulsen and Gjessing [19] simplified the version of the model which was used by Paulsen [16] through incorporating a stochastic rate of return on investments. Then they computed the probability of eventual ruin and the infinitesimal generator of the risk process. Gerber and Yang [11] proposed the general risk model with investment, in which the insurance company can invest its surplus in a risky asset and risk-free asset. If the company invests money in the bank or borrows money from the bank, the rate of return can be described as

$$dR_t = \begin{cases} rR_t dt, & \text{if surplus is nonnegative,} \\ \alpha R_t dt, & \text{if surplus is negative,} \end{cases} \tag{1.1}$$

where r is the company's lending rate, α is the borrowing rate which satisfies $\alpha \geq r > 0$. The model of the risky asset satisfies the stochastic differential equation

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$$dS_t = \mu S_t dt + \sigma S_t dB_{0t}, \tag{1.2}$$

where $\mu t + \sigma B_{0t}$ is the return on investment S_t to the risky asset, $\{B_{0t}\}$ is the standard Brownian motion. Yin and Wen [22] extended the model of Paulsen [17] to the following form:

$$U_t = u + P_t + \int_0^t U_{s-} dR_s \quad \text{with} \quad P_0 = R_0 = 0,$$

where $\{P_t\}$, $\{R_t\}$ were independent Lévy processes, u was the initial value, $\{P_t\}$ was a general two-sided jump-diffusion risk model.

Dividend strategies were first introduced by De Finetti [6]. He formulated the problem and solved it under the assumption that the surplus was a discrete process without investment. The study of the dividend strategy was also be conducted by Bühlmann [3], Paulsen [16], Jeanblanc-Picqué and Shiryaev [12], Asmussen and Taksar [2], Gerber and Shiu [10], Wan [20], Lu and Wu [14], Yin and Yuen [21]. Many results were obtained using the property of stationary and independent increments of the Poisson process and Brownian motion. Recently, Yin and Wen [22] extended the model raised by Paulsen [17] and obtained the integrodifferential equations of the Gerber – Shiu functions and total discounted dividends, respectively. For further references, see two survey papers Paulsen [17, 18]. In recent papers the model was extended to renewal risk model with stochastic return, see Yin and Wen [22] and Li [13].

Motivated by the previously mentioned papers, we are going to study the moment generating function of dividend value under barrier and threshold dividend strategies on a perturbed compound Poisson model with stochastic investment and debit interest. The rest of the paper is structured as follows. In Section 2, the perturbed compound Poisson model with stochastic investment and debit interest is introduced. In Section 3, the second-order integrodifferential equations for the moment generating function of aggregate dividends under the barrier dividend strategy are established. In the case of the exponential claim size, the exact solution of the third-order differential equations and the closed-form of the expected dividend value are obtained. In Section 4, the threshold dividend strategy is discussed. The integrodifferential equations for the moment generating function of aggregate dividends and the expected dividend value are obtained.

2. The insurance risk model with investment income. We consider a company with initial surplus u. If no dividends are paid, the surplus at time t is

$$P_t = u + ct - \sum_{i=1}^{N_t} X_i + \sigma_1 W_t,$$

where $c > 0$ is the premium rate, σ_1 is a constant. The aggregate claim process $\left\{ \sum_{i=1}^{N_t} X_i \right\}$ is the compound Poisson process with parameter λ, $\{X_i\}$ is a sequence of independent and identically distributed random variables, the density function is defined as $p(x)$. $\{W_t\}$ is the standard Brownian motion independent of the aggregate claim process. Now, suppose that the surplus is nonnegative, then the company can invest its money in a risk-free asset and a risky asset. The risk-free asset price is assumed to follow the stochastic differential equation (1.1), and the risky asset price is the same as (1.2). W_t and B_{0t} are correlated with $dW_t dB_{0t} = \rho dt$.

Assume that if the surplus $U_t > 0$ then the insurance company invests the surplus by a fixed proportion τ to the risk-free asset and the rest proportion $1-\tau$ of the surplus to the risky asset. If the surplus $U_t \le 0$, then the insurance company borrows money from the bank under the debt rate α. Hence, the modified surplus risk process $\{U_t\}$ satisfies the following equation. Let $t>0$, for small enough Δt,

$$\Delta U_t = U_t \Delta t \times \begin{cases} \tau r, & U_t > 0, \\ \alpha, & U_t \le 0, \end{cases} + (1-\tau) U_t (\mu \Delta t + \sigma \Delta B_{0t}) \times \begin{cases} 1, & U_t > 0, \\ 0, & U_t \le 0, \end{cases} + \Delta P_t.$$

Let initial $u>0$, for small enough t and $\Delta t = dt$, we have

$$\begin{aligned} dU_s &= \tau r U_{s-} ds + (1-\tau)\mu U_{s-} ds + (1-\tau)\sigma U_{s-} dB_{0s} + dP_s = \\ &= [\tau r + (1-\tau)\mu] U_{s-} ds + (1-\tau)\sigma U_{s-} dB_{0s} + dP_s = \\ &= U_{s-} dL_s + dP_s, \qquad 0 \le s < t, \end{aligned} \tag{2.1}$$

where $L_t = adt + bdB_{0t}$. We use a and b that are defined below

$$a = \tau r + (1-\tau)\mu, \qquad b = (1-\tau)\sigma.$$

From (2.5) in Yin and Wen [22] as a special case we get the another form of the equation (2.1) listed as follows:

$$U_t = u + \int_0^t (c + aU_s)\, ds + \int_0^t \sqrt{(\sigma_1 + \rho b U_{s-})^2 + b^2(1-\rho^2) U_{s-}^2}\, dB_s - \sum_{i=1}^{N_t} X_i, \quad u > 0,$$

where $\{B_t, t \ge 0\}$ is a standard Brownian motion independent of the compound Poisson processes involved.

If $u \le 0$, for small enough t and $\Delta t = dt$, then $\{U_t\}$ satisfies the equation

$$dU_t = dP_t + \alpha U_t dt. \tag{2.2}$$

To solve the stochastic differential equation (2.2) we consider the process $V_t = U_t e^{-\alpha t}$. By using the product rule, we have $dV_t = e^{-\alpha t} dU_t - \alpha e^{-\alpha t} U_t dt$. Combining with (2.2) we obtain $dV_t = e^{-\alpha t} dP_t$. This gives $V_t = V_0 + \int_0^t e^{-\alpha s} dP_s$. The solution for stochastic differential equation (2.2) is

$$U_t = e^{\alpha t}\left(u + c\int_0^t e^{-\alpha s} ds\right) - e^{\alpha t}\int_0^t e^{-\alpha s} d\sum_{i=1}^{N_s} X_i + \sigma_1 e^{\alpha t}\int_0^t e^{-\alpha s} dW(s), \quad u \le 0.$$

According to the above results, for small enough $t>0$, we get the model

$$U_t = \begin{cases} u + \int_0^t (c + aU_s)\, ds + \int_0^t \sqrt{(\sigma_1 + \rho b U_{s-})^2 + b^2(1-\rho^2) U_{s-}^2}\, dB_s - \sum_{i=1}^{N_t} X_i, & u > 0, \\ e^{\alpha t}\left(u + c\int_0^t e^{-\alpha s} ds\right) - e^{\alpha t}\int_0^t e^{-\alpha s} d\sum_{i=1}^{N_s} X_i + \sigma_1 e^{\alpha t}\int_0^t e^{-\alpha s} dW(s), & u \le 0. \end{cases} \tag{2.3}$$

We assume that the company can still run as long as the surplus is above the level $-c/\alpha$, that is, if the surplus is below $-c/\alpha$, then the company's interest can no longer pay off the debt and absolute ruin occurs immediately.

3. The barrier dividend strategy of the model. In this section we focus on the barrier dividend strategy. If the surplus $U_t \geq \eta,\ \eta > 0$, then the excess $D(t) = U_t - \eta$ all will be paid to the shareholders as dividends. However, if the surplus $U_t < \eta$, then no dividends are paid. If the surplus is below $-\dfrac{c}{\alpha}$, then absolute ruin happens and the surplus process stops. We define the dividend value until time t:

$$D_t(\eta) = \int\limits_0^t I(U_s > \eta) dD(s).$$

Then the modified surplus is

$$U_\eta(t) = U_t - D_t(\eta),$$

where $\{U_t\}$ is modeled by (2.3).

For simplicity, we let

$$E[\cdot|U_0 = u] = E_u(\cdot).$$

We denote $V(u, \eta)$ as the dividend value function of the strategy, which is

$$V(u, \eta) = E_u[D_{T_1}],$$

where $D_{T_1} = \displaystyle\int_0^{T_1} e^{-\delta t} dD_t(\eta),\ \delta > 0$ is the force of interest for valuation, and T_1 is the absolute ruin time defined by

$$T_1 = \inf \left\{ t \geq 0, U_t(\eta) \leq -\frac{c}{\alpha} \right\}.$$

Let $M(u, y, \eta) = E_u[e^{yD_{T_1}}]$ be the moment generating function of D_{T_1} and

$$M(u, y, \eta) = \begin{cases} e^{y[(u-\eta)+V(\eta,\eta)]}, & u > \eta, \\ M_1(u, y, \eta), & 0 \leq u \leq \eta, \\ M_2(u, y, \eta), & -\dfrac{c}{\alpha} < u < 0. \end{cases}$$

We assume that, for any $y < \infty$, $M(u, y, \eta)$ exists, $M_1(u, y, \eta)$ and $M_2(u, y, \eta)$ are twice continuously differentiable for $u \in \left(-\dfrac{c}{\alpha}, \eta\right)$.

Theorem 3.1. *If $0 < u < \eta$, then $M_1(u, y, \eta)$ satisfies the integrodifferential equation*

$$\lambda M_1(u, y, \eta) = (au + c)\frac{\partial M_1(u, y, \eta)}{\partial u} - \delta y \frac{\partial M_1(u, y, \eta)}{\partial y} +$$
$$+ \frac{1}{2}(\sigma_1^2 + 2\rho b \sigma_1 u + b^2 u^2)\frac{\partial^2 M_1(u, y, \eta)}{\partial u^2} +$$
$$+ \lambda \int\limits_0^u M_1(u - x, y, \eta) p(x)\, dx +$$
$$+ \lambda \int\limits_u^{u+\frac{c}{\alpha}} M_2(u - x, y, \eta) p(x)\, dx + \lambda \int\limits_{u+\frac{c}{\alpha}}^{\infty} p(x)\, dx. \tag{3.1}$$

If $-\dfrac{c}{\alpha} < u < 0$, *then* $M_2(u,y,\eta)$ *satisfies the following equation*:

$$\lambda M_2(u,y,\eta) = \frac{1}{2}\sigma_1^2 \frac{\partial^2 M_2(u,y,\eta)}{\partial u^2} + (\alpha u + c)\frac{\partial M_2(u,y,\eta)}{\partial u} - \delta y \frac{\partial M_2(u,y,\eta)}{\partial y} +$$

$$+\lambda \int\limits_0^{u+\frac{c}{\alpha}} M_2(u-x,y,\eta)p(x)\,dx + \lambda \int\limits_{u+\frac{c}{\alpha}}^{\infty} p(x)\,dx \tag{3.2}$$

with boundary conditions

$$M_2\left(-\frac{c}{\alpha},y,\eta\right) = 1, \qquad M_1(0+,y,\eta) = M_2(0-,y,\eta),$$
$$\left.\frac{\partial M(u,y,\eta)}{\partial u}\right|_{u=\eta} = yM_1(\eta,y,\eta). \tag{3.3}$$

Proof. If $0 \le u < \eta$, we consider an infinitesimal time interval $(0,t)$. By the Markov property of the process $\{U_t\}$, we have

$$M_i(u,y,\eta) = E_u M_i(U_t, ye^{-\delta t},\eta) + o(t), \qquad i = 1,2, \quad 0 \le u < \eta. \tag{3.4}$$

Let $A \overset{d}{=} B$ means that A and B have the same distribution. Set

$$Y_t = u + \int\limits_0^t (c + aU_s)ds + \int\limits_0^t \sqrt{(\sigma_1 + \rho b U_{s-})^2 + b^2(1-\rho^2)U_{s-}^2 dB_s}.$$

In the infinitesimal time interval $(0,t)$, the insurance risk process (2.3) has three possible cases.

(i) If no jump happen in $(0,t)$, then its probability is $e^{\lambda t}$ and $U_t \overset{d}{=} Y_t$.

(ii) If there is only one jump in $(0,t)$ and the claim size is X_1, then its probability is $\lambda t e^{-\lambda t}$ and $U_t \overset{d}{=} Y_t - X_1$.

Further, according to the amount of the claim, there are the following situations:

(1) if $0 < X_1 < Y_t$, then ruin do not occur,

(2) if $Y_t < X_1 < Y_t + \dfrac{c}{\alpha}$, then the ruin occur,

(3) if $X_1 \ge Y_t + \dfrac{c}{\alpha}$, then the absolute ruin occur.

(iii) If there are two or more jumps, then its probability is $o(t)$.

So, we have

$$M_1(u,y,\eta) = e^{-\lambda t}E_u\big[M_1(Y_t, ye^{-\delta t},\eta)\big] +$$

$$+\lambda t e^{-\lambda t} E_u \int\limits_0^{Y_t} M_1(Y_t - x, ye^{-\delta t},\eta)p(x)\,dx +$$

$$+\lambda t e^{-\lambda t} E_u \int\limits_{Y_t}^{Y_t+\frac{c}{\alpha}} M_2(Y_t - x, ye^{-\delta t},\eta)p(x)\,dx +$$

$$+\lambda te^{-\lambda t}E_u\int\limits_{Y_t+\frac{c}{\alpha}}^{\infty}p(x)\,dx+o(t). \tag{3.5}$$

By Itô's formula, we obtain

$$dM_1(Y_t,ye^{-\delta t},\eta)=\frac{\partial M_1(Y_t,ye^{-\delta t},\eta)}{\partial u}dY_t+y\frac{\partial M_1(Y_t,ye^{-\delta t},\eta)}{\partial y}de^{-\delta t}+$$
$$+\frac{1}{2}\frac{\partial^2 M_1(Y_t,ye^{-\delta t},\eta)}{\partial u^2}(dY_t)^2.$$

Then

$$E_uM_1(Y_t,ye^{-\delta t},\eta)=M_1(u,y,\eta)+$$
$$+E_u\left[\int\limits_0^t(c+aU_s)\frac{\partial M_1(Y_s,ye^{-\delta s},\eta)}{\partial u}ds-\delta y\int\limits_0^t\frac{\partial M_1(Y_s,ye^{-\delta s},\eta)}{\partial y}e^{-\delta s}ds\right]+$$
$$+\frac{1}{2}E_u\int\limits_0^t\frac{\partial^2 M_1(Y_s,ye^{-\delta s},\eta)}{\partial u^2}\left[(\sigma_1+\rho bU_{s-})^2+b^2(1-\rho^2)U_{s-}^2\right]ds. \tag{3.6}$$

Substituting (3.6) into (3.5) and dividing both sides of (3.4) by t, then letting $t\to 0$ and rearranging it we get (3.1).

If $-\dfrac{c}{\alpha}<u<0$, we let that

$$h_\alpha(t,u)=ue^{\alpha t}+\frac{c(e^{\alpha t}-1)}{\alpha}+\sigma_1e^{\alpha t}\int\limits_0^t e^{-\alpha s}\,dW_s.$$

By full probability formula we have

$$M_2(u,y,\eta)=e^{-\lambda t}M_2(h_\alpha(t,u),ye^{-\delta t},\eta)+$$
$$+\lambda te^{-\lambda t}E_u\int\limits_0^{h_\alpha+\frac{c}{\alpha}}M_2(h_\alpha(t,u)-x,ye^{-\delta t},\eta)p(x)\,dx+$$
$$+\lambda te^{-\lambda t}E_u\int\limits_{h_\alpha+\frac{c}{\alpha}}^{\infty}p(x)\,dx+o(t). \tag{3.7}$$

By using Itô's formula, we obtain

$$dM_2(h_\alpha(t,u),ye^{-\delta t},\eta)=\frac{\partial M_2(h_\alpha(t,u),ye^{-\delta t},\eta)}{\partial u}dh_\alpha(t,u)+$$
$$+\frac{\partial M_2(h_\alpha(t,u),ye^{-\delta t},\eta)}{\partial u}dye^{-\delta t}+\frac{1}{2}\frac{\partial^2 M_2(h_\alpha(t,u),ye^{-\delta t},\eta)}{\partial u^2}(dh_\alpha)^2. \tag{3.8}$$

By using (3.7) and (3.8), we get (3.2).

Noting that if $u = -\dfrac{c}{\alpha}$, then $T_1 = 0$ and $D_{T_1} = 0$. So we have

$$M_2\left(-\frac{c}{\alpha}, y, \eta\right) = E_{-\frac{c}{\alpha}}\left(e^{yD_{T_1}}\right) = 1, \qquad M_1(0+, y, \eta) = M_2(0-, y, \eta),$$

due to $M_i(u, y, \eta)$, $i = 1, 2$, are continuous at $u = 0$.

By the definition of $M(u, y, \eta)$, we get $\dfrac{\partial M(u, y, \eta}{\partial u}|_{u=\eta} = yM_1(\eta, y, \eta)$. The boundary conditions (3.3) are obtained.

Theorem 3.1 is proved.

Let $M(u, y, \eta) = 1 + yV(u, \eta) + o(y)$, for small enough y, where

$$V(u, \eta) = E_u[D_{T_1}] = \begin{cases} V_1(u, \eta), & 0 < u < \eta, \\ V_2(u, \eta), & -\dfrac{c}{\alpha} < u < 0. \end{cases} \tag{3.9}$$

Substituting (3.9) into (3.1) and (3.2), comparing the coefficients of y, we get the following results.

Theorem 3.2. *If* $0 < u < \eta$, *then* $V_i(u, \eta), i = 1, 2$, *satisfy the equation*

$$(\lambda + \delta)V_1(u, \eta) = \frac{1}{2}\left(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2\right)V_1''(u, \eta) + (au + c)V_1'(u, \eta) +$$

$$+\lambda \int\limits_0^u V_1(u - x, \eta)p(x)dx + \lambda \int\limits_u^{u+\frac{c}{\alpha}} V_2(u - x, \eta)p(x)dx. \tag{3.10}$$

If $-\dfrac{c}{\alpha} < u < 0$, *then* $V_2(u, \eta)$ *satisfies the equation*

$$(\lambda + \delta)V_2(u, \eta) = \frac{1}{2}\sigma_1^2 V_2''(u, \eta) + (\alpha u + c)V_2'(u, \eta) +$$

$$+\lambda \int\limits_0^{u+\frac{c}{\alpha}} V_2(u - x, \eta)p(x)\, dx \tag{3.11}$$

with boundary conditions

$$V_1'(\eta) = 1, \qquad V_1'(0+) = V_2'(0-), \qquad V_1(0+) = V_2(0-). \tag{3.12}$$

Example **3.1.** Assume that the claim size has an exponential distribution with the density function $p(x) = e^{-x}$. Applying the operator $\left(\dfrac{d}{du} + 1\right)$ on (3.10) and (3.11), we get the following results.

If $0 < u < \eta$, then $V_1(u, \eta)$ satisfies the equation

$$\frac{1}{2}\left(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2\right)V_1'''(u, \eta) +$$

$$+\left[\frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2) + (a + b^2)u + \rho b\sigma_1 + c\right]V_1''(u, \eta) +$$

$$+(au + c + a - \delta - \lambda)V_1'(u, \eta) - \delta V_1(u, \eta) = 0. \tag{3.13}$$

If $-\dfrac{c}{\alpha} < u < 0$, then $V_2(u,\eta)$ satisfies the equation

$$\frac{1}{2}\sigma_1^2 V_2'''(u,\eta) + \left(\alpha u + c + \frac{1}{2}\sigma_1^2\right) V_2''(u,\eta) + (\alpha u - \lambda - \delta + \alpha) V_2'(u,\eta) - \delta V_2(u,\eta) = 0. \quad (3.14)$$

Letting $\delta = b^2 = 0,\ \lambda = a + c$, then (3.13) gives the equation

$$\frac{1}{2}\sigma_1^2 V_1'''(u,\eta) + \left(au + c + \frac{1}{2}\sigma_1^2\right) V_1''(u,\eta) + auV_1'(u,\eta) = 0.$$

Using substitutions $x = au$, $V_1'(u,\eta) = e^{-u}g(x)$; $z = ax + ac - \dfrac{a\sigma_1^2}{2}$, $g(x) = \dfrac{2}{a^2\sigma_1^2} f(x)$; $t = -\dfrac{z^2}{2}$, $f(z) = h(t)$, (3.13) yields

$$th''(t) + \left(\frac{1}{2} - \frac{2}{a^3\sigma_1^2} t\right) h'(t) + ch(t) = 0. \quad (3.15)$$

For convenience, we let $\sigma_1^2 a^3 = 2$, thus (3.15) changes into a confluent hypergeometric equation

$$th''(t) + \left(\frac{1}{2} - t\right) h'(t) + ch(t) = 0.$$

The general solution to the above equation is a linear combination of two independent solutions

$$h(t) = c_5 H\left(-c, \frac{1}{2}, t\right) + c_6 L\left(-c, \frac{1}{2}, t\right),$$

where $H\left(-c, \dfrac{1}{2}, t\right)$ and $L\left(-c, \dfrac{1}{2}, t\right)$ are the first and second kind of confluent hypergeometric functions respectively, c_5 and c_6 are constants. Transforming back to the original variables, we get the following solution:

$$V_1'(u,\eta) = \frac{2c_5}{a^2\sigma_1^2} e^{-u} H\left(-c, \frac{1}{2}, \frac{-a^2(au + c - \sigma_1^2/2)^2}{2}\right) +$$
$$+ \frac{2c_6}{a^2\sigma_1^2} e^{-u} L\left(-c, \frac{1}{2}, \frac{-a^2(au + c - \sigma_1^2/2)^2}{2}\right).$$

So $V_1(u,\eta) = \displaystyle\int_0^u V_1'(x)\,dx + V_1(0+,\eta)$. Next, we compute $c_5,\ c_6$. For (3.14), let $\sigma_1 = 0$, we have

$$(\alpha u + c) V_2''(u,\eta) + (\alpha u + c - \lambda - \delta + \alpha) V_2'(u,\eta) - \delta V_2(u,\eta) = 0. \quad (3.16)$$

With change of variable $y = -u - \dfrac{c}{\alpha}$, $g(y) = V_2(u,\eta)$, (3.16) becomes

$$yg''(y) + \left(\frac{\alpha - \lambda - \delta}{\alpha} - y\right) g'(y) + \frac{\delta}{\alpha} g(y) = 0.$$

This is an confluent hypergeometric equation. The general solution is

$$g(y)=c_7(-y)^{\frac{\lambda+\delta}{\alpha}}e^yH\left(1+\frac{\delta}{\lambda},1+\frac{\lambda+\delta}{\alpha},-y\right)+c_8e^yL\left(1-\frac{\delta}{\lambda},1-\frac{\lambda+\delta}{\alpha},-y\right).$$

Transforming back to the original variable, we have

$$V_2(u,\eta)=g\left(-u-\frac{c}{\alpha}\right)=c_7\left(u+\frac{c}{\alpha}\right)^{\frac{\lambda+\delta}{\alpha}}e^{-u-\frac{c}{\alpha}}H\left(1+\frac{\delta}{\lambda},1+\frac{\lambda+\delta}{\alpha},u+\frac{c}{\alpha}\right)+$$

$$+c_8e^{-u-\frac{c}{\alpha}}L\left(1-\frac{\delta}{\lambda},1-\frac{\lambda+\delta}{\alpha},u+\frac{c}{\alpha}\right). \tag{3.17}$$

As

$$\lim_{u\downarrow-\frac{c}{\alpha}}V_2(u,\eta)=c_8L\left(1-\frac{\delta}{\lambda},1-\frac{\lambda+\delta}{\alpha},0\right)=c_8\frac{\Gamma\Big(\dfrac{\delta+\delta}{\alpha}\Big)}{\Gamma\Big(1+\dfrac{\delta}{\alpha}\Big)}=0,$$

so $c_8=0$.

Letting $\delta=0$ in (3.17), then we obtain

$$V_2(u,\eta)=c_7\left(u+\frac{c}{\alpha}\right)^{\frac{\lambda}{\alpha}}e^{-u-\frac{c}{\alpha}}H\left(1,1+\frac{\lambda}{\alpha},u+\frac{c}{\alpha}\right).$$

By using the boundary condition (3.12), we get

$$\frac{2c_5}{a^2\sigma_1^2}e^{-\eta}H\left(-c,\frac{1}{2},-\frac{a^2(a\eta-\sigma_1^2/2)^2}{2}\right)+\frac{2c_6}{a^2\sigma_1^2}e^{-\eta}L\left(-c,\frac{1}{2},\frac{a^2(a\eta+c+\sigma_1^2/2)}{2}\right)=1, \tag{3.18}$$

$$V_1'(0+)=\frac{2c_5}{a^2\sigma_1^2}H\left(-c,\frac{1}{2},-\frac{a^2(c-\sigma_1^2/2)^2}{2}\right)+\frac{2c_6}{a^2\sigma_1^2}L\left(-c,\frac{1}{2},-\frac{a^2(c-\sigma_1^2/2)^2}{8}\right)=$$

$$=c_7\left(\frac{c}{\alpha}\right)^{\frac{\lambda}{\alpha}}e^{-\frac{c}{\alpha}}\left[\frac{\lambda}{c}H\left(1,1+\frac{\lambda}{\alpha},\frac{c}{\alpha}\right)-H\left(1,1+\frac{\lambda}{\alpha},\frac{c}{\alpha}\right)+\frac{\alpha}{\alpha+\lambda}H\left(2,2+\frac{\lambda}{\alpha},\frac{c}{\alpha}\right)\right]=$$

$$=V_2'(0-), \tag{3.19}$$

and

$$V_1(0+)=\frac{2c_5}{a^2\sigma_1^2}\int_0^\eta e^{-x}H\left(-c,\frac{1}{2},-\frac{a^2(ax-\sigma_1^2/2)^2}{2}\right)dx+$$

$$+\frac{2c_6}{a^2\sigma_1^2}\int_0^\eta e^{-x}L\left(-c,\frac{1}{2},\frac{a^2(ax+c+\sigma_1^2/2)}{2}\right)dx=$$

$$=c_7\left(\frac{c}{\alpha}\right)^{\frac{\lambda}{\alpha}}H\left(1,1+\frac{\lambda}{\alpha};\frac{c}{\alpha}\right)=V_2(0-). \tag{3.20}$$

Let

$$P=\frac{\lambda}{c}H\left(1,1+\frac{\lambda}{\alpha},\frac{c}{\alpha}\right)-H\left(1,1+\frac{\lambda}{\alpha},\frac{c}{\alpha}\right)+\frac{\alpha}{\alpha+\lambda}H\left(2,2+\frac{\lambda}{\alpha},\frac{c}{\alpha}\right),$$

$$A_1 = \int_0^{\eta} e^{-x} H\left(-c, \frac{1}{2}, -\frac{a^2(ax - \sigma_1^2/2)^2}{2}\right) dx,$$

$$B_1 = \int_0^{\eta} e^{-x} L\left(-c, \frac{1}{2}, \frac{a^2(ax + c + \sigma_1^2/2)}{2}\right) dx,$$

$$A_2 = H\left(-c, \frac{1}{2}, -\frac{a^2(a\eta - \sigma_1^2/2)^2}{2}\right),$$

$$B_2 = L\left(-c, \frac{1}{2}, -\frac{a^2(c - \sigma_1^2/2)^2}{8}\right),$$

$$A_3 = H\left(-c, \frac{1}{2}, -\frac{a^2(c - \sigma_1^2/2)^2}{2}\right),$$

$$B_3 = L\left(-c, \frac{1}{2}, -\frac{a^2(c - \sigma_1^2/2)^2}{8}\right),$$

then (3.18), (3.19) and (3.20) can be rewritten by

$$\frac{2c_5}{a^2\sigma_1^2} A_2 + \frac{2c_6}{a^2\sigma_1^2} B_2 = e^{\eta},$$

$$\frac{2c_5}{a^2\sigma_1^2} A_3 + \frac{2c_6}{a^2\sigma_1^2} B_3 = c_7 \left(\frac{c}{\alpha}\right)^{\frac{\lambda}{\alpha}} e^{-\frac{c}{\alpha}} P,$$

$$\frac{2c_5}{a^2\sigma_1^2} A_1 + \frac{2c_6}{a^2\sigma_1^2} B_1 = c_7 \left(\frac{c}{\alpha}\right)^{\frac{\lambda}{\alpha}} e^{-\frac{c}{\alpha}} H\left(1, 1 + \frac{\lambda}{\alpha}, \frac{c}{\alpha}\right).$$

The solutions to the above three equations are

$$c_5 = \frac{a^2\sigma_1^2 e^{\eta}}{2} \, \frac{B_1 P - B_3 H\left(1, 1 + \frac{\lambda}{\alpha}, \frac{c}{\alpha}\right)}{(A_2B_1 - A_1B_2)P + H\left(1, 1 + \frac{\lambda}{\alpha}, \frac{c}{\alpha}\right)(B_2A_3 - A_2B_3)},$$

$$c_6 = \frac{a^2\sigma_1^2 e^{\eta}}{2} \, \frac{A_3 H\left(1, 1 + \frac{\lambda}{\alpha}, \frac{c}{\alpha}\right) - A_1 P}{(A_2B_1 - A_1B_2)P + H\left(1, 1 + \frac{\lambda}{\alpha}, \frac{c}{\alpha}\right)(B_2A_3 - A_2B_3)},$$

$$c_7 = e^{\frac{c}{\alpha} + \eta} \left(\frac{\alpha}{c}\right)^{\frac{\lambda}{\alpha}} \, \frac{P(B_1A_3 - A_1B_3)}{(A_2B_1 - A_1B_2)P + H\left(1, 1 + \frac{\lambda}{\alpha}, \frac{c}{\alpha}\right)(B_2A_3 - A_2B_3)}.$$

4. The threshold dividend strategy. In this section, we consider the threshold dividend strategy for surplus U_t, where U_t is defined by (2.3). The company will pay dividends to its shareholders according to the following strategy governed by parameters $\eta > 0$ and $\beta > 0$. Whenever the (modified) surplus is below the level η, no dividends are paid. However, when the modified surplus is above η, dividends are paid continuously at a constant rate β. If the surplus is below the level $-\dfrac{c}{\alpha}$, the absolute ruin happens and the dividend process stops.

Let $\tilde{D}_t(\eta)$ denote the aggregate dividends paid by time $t \geq 0$. The modified surplus process $\tilde{U}_\eta(t)$ is given by

$$\tilde{U}_\eta(t) = U_t - \tilde{D}_t(\eta).$$

Let $\delta > 0$ be the force of interest for valuation, and let D_{T_2} denote the present value of all dividends until the absolute ruin,

$$D_{T_2} = \int\limits_0^{T_2} e^{-\delta t} d\tilde{D}_t(\eta),$$

where

$$T_2 = \inf\left\{t > 0, \tilde{U}_t(\eta) \leq -\frac{c}{\alpha}\right\}.$$

We denote $\tilde{V}(u,\eta)$ as the expected dividend value function of the strategy, that is

$$\tilde{V}(u,\eta) = E_u[D_{T_2}].$$

Let $\tilde{M}(u,y,\eta) = E_u[e^{yD_{T_2}}]$ be the moment generating function of D_{T_2}. For any $y < \infty$, we have $0 < \tilde{M}(u,y,\eta) \leq \lim_{u\uparrow\infty} \tilde{M}(u,y,\eta) = e^{y\frac{\beta}{\delta}} < +\infty$, then $\tilde{M}(u,y,\eta)$ exists.

We define

$$\tilde{M}(u,y,\eta) = \begin{cases} \tilde{M}_1(u,y,\eta), & \text{if} \quad u \geq 0, \\ \tilde{M}_2(u,y,\eta), & \text{if} \quad -\dfrac{c}{\alpha} < u < 0. \end{cases}$$

Theorem 4.1. *Suppose that $\tilde{M}(u,y,\eta)$ is twice continuously differentiable $u \in \left(-\dfrac{c}{\alpha}, \eta\right)$, then $\tilde{M}_1(u,y,\eta)$ and $\tilde{M}_2(u,y,\eta)$ satisfy the following equations*:

$$\lambda\tilde{M}_2(u,y,\eta) = (\alpha u + c)\frac{\partial\tilde{M}_2(u,y,\eta)}{\partial u} - \delta y\frac{\partial\tilde{M}_2(u,y,\eta)}{\partial y} + \frac{1}{2}\sigma_1^2\frac{\partial^2\tilde{M}_2(u,y,\eta)}{\partial u^2} +$$

$$+\lambda\int\limits_0^{u+\frac{c}{\alpha}} \tilde{M}_2(u-x,y,\eta)p(x)\,dx + \lambda\int\limits_{u+\frac{c}{\alpha}}^{\infty} p(x)\,dx, \quad -\frac{c}{\alpha} < u < 0, \tag{4.1}$$

$$\lambda\tilde{M}_1(u,y,\eta) = (au + c)\frac{\partial\tilde{M}_1(u,y,\eta)}{\partial u} - \delta y\frac{\partial\tilde{M}_1(u,y,\eta)}{\partial y} +$$

$$+\frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2)\frac{\partial^2\tilde{M}_1(u,y,\eta)}{\partial u^2} + \lambda\int\limits_0^u \tilde{M}_1(u-x,y,\eta)p(x)\,dx +$$

$$+\lambda\int\limits_u^{u+\frac{c}{\alpha}} \tilde{M}_2(u-x,y,\eta)p(x)\,dx + \lambda\int\limits_{u+\frac{c}{\alpha}}^{\infty} p(x)\,dx, \quad 0 \leq u < \eta, \tag{4.2}$$

$$(\lambda - y\beta)\tilde{M}_1(u,y,\eta) = \frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2)\frac{\partial^2\tilde{M}_1(u,y,\eta)}{\partial u^2} +$$

$$+\big[(c-\beta)+au-\delta y\big]\frac{\partial\tilde{M}_1(u,y,\eta)}{\partial y}+\lambda\int\limits_0^u\tilde{M}_1(u-x,y,\eta)p(x)\,dx+$$

$$+\lambda\int\limits_u^{u+\frac{c}{\alpha}}\tilde{M}_2(u-x,y,\eta)p(x)\,dx+\lambda\int\limits_{u+\frac{c}{\alpha}}^{\infty}p(x)\,dx,\quad u\geq\eta, \tag{4.3}$$

with boundary conditions

$$\tilde{M}_2\left(-\frac{c}{\alpha},y,\eta\right)=1,\qquad \lim_{u\uparrow\infty}\tilde{M}_1(u,y,\eta)=e^{y\frac{\beta}{\delta}},\qquad \tilde{M}_1(0+,y,\eta)=\tilde{M}_2(0-,y,\eta).$$

Proof. The proof of (4.1), (4.2) are similar to (3.1), (3.2), and are omitted. Next, we prove (4.3). By Markovian property, we get

$$\tilde{M}(u,y,\eta)=E_u[\tilde{M}(U_t,ye^{-\delta t},\eta)]+o(t).$$

Hence,

$$\tilde{M}_1(u,y,\eta)=(1-\lambda t)e^{-\lambda t}e^{y\beta t}E_u[\tilde{M}_1(u,ye^{-\delta t},\eta)]+$$

$$+\lambda te^{-\lambda t}e^{y\beta t}E_u\int\limits_0^{Z_t}\tilde{M}_1(Z_t-x,ye^{-\delta t},\eta)p(x)\,dx+$$

$$+\lambda te^{-\lambda t}e^{y\beta t}E_u\int\limits_{Z_t}^{Z_t+\frac{c}{\alpha}}\tilde{M}_2(Z_t-x,ye^{-\delta t},\eta)p(x)\,dx+$$

$$+\lambda te^{-\lambda t}e^{y\beta t}E_u\int\limits_{Z_t+\frac{c}{\alpha}}^{\infty}p(x)\,dx+o(t),$$

where

$$Z_t=u+\int\limits_0^t(c-\beta+aU_s)ds+\int\limits_0^t\sqrt{(\sigma_1+\rho bU_{s-})^2+b^2(1-\rho^2)U_{s-}^2}dB_s.$$

By using the same arguments as in the proof of Theorem 3.1, we get (4.3).

Noting that if $u=-\dfrac{c}{\alpha}$, then $T_2=0$ and $D_{T_2}=0$. So we have

$$\tilde{M}_2\left(-\frac{c}{\alpha},y,\eta\right)=E_{-\frac{c}{\alpha}}\left(e^{yD_{T_2}}\right)=1.$$

$\tilde{M}_1(0+,y,\eta)=\tilde{M}_2(0-,y,\eta)$ due to $\tilde{M}(u,y,\eta)$ is continuous at $u=0$.

By the definition of $\tilde{M}_1(u,y,\eta)$, we get $\lim_{u\uparrow\infty}\tilde{M}_1(u,y,\eta)=e^{y\frac{\beta}{\delta}}$. The boundary conditions for $\tilde{M}_i,\ i=1,2$, are obtained.

Theorem 4.1 is proved.

Set $\tilde{M}(u,y,\eta)=1+y\tilde{V}(u,\eta)+o(y)$, where

$$\tilde{V}(u,\eta) = E_u[D_{T_2}] = \begin{cases} \tilde{V}_1(u,\eta), & u \geq 0, \\ \tilde{V}_2(u,\eta), & -\dfrac{c}{\alpha} < u < 0. \end{cases} \tag{4.4}$$

Substituting (4.4) into (4.1), (4.2) and (4.3), comparing the coefficients of y, we get the following results.

Theorem 4.2. *If* $-\dfrac{c}{\alpha} < u < 0$, *then* $\tilde{V}_2(u,\eta)$ *in* (4.4) *satisfies the equation*

$$(\delta+\lambda)\tilde{V}_2(u,\eta) = (\alpha u + c)\tilde{V}_2'(u,\eta) + \frac{1}{2}\sigma_1^2\tilde{V}_2''(u,\eta) +$$
$$+\lambda\int_0^{u+\frac{c}{\alpha}} \tilde{V}_2(u-x,y,\eta)p(x)\,dx + \lambda\int_{u+\frac{c}{\alpha}}^{\infty} p(x)\,dx. \tag{4.5}$$

If $0 \leq u < \eta$, *then* $\tilde{V}_i(u,\eta), i = 1,2,$ *in* (4.4) *satisfy the equation*

$$(\delta+\lambda)\tilde{V}_1(u,\eta) = (au + c)\tilde{V}_1'(u,\eta) + \frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2)\tilde{V}_1''(u,\eta) +$$
$$+\lambda\int_0^{u} \tilde{V}_1(u-x,\eta)p(x)\,dx + \lambda\int_{u}^{u+\frac{c}{\alpha}} \tilde{V}_2(u-x,\eta)p(x)\,dx + \lambda\int_{u+\frac{c}{\alpha}}^{\infty} p(x)\,dx. \tag{4.6}$$

If $u \geq \eta$, *then* $\tilde{V}_i(u,\eta), i = 1,2,$ *in* (4.4) *satisfy the equation*

$$(\delta+\lambda)\tilde{V}_1(u,\eta) = \frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2)\tilde{V}_1''(u,\eta) + (c-\beta+au)\tilde{V}_1'(u,\eta) +$$
$$+\lambda\int_0^{u} \tilde{V}_1(u-x,\eta)p(x)\,dx + \lambda\int_{u}^{u+\frac{c}{\alpha}} \tilde{V}_2(u-x,\eta)p(x)\,dx + \beta \tag{4.7}$$

with boundary conditions

$$\tilde{V}_2\left(-\frac{c}{\alpha}+,\eta\right) = 0, \qquad \lim_{u\uparrow\infty}\tilde{V}_1(u,\eta) = \frac{\beta}{\delta},$$
$$\tilde{V}_1(0+,\eta) = \tilde{V}_2(0-,\eta), \qquad \tilde{V}_1'(0+,\eta) = \tilde{V}_2'(0-,\eta),$$
$$\tilde{V}_1'(\eta+,\eta) = \tilde{V}_1'(\eta-,\eta), \qquad \tilde{V}_1(\eta+,\eta) = \tilde{V}_1(\eta-,\eta).$$

***Remark* 4.1.** Letting $\lambda = 0$ in (4.5), (4.6) and (4.7), then they can be changed to

$$\delta\tilde{V}_2(u,\eta) = (\alpha u + c)\tilde{V}_2'(u,\eta) + \frac{1}{2}\sigma_1^2\tilde{V}_2''(u,\eta), \qquad -\frac{c}{\alpha} < u < 0, \tag{4.8}$$

$$\delta\tilde{V}_1(u,\eta) = (au + c)\tilde{V}_1'(u,\eta) + \frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2)\tilde{V}_1''(u,\eta), \qquad 0 \leq u < \eta, \tag{4.9}$$

$$\delta\tilde{V}(u,\eta) = \frac{1}{2}(b^2u^2 + 2\rho b\sigma_1 u + \sigma_1^2)\tilde{V}_1''(u,\eta) + (c-\beta+au)\tilde{V}'(u,\eta) + \beta, \qquad u \geq \eta. \tag{4.10}$$

In the case of $u \geq 0$, (4.9) and (4.10) are obtained by Yin and Wen [22], where they got the same results by using different methods. We extend their results for the more general case.

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