UDC 517.5

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CLASSICAL KANTOROVICH OPERATORS REVISITED*

ЗНОВУ ПРО КЛАСИЧНІ ОПЕРАТОРИ КАНТОРОВИЧА

The main object of this paper is to improve some known estimates for the classical Kantorovich operators. We obtain a quantitative Voronovskaya-type result in terms of the second moduli of continuity, which improves some previous results. In order to explain the nonmultiplicativity of the Kantorovich operators, we present a Chebyshev – Grüss inequality. Two Grüss – Voronovskaya theorems for Kantorovich operators are also considered.

Основним об’єктом дослідження є поліпшення деяких відомих оцінок для класичних операторів Канторовича. Отримано кількісний результат типу Вороновської в термінах других модулів неперервності, що поліпшує деякі попередні результати. Щоб пояснити немультиплікативність операторів Канторовича, ми наводимо нерівність Чебишова – Грюсса. Також розглянуто теореми Грюсса – Вороновської для операторів Канторовича.

1. Introduction. In 1930 L. V. Kantorovich [11] introduced a significant modification of the classical Bernstein operators given by

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int\limits_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt.$$

Here $n \geq 1,\ f \in L_1[0,1],\ x \in [0,1]$ and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \leq k \leq n,$$

$$p_{n,k} \equiv 0, \qquad \text{if} \quad k < 0 \quad \text{or} \quad k > n.$$

These mappings are relevant since they provide a constructive tool to approximate any function in $L_p[0,1],\ 1 \leq p < \infty$, in the L_p-norm. For $p = \infty$, $C[0,1]$ has to be used instead of $L_\infty[0,1]$.

These classical Kantorovich operators have been attracting a lot of attention since then, but results on them are somehow scattered in the literature. They share this with other relevant variations of the Bernstein-type: Durrmeyer, genuine Bernstein – Durrmeyer and, last but not least, variation-diminishing Schoenberg splines.

In the present note we first collect and improve some of the known estimates by giving quite a precise inequality for $f \in C^r[0,1],\ r \in \mathbb{N} \cup \{0\}$, a new Voronovskaya result in terms of ω_2 and a Chebyshev – Grüss inequality giving an explanation of their nonmultiplicativity. The last part of this article deals with two Grüss – Voronovskaya theorems for Kantorovich operators.

Most estimates in this article will be given in terms of moduli of smoothness of higher order. In the background, but not explicitly mentioned, is always the K-functional technique. In this sense we were very much influenced by the work of Zygmund (see, e.g., [16]), a hardly accessible conference contribution of Peetre [12] and also by the book of Dzyadyk [4].

* The first author was supported by Lucian Blaga University of Sibiu research under grant LBUS-IRG-2018-04. The second appreciates financial support of the University of Duisburg-Essen.

2. Some previous results. In this section we collect some results given earlier. Quite a strong general result was given by the second author and Xin-long Zhou [10] in 1995.

Let $\varphi(x) = \sqrt{x(1-x)}$ and $P(D)$ be the differential operator given by

$$P(D)f := (\varphi^2 f')', \qquad f \in C^2[0,1].$$

For $f \in L_p[0,1]$, $1 \le p \le \infty$, the functional $K(f,t)_p$ is defined as below

$$K(f,t)_p := \inf \left\{ \|f-g\|_p + t^2 \|P(D)g\|_p : g \in C^2[0,1] \right\}.$$

Using the above functional in [10] the following theorem was proved.

Theorem 2.1. *There exists an absolute positive constant C such that for all $f \in L_p[0,1]$, $1 \le \le p \le \infty$, there holds*

$$C^{-1}K(f,n^{-1/2})_p \le \|f - K_n f\|_p \le CK(f,n^{-1/2})_p.$$

Also, in order to characterize the K-functional used in Theorem 2.1, the next result was given in [10].

Theorem 2.2. *We have*

$$K(f,t)_p \sim \omega_\varphi^2(f,t)_p + t^2 E_0(f)_p, \qquad 1 < p < \infty,$$

and

$$K(f,t)_\infty \sim \omega_\varphi^2(f,t)_\infty + \omega(f,t^2)_\infty.$$

Here $\omega(f,t)_p$ is the classical modulus, $\omega_\varphi^2(f,t)_\infty$ denotes the second order modulus of smoothness with weight function φ and $E_0(f)_p$ is the best approximation constant of f defined by

$$E_0(f)_p = \inf_c \|f - c\|_p.$$

Moreover, all quantities subscripted by ∞ are taken with respect to the uniform norm in $C[0,1]$. The following theorem of Păltănea [13] is the key to give a more explicit result in terms of classical moduli for continous functions. (See [8] for details.)

Theorem 2.3 [13]. *If $L\colon C[0,1] \to C[0,1]$ is a positive linear operator, then for $f \in C[0,1]$, $x \in [0,1]$ and each $0 < h \le \dfrac{1}{2}$ the following holds*:

$$|L(f;x) - f(x)| \le |L(e_0;x) - 1|\,|f(x)| + \frac{1}{h}|L(e_1 - x;x)|\omega(f;h) +$$

$$+ \left[(Le_0)(x) + \frac{1}{2h^2} L((e_1 - x)^2;x) \right] \omega_2(f;h).$$

The condition $h \le 1/2$ can be eliminated for operators L reproducing linear functions.

Theorem 2.4. *For all $f \in C[0,1]$ and all $n \ge 4$,*

$$\|K_n f - f\|_\infty \le \frac{1}{2\sqrt{n}} \omega_1 \left(f; \frac{1}{\sqrt{n}} \right) + \frac{9}{8} \omega_2 \left(f; \frac{1}{\sqrt{n}} \right).$$

This result can be extended to simultaneous approximation, see again [8].

Theorem 2.5. *Let $r \in \mathbb{N}_0, n \ge 4,\ f \in C^r[0,1]$. Then*

$$\|D^r K_n f - D^r f\|_\infty \le \frac{(r+1)r}{2n} \|D^r f\|_\infty + \frac{r+1}{2\sqrt{n}} \omega_1 \left(D^r f; \frac{1}{\sqrt{n}} \right) + \frac{9}{8} \omega_2 \left(D^r f; \frac{1}{\sqrt{n}} \right).$$

3. A quantitative Voronovskaya result. This part has its predecessor in a hardly known booklet of Videnskij in which a quantitative version of the well-known Voronovskaya theorem for the classical Bernstein operators can be found (see [15]). This estimate was generalized and improved in [9]. An application for Kantorovich operators was given in [8]. Here we improve it as follows:

Theorem 3.1. *For* $n \geq 1$ *and* $f \in C^2[0,1]$, *one has*

$$\left\| n\left(K_n f - f\right) - \frac{1}{2}\left(Xf'\right)'\right\|_\infty \leq \frac{2}{3(n+1)}\left(\frac{3}{4}\|f'\|_\infty + \|f''\|_\infty\right) +$$

$$+\frac{9}{32}\left\{\frac{2}{\sqrt{n+1}}\omega_1\left(f''; \frac{1}{\sqrt{n+1}}\right) + \omega_2\left(f''; \frac{1}{\sqrt{n+1}}\right)\right\}, \tag{1}$$

where $X = x(1-x)$ *and* $X' = 1-2x,\ x \in [0,1]$.

Proof. From [9] (Theorem 3) we get

$$\left| K_n(f;x) - f(x) - K_n(t-x;x)f'(x) - \frac{1}{2}K_n\left((e_1-x)^2;x\right)f''(x)\right| \leq$$

$$\leq K_n((e_1-x)^2;x)\left\{\frac{|K_n((e_1-x)^3;x)|}{K_n((e_1-x)^2;x)}\frac{5}{6h}\omega_1(f'';h) + \left(\frac{3}{4} + \frac{K_n((e_1-x)^4;x)}{K_n((e_1-x)^2;x)}\frac{1}{16h^2}\right)\omega_2(f'';h)\right\}.$$

Using the central moments up to order 4 for Kantorovich operators, namely

$$K_n\left(t-x;x\right) = \frac{1-2x}{2(n+1)},$$

$$K_n\left((t-x)^2;x\right) = \frac{1}{(n+1)^2}\left\{x(1-x)(n-1) + \frac{1}{3}\right\},$$

$$K_n\left((t-x)^3;x\right) = \frac{1-2x}{4(n+1)^3}\left\{10x(1-x)n + 2x^2 - 2x + 1\right\},$$

$$K_n\left((t-x)^4;x\right) = \frac{1}{(n+1)^4}\left\{3x^2(1-x)^2n^2 + 5x(1-x)(1-2x)^2n + x^4 - 2x^3 + 2x^2 - x + \frac{1}{5}\right\},$$

we have

$$\frac{|K_n\left((t-x)^3;x\right)|}{K_n\left((t-x)^2;x\right)} \leq \frac{5}{2(n+1)}, \qquad \frac{|K_n\left((t-x)^4;x\right)|}{K_n\left((t-x)^2;x\right)} \leq \frac{3(n+2)}{(n+1)^2}.$$

Therefore, the following inequality holds:

$$\left| K_n(f;x) - f(x) - \frac{1-2x}{2(n+1)}f'(x) - \frac{1}{2}\left[\frac{x(1-x)(n-1)}{(n+1)^2} + \frac{1}{3(n+1)^2}\right]f''(x)\right| \leq$$

$$\leq \left[x(1-x)\frac{n-1}{(n+1)^2} + \frac{1}{3(n+1)^2}\right]\left\{\frac{25}{12h(n+1)}\omega_1(f'';h) + \left(\frac{3}{4} + \frac{3(n+2)}{16h^2(n+1)^2}\right)\omega_2(f'';h)\right\}$$

and for $h = \dfrac{1}{\sqrt{n+1}}$ we obtain, after multiplying both sides by n,

$$\left| n\left[K_n(f;x)-f(x)\right]-\frac{n}{n+1}\left(\frac{1}{2}-x\right)f'(x)-\frac{1}{2}\left[x(1-x)\frac{n(n-1)}{(n+1)^2}+\frac{n}{3(n+1)^2}\right]f''(x)\right| \le$$

$$\le \frac{9}{32}\left\{\frac{2}{\sqrt{n+1}}\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right)+\omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\}.$$

We can write

$$\left| n\left[K_n(f;x)-f(x)\right]-\frac{1-2x}{2}f'(x)-\frac{1}{2}x(1-x)f''(x)\right| \le$$

$$\le \left| n\left[K_n(f;x)-f(x)\right]-\frac{n}{n+1}\left(\frac{1}{2}-x\right)f'(x)-\frac{1}{2}\left[x(1-x)\frac{n(n-1)}{(n+1)^2}+\frac{n}{3(n+1)^2}\right]f''(x)\right| +$$

$$+\left|\frac{1-2x}{2}\frac{1}{n+1}f'(x)+\frac{1}{2}x(1-x)\frac{3n+1}{(n+1)^2}f''(x)-\frac{n}{6(n+1)^2}f''(x)\right| \le$$

$$\le \frac{9}{32}\left\{\frac{2}{\sqrt{n+1}}\omega_1\left(f'';\frac{1}{\sqrt{n+1}}\right)+\omega_2\left(f'';\frac{1}{\sqrt{n+1}}\right)\right\}+\frac{2}{3(n+1)}\left(\frac{3}{4}\|f'\|_\infty+\|f''\|_\infty\right).$$

4. Chebyshev – Grüss inequality for Kantorovich operators. In a 2011 paper Raşa and the present authors [1] published the following Grüss-type inequality for positive linear operators reproducing constant functions. We give below the improved form of Rusu given in [14]:

Theorem 4.1. *Let $H: C[a,b]\to C[a,b]$ be positive, linear and satisfy $He_0=e_0$. Put*

$$D(f,g;x):=H(fg;x)-H(f;x)\,H(g;x).$$

Then, for $f,g\in C[a,b]$ and $x\in[a,b]$ fixed, one has

$$|D(f,g;x)|\le\frac{1}{4}\tilde{\omega}\left(f;2\sqrt{H\left((e_1-x)^2;x\right)}\right)\tilde{\omega}\left(g;2\sqrt{H\left((e_1-x)^2;x\right)}\right).$$

Here $\tilde{\omega}$ is the least concave majorant of the first order modulus ω_1 given by

$$\tilde{\omega}(f;t)=\sup\left\{\frac{(t-x)\omega_1(f;y)+(y-t)\omega_1(f;x)}{y-x}:0\le x\le t\le y\le b-a, x\ne y\right\}.$$

***Remark* 4.1.** For an accesible proof of the equality between $\tilde{\omega}$ and a certain K-functional used in the proof of the above theorem see [13].

Hence the nonmultiplicativity of Kantorovich operators can be explained as in the following theorem.

Theorem 4.2. *For the classical Kantorovich operators $K_n: C[0,1]\to C[0,1]$ one has the uniform inequality*

$$\|K_n(fg)-K_nfK_ng\|_\infty\le\frac{1}{4}\tilde{\omega}\left(f;2\sqrt{\frac{1}{2(n+1)}}\right)\tilde{\omega}\left(g;2\sqrt{\frac{1}{2(n+1)}}\right),\quad n\ge 1, \qquad (2)$$

for all $f,\ g\in C[0,1]$.

Proof. The most precise upper bound is obtained if we use the exact representation

$$K_n\left((t-x)^2;x\right)=\frac{1}{(n+1)^2}\left\{(n-1)x(1-x)+\frac{1}{3}\right\}.$$

Close to $x=0,1$ this shows the familiar endpoint improvement. For shortness we use the estimate

$$K_n\left((t-x)^2;x\right)\le\frac{1}{2(n+1)}.$$

5. Grüss – Voronovskaya theorems. The first Grüss – Voronovskaya theorem for classical Bernstein operators was given by Gal and Gonska [5]. In Theorem 2.1 of this paper a quantitative form was given (see also Theorem 2.5 there). The other examples in [5] deal with operators reproducing linear functions; this is not the case for the Kantorovich mappings. The limit for K_n was identified recently in [2] to be the same as in the Bernstein case, namely

$$f'(x)g'(x)x(1-x) \text{ for } f,g\in C^2[0,1].$$

Our first quantitative version is given in the following theorem.

Theorem 5.1. *Let* $f,\ g\in C^2[0,1]$. *Then, for each* $x\in[0,1]$,

$$\left\|n\left[K_n(fg)-K_nf\cdot K_ng\right]-Xf'g'\right\|_\infty=\begin{cases}o(1), & f,g\in C^2[0,1],\\ \mathcal{O}\left(\dfrac{1}{\sqrt{n}}\right), & f,g\in C^3[0,1],\\ \mathcal{O}\left(\dfrac{1}{n}\right), & f,g\in C^4[0,1].\end{cases}$$

Proof. We proceed as in [5] by creating first three Voronovskaya-type expressions from the difference in question plus the remaining quantities. Recall that the Voronovskaya limit for Kantorovich operators is

$$\frac{1}{2}(Xf')'=\frac{1}{2}Xf''(x)+\frac{1}{2}X'f'(x),$$

where $X:=x(1-x)$, so $X'=1-2x$.

For $f,\ g\in C^2[0,1]$, one has

$$K_n(fg;x)-K_n(f;x)K_n(g;x)-\frac{1}{n}Xf'(x)g'(x)=$$

$$=K_n(fg;x)-(fg)(x)-\frac{1}{2n}\left(X(fg)'\right)'-$$

$$-f(x)\left[K_n(g;x)-g(x)-\frac{1}{2n}(Xg')'\right]-g(x)\left[K_n(f;x)-f(x)-\frac{1}{2n}(Xf')'\right]+$$

$$+\left[g(x)-K_n(g;x)\right]\left[K_n(f;x)-f(x)\right]-$$

$$-K_n(f;x)K_n(g;x)-\frac{1}{n}Xf'g'+(fg)(x)+\frac{1}{2n}(X(fg)')'+$$

$$+f(x)\left[K_n(g;x)-g(x)-\frac{1}{2n}(Xg')'\right]+g(x)\left[K_n(f;x)-f(x)-\frac{1}{2n}(Xf')'\right]-$$

$$-\left[g(x)-K_n(g;x)\right]\left[K_n(f;x)-f(x)\right].$$

The first three lines will be estimated below. First we will show that the sum of the following three lines equals 0.

For the time being we will leave out the argument x. One has

$$-K_nf\cdot K_ng-\frac{1}{n}Xf'g'+fg+\frac{1}{2n}\left(X'(fg)'+X(fg)''\right)+$$

$$+fK_ng-fg-\frac{1}{2n}f(X'g'+Xg'')+gK_nf-fg-\frac{1}{2n}g(X'f'+Xf'')-$$

$$-[g-K_ng]\,[K_nf-f]=$$

$$=-K_nf\cdot K_ng-\frac{1}{n}Xf'g'+fg+\frac{1}{2n}\left(X'f'g+X'fg'\right)+\frac{1}{2n}X\left(f''g+2f'g'+fg''\right)+$$

$$+fK_ng-fg-\frac{1}{2n}(fX'g'+fXg'')+gK_nf-fg-\frac{1}{2n}(gX'f'+gXf'')-$$

$$-gK_nf+K_ng\cdot K_nf+fg-fK_ng=0.$$

For the first two lines above we will use the Voronovskaya estimate given earlier, namely that for $h\in C^2[0,1]$ one has

$$\left\|n\left(K_nh-h\right)-\frac{1}{2}\left(Xh'\right)'\right\|_\infty\le\frac{2}{3(n+1)}\left(\frac{3}{4}\|h'\|_\infty+\|h''\|_\infty\right)+$$

$$+\frac{9}{32}\left\{\frac{2}{\sqrt{n+1}}\omega_1\left(h'';\frac{1}{\sqrt{n+1}}\right)+\omega_2\left(h'';\frac{1}{\sqrt{n+1}}\right)\right\}=:U(h,n).$$

For the third line we use Theorem 2.4 showing that for $h\in C^2[0,1]$ we get

$$\|K_nh-h\|_\infty\le\frac{1}{2n}\|h'\|_\infty+\frac{9}{8n}\|h''\|_\infty=\mathcal{O}\left(\frac{1}{n}\right).$$

Collecting these inequalities gives

$$\left\|n\left[K_n(fg)-K_nf\cdot K_ng\right]-Xf'g'\right\|_\infty\le$$

$$\le U(fg,n)+\|f\|_\infty U(g,n)+\|g\|_\infty U(f,n)+\mathcal{O}\left(\frac{1}{n}\right)=$$

$$=\begin{cases}o(1), & f,g\in C^2[0,1],\\ \mathcal{O}\left(\dfrac{1}{\sqrt{n}}\right), & f,g\in C^3[0,1],\\ \mathcal{O}\left(\dfrac{1}{n}\right), & f,g\in C^4[0,1].\end{cases}$$

In the following we give a Grüss – Voronovskaya type theorem when f and g are only in $C^1[0,1]$.

Theorem 5.2. *Let $f,\ g \in C^1[0,1]$ and $n \geq 1$. Then there is a constant C independent of n, f, g and x, such that*

$$\left\| K_n(fg) - K_nf \cdot K_ng - \frac{X}{n} f'g' \right\|_\infty \leq \frac{C}{n} \left\{ \omega_3\left(f', n^{-\frac{1}{6}}\right) \omega_3\left(g', n^{-\frac{1}{6}}\right) + \right.$$

$$+ \|f'\|_\infty \omega_3\left(g', n^{-\frac{1}{6}}\right) + \|g'\|_\infty \omega_3\left(f', n^{-\frac{1}{6}}\right) +$$

$$\left. + \max\left\{ \frac{\|f'\|_\infty}{n^{\frac{1}{2}}}, \omega_3\left(f', n^{-\frac{1}{6}}\right) \right\} \max\left\{ \frac{\|g'\|_\infty}{n^{\frac{1}{2}}}, \omega_3\left(g', n^{-\frac{1}{6}}\right) \right\} \right\}.$$

Proof. Let

$$E_n(f,g;x) = K_n(fg;x) - K_n(f;x)K_n(g;x) - \frac{x(1-x)}{n} f'(x)g'(x), \qquad (3)$$

and denote C a constant independent of n, f, g and x, which may change its values during the course of the proof.

For $f, g \in C^1[0,1]$ fixed and $u, v \in C^4[0,1]$ arbitrary, one has

$$|E_n(f,g;x)| = |E_n(f-u+u, g-v+v;x)| \leq$$

$$\leq |E_n(f-u, g-v;x)| + |E_n(u, g-v;x)| + |E_n(f-u, v;x)| + |E_n(u,v;x)|. \qquad (4)$$

Let $h(x) = x,\ x \in [0,1]$. Applying [1] (Theorem 4) there exists $\eta,\ \theta \in [0,1]$ such that

$$K_n(fg;x) - K_n(f;x)K_n(g;x) = f'(\eta)g'(\theta)\left[K_n(h^2;x) - (K_n(h;x))^2\right] =$$

$$= f'(\eta)g'(\theta)\left\{ x(1-x)\frac{n}{(n+1)^2} + \frac{1}{12(n+1)^2} \right\}. \qquad (5)$$

From (3) and (5) we get

$$|nE_n(f,g;x)| \leq \left[x(1-x)\frac{n^2}{(n+1)^2} + \frac{n}{12(n+1)^2} + x(1-x) \right] \|f'\|_\infty \|g'\|_\infty \leq$$

$$\leq 2\left[x(1-x) + \frac{1}{24(n+1)} \right] \|f'\|_\infty \|g'\|_\infty. \qquad (6)$$

Using Theorem 3.1, for $f \in C^4[0,1]$, we have

$$\left| n\left[K_n(f;x) - f(x)\right] - \frac{1}{2}\left(Xf'\right)'(x) \right| \leq C\frac{1}{n}\left(\|f'\|_\infty + \|f''\|_\infty + \|f'''\|_\infty + \|f^{(4)}\|_\infty \right).$$

But, for $f \in C^n[a,b],\ n \in \mathbb{N}$, one has (see [6], Remark 2.15)

$$\max_{0 \leq k \leq n} \left\{ \|f^{(k)}\| \right\} \leq C \max\left\{ \|f\|_\infty, \|f^{(n)}\|_\infty \right\}.$$

Therefore,

$$\left|n\left[K_n(f;x)-f(x)\right]-\frac{1}{2}\left(Xf'\right)'(x)\right|\leq\frac{C}{n}\max\left\{\|f'\|_\infty,\|f^{(4)}\|_\infty\right\}. \tag{7}$$

For $u,v\in C^4[0,1]$ using the same decomposition as in proof of Theorem 5.1, the relation (7) and Theorem 2.4, we get

$$|E_n(u,v;x)|\leq\left|K_n(uv;x)-(uv)(x)-\frac{1}{2n}\left(X(uv)'\right)'\right|+$$

$$+|u(x)|\left|K_n(v;x)-v(x)-\frac{1}{2n}(Xv')'\right|+|v(x)|\left|K_n(u;x)-u(x)-\frac{1}{2n}(Xu')'\right|+$$

$$+|v(x)-K_n(v;x)|\,|K_n(u;x)-u(x)|\leq$$

$$\leq\frac{C}{n^2}\max\left\{\|u'\|_\infty,\|u^{(4)}\|_\infty\right\}\max\left\{\|v'\|_\infty,\|v^{(4)}\|_\infty\right\}. \tag{8}$$

From the relations (4), (6) and (8) we obtain

$$|E_n(f,g;x)|\leq\frac{2}{n}\left[x(1-x)+\frac{1}{24(n+1)}\right]\left\{\|(f-u)'\|_\infty\|(g-v)'\|_\infty+\|u'\|_\infty\|(g-v)'\|_\infty+\right.$$

$$\left.+\|(f-u)'\|_\infty\|v'\|_\infty\right\}+\frac{C}{n^2}\max\left\{\|u'\|_\infty,\|u^{(4)}\|_\infty\right\}\max\left\{\|v'\|_\infty,\|v^{(4)}\|_\infty\right\}.$$

Using [7] (Lemma 3.1) for $r=1,\ s=2,\ f_{h,3}=u$ and $g_{h,3}=v$, for all $h\in(0,1]$ and $n\in\mathbb{N}$, we have

$$|E_n(f,g;x)|\leq\frac{C}{n}\left\{\omega_3(f',h)\omega_3(g',h)+\frac{1}{h}\omega_1(f,h)\omega_3(g',h)+\frac{1}{h}\omega_1(g,h)\omega_3(f',h)\right\}+$$

$$+\frac{C}{n^2}\max\left\{\frac{1}{h}\omega_1(f,h),\frac{1}{h^3}\omega_3(f',h)\right\}\max\left\{\frac{1}{h}\omega_1(g,h),\frac{1}{h^3}\omega_3(g',h)\right\}\leq$$

$$\leq\frac{C}{n}\left\{\omega_3(f',h)\omega_3(g',h)+\|f'\|_\infty\omega_3(g',h)+\|g'\|_\infty\omega_3(f',h)\right\}+$$

$$+\frac{C}{n^2}\max\left\{\|f'\|_\infty,\frac{1}{h^3}\omega_3(f',h)\right\}\max\left\{\|g'\|_\infty,\frac{1}{h^3}\omega_3(g',h)\right\}.$$

Choosing $h=n^{-\frac{1}{6}}$, we obtain

$$|E_n(f,g;x)|\leq\frac{C}{n}\left\{\omega_3\left(f',n^{-\frac{1}{6}}\right)\omega_3\left(g',n^{-\frac{1}{6}}\right)+\right.$$

$$+\|f'\|_\infty\omega_3\left(g',n^{-\frac{1}{6}}\right)+\|g'\|_\infty\omega_3\left(f',n^{-\frac{1}{6}}\right)+$$

$$\left.+\max\left\{\frac{\|f'\|_\infty}{n^{\frac{1}{2}}},\omega_3\left(f',n^{-\frac{1}{6}}\right)\right\}\max\left\{\frac{\|g'\|_\infty}{n^{\frac{1}{2}}},\omega_3\left(g',n^{-\frac{1}{6}}\right)\right\}\right\}.$$

This implies the theorem.

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Received 28.08.18