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FINE SPECTRA OF TRIDIAGONAL TOEPLITZ MATRICES
ТОНКІ СПЕКТРИ ТРИДІАГОНАЛЬНИХ МАТРИЦЬ ТЬОПЛІЦА

The fine spectra of n-banded triangular Toeplitz matrices and $(2n+1)$-banded symmetric Toeplitz matrices were computed in (*M. Altun*, Appl. Math. and Comput. – 2011. – **217**. – P. 8044 – 8051) and (*M. Altun*, Abstr. and Appl. Anal. – 2012. – Article ID 932785). As a continuation of these results, we compute the fine spectra of tridiagonal Toeplitz matrices. These matrices are, in general, not triangular and not symmetric.

Тонкі спектри n-смугових трикутних матриць Тьопліца та $(2n+1)$-смугових симетричних матриць Тьопліца було отримано в (*M. Altun*, Appl. Math. and Comput. – 2011. – **217**. – P. 8044 – 8051) та (*M. Altun*, Abstr. and Appl. Anal. – 2012. – Article ID 932785). Як продовження цих результатів розраховано тонкі спектри тридіагональних матриць Тьопліца. В загальному випадку ці матриці не є ані трикутними, ані симетричними.

1. Introduction and preliminaries. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. Some other parts also arise by examining the surjectivity of the operator and continuity of the inverse operator. Such subparts of the spectrum are called the fine spectra of the operator.

The spectra and fine spectra of linear operators defined by some particular limitation matrices over some sequence spaces were studied by several authors. We introduce the knowledge in the existing literature concerning the spectrum and the fine spectrum. Wenger [21] examined the fine spectrum of the integer power of the Cesàro operator over c and Rhoades [17] generalized this result to the weighted mean methods. Reade [16] worked on the spectrum of the Cesàro operator over the sequence space c_0. Gonzàlez [12] studied the fine spectrum of the Cesàro operator over the sequence space ℓ_p. Okutoyi [15] computed the spectrum of the Cesàro operator over the sequence space bv. Recently, Rhoades and Yıldırım [18] examined the fine spectrum of factorable matrices over c_0 and c. Akhmedov and Başar [1, 2] have determined the fine spectrum of the Cesàro operator over the sequence spaces c_0, ℓ_∞, and ℓ_p. Altun and Karakaya [8] computed the fine spectra of lacunary matrices over c_0 and c. Furkan, Bilgiç and Altay [10] determined the fine spectrum of $B(r,s,t)$ over the sequence spaces c_0 and c, where $B(r,s,t)$ is a lower triangular triple-band matrix. Later, Altun [6, 7] computed the fine spectra of triangular and symmetric Toeplitz matrices over c_0 and c.

Recently, Akhmedov and El-Shabrawy [3] have obtained the fine spectrum of the generalized difference operator $\Delta_{a,b}$, defined as a double band matrix with the convergent sequences $\tilde{a} = (a_k)$ and $\tilde{b} = (b_k)$ having certain properties, over c. In 2010, Srivastava and Kumar [19] have determined the spectra and the fine spectra of the generalized difference operator Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_n)$ and they have just generalized these results by the generalized difference operator Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ (see [20]).

In this work, our purpose is to determine the spectra of the operator, for which the corresponding matrix is a tridiagonal Toeplitz matrix, over the sequence spaces ℓ_1, c_0, c, and ℓ_∞. We will also give the fine spectral results for the spaces ℓ_1, c_0, and c.

Let X and Y be Banach spaces and $U\colon X\to Y$ be a bounded linear operator. By $\mathcal{R}(U)$ we denote the range of U, i.e.,

$$\mathcal{R}(U)=\{y\in Y: y=Ux; x\in X\}.$$

By $B(X)$ we denote the set of all bounded linear operators on X into itself. If X is any Banach space and $U\in B(X)$, then the *adjoint* U^* of U is a bounded linear operator on the dual X^* of X defined by $(U^*\phi)(x)=\phi(Ux)$ for all $\phi\in X^*$ and $x\in X$. Let $X\neq\{\theta\}$ be a complex normed space and $U:\mathcal{D}(U)\to X$ be a linear operator with domain $\mathcal{D}(U)\subseteq X$. With U we associate the operator

$$U_\lambda=U-\lambda I,$$

where λ is a complex number and I is the identity operator on $\mathcal{D}(U)$. If U_λ has an inverse, which is linear, we denote it by U_λ^{-1}, that is

$$U_\lambda^{-1}=(U-\lambda I)^{-1}$$

and call it the *resolvent operator* of U_λ. If $\lambda=0$ we will simply write U^{-1}. Many properties of U_λ and U_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that U_λ^{-1} exists. Boundedness of U_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of U_λ^{-1} is dense in X. For our investigation of U, U_λ, and U_λ^{-1}, we need some basic concepts in spectral theory which are given as follows (see [14, p. 370, 371]):

Let $X\neq\{\theta\}$ be a complex normed space and $U:\mathcal{D}(U)\to X$ be a linear operator with domain $\mathcal{D}(U)\subseteq X$. A *regular value* λ of U is a complex number such that

$(\mathbf{R_1})$ U_λ^{-1} exists,

$(\mathbf{R_2})$ U_λ^{-1} is bounded,

$(\mathbf{R_3})$ U_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* $\rho(U)$ of U is the set of all regular values λ of U. Its complement $\sigma(U)=\mathbb{C}\setminus\rho(U)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of U. Furthermore, the spectrum $\sigma(U)$ is partitioned into three disjoint sets as follows: The *point spectrum* $\sigma_p(U)$ is the set such that U_λ^{-1} does not exist. A $\lambda\in\sigma_p(U)$ is called an *eigenvalue* of U. The *continuous spectrum* $\sigma_c(U)$ is the set such that U_λ^{-1} exists and satisfies (R_3) but not (R_2). The *residual spectrum* $\sigma_r(U)$ is the set such that U_λ^{-1} exists but does not satisfy (R_3).

A triangle is a lower triangular matrix with all of the principal diagonal elements nonzero. We shall write ℓ_∞, c, and c_0 for the spaces of all bounded, convergent and null sequences, respectively, that is

$$\ell_\infty=\left\{x=(x_k):\ \sup_k|x_k|<\infty\right\},$$

$$c=\left\{x=(x_k):\ \lim_k x_k \text{ exists}\right\},$$

$$c_0=\left\{x=(x_k):\ \lim_k x_k=0\right\}.$$

By ℓ_p we denote the space of all p-absolutely summable sequences, where $1 \leq p < \infty$. In particular ℓ_1 denotes the space of all absolutely summable sequences, that is

$$\ell_1 = \left\{ x = (x_k) \colon \sum_k |x_k| < \infty \right\},$$

$$\ell_p = \left\{ x = (x_k) \colon \sum_k |x_k|^p < \infty \right\}.$$

Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A \colon \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in γ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad n \in \mathbb{N}. \tag{1}$$

By (μ, γ) we denote the class of all matrices A such that $A \colon \mu \to \gamma$. Thus, $A \in (\mu, \gamma)$ if and only if the series on the right-hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \gamma$ for all $x \in \mu$.

A tridiagonal nonsymmetric infinite matrix is of the form

$$T = T(q, r, s) = \begin{bmatrix} q & r & 0 & 0 & 0 & 0 & \cdots \\ s & q & r & 0 & 0 & 0 & \cdots \\ 0 & s & q & r & 0 & 0 & \cdots \\ 0 & 0 & s & q & r & 0 & \cdots \\ 0 & 0 & 0 & s & q & r & \cdots \\ 0 & 0 & 0 & 0 & s & q & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The spectral results when T is triangular can be found in [6], so for the sequel we will have $s \neq 0$ and $r \neq 0$.

Let R be the right shift operator

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and L be the left shift operator

$$L = R^t = R^{-1}.$$

Let us call $Q(z) = sz + q + rz^{-1}$ as the associated function of the operator T. Let P be the function $P(z) = rz + q + sz^{-1}$. Clearly, the roots of $Q(z)$ are nonzero. Let α_1 and α_2 be roots of $Q(z)$. It is easy to verify that α_1^{-1} and α_2^{-1} are roots of $P(z)$. We also have

$$T = s(I - \alpha_1 L)(R - \alpha_2 I). \tag{2}$$

Let D be the unit disc $\{z \in \mathbb{C} \colon |z| \leq 1\}$; ∂D be the unit circle $\{z \in \mathbb{C} \colon |z| = 1\}$ and D° be the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$.

Theorem 1.1 (cf. [22]). *Let U be an operator with the associated matrix $A = (a_{nk})$.*

(i) $U \in B(c)$ *if and only if*

$$||A|| := \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty, \tag{3}$$

$$a_k := \lim_{n\to\infty} a_{nk} \quad \text{exists for each } k, \tag{4}$$

$$a := \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \quad \text{exists.} \tag{5}$$

(ii) $U \in B(c_0)$ *if and only if* (3) *and* (4) *with* $a_k = 0$ *for each* k.

(iii) $U \in B(\ell_\infty)$ *if and only if* (3).

In these cases, the operator norm of U *is*

$$||U||_{(\ell_\infty,\ell_\infty)} = ||U||_{(c,c)} = ||U||_{(c_0,c_0)} = ||A||.$$

(iv) $U \in B(\ell_1)$ *if and only if*

$$||A^t|| = \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty. \tag{6}$$

In this case the operator norm of U *is* $||U||_{(\ell_1,\ell_1)} = ||A^t||$.

***Corollary* 1.1.** $T \in B(\mu)$ *for* $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$ *and*

$$\|T\|_{(\mu,\mu)} = |q| + |r| + |s|.$$

Lemma 1.1. *For the linear system of equations*

$$\begin{aligned} qx_0 + rx_1 &= 0, \\ sx_0 + qx_1 + rx_2 &= 0, \\ sx_1 + qx_2 + rx_3 &= 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{7}$$

the general solution is

$$x_n = \begin{cases} C\left(\dfrac{\alpha_2}{\alpha_1^n} - \dfrac{\alpha_1}{\alpha_2^n}\right), & \text{if } \alpha_1 \neq \alpha_2, \\ C\dfrac{1+n}{\alpha_1^n}, & \text{if } \alpha_1 = \alpha_2, \end{cases} \tag{8}$$

where $C \in \mathbb{C}$ is a general constant.

Proof. To solve (7) we have $x_1 = -(q/r)x_0$ and

$$rx_n + qx_{n-1} + sx_{n-2} = 0 \quad \text{for} \quad n \geq 2.$$

This is a linear recurrence relation with the characteristic equation

$$0 = rz^2 + qz + s = zP(z) = r(z - \alpha_1^{-1})(z - \alpha_2^{-1})$$

which has roots α_1^{-1} and α_2^{-1}. By the theory of recurrence relations, the general solution of (7) is

$$x_n = \begin{cases} \dfrac{C_1}{\alpha_1^n} + \dfrac{C_2}{\alpha_2^n}, & \text{if } \alpha_1 \neq \alpha_2, \\ \dfrac{C_3 + C_4 n}{\alpha_1^n}, & \text{if } \alpha_1 = \alpha_2, \end{cases}$$

with the restriction from the first line, that is $x_1 = -(q/r)x_0$. Notice that, since α_1 and α_2 are roots of Q, we have $\alpha_1 + \alpha_2 = -q/s$ and $\alpha_1\alpha_2 = r/s$.

If $\alpha_1 \neq \alpha_2$, then we obtain

$$\frac{C_1}{\alpha_1} + \frac{C_2}{\alpha_2} = x_1 = -\frac{q}{r}x_0 = -\frac{q}{r}(C_1 + C_2) = \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2}(C_1 + C_2).$$

Therefore, $C_1\alpha_1 + C_2\alpha_2 = 0$ which implies $C_1 = C\alpha_2$ and $C_2 = -C\alpha_1$ for a general constant C.

If $\alpha_1 = \alpha_2$, then we get

$$\frac{C_3 + C_4}{\alpha_1} = x_1 = -\frac{q}{r}x_0 = -\frac{q}{r}C_3 = \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2}C_3 = \frac{2}{\alpha_1}C_3.$$

Therefore, $C_3 = C_4 = C$ for a general constant C.

Theorem 1.2.

(i) $T \in (c_0, c_0)$ *is one-to-one if and only if* Q *has a root in the unit disc.*

(ii) $T \in (\ell_1, \ell_1)$ *is one-to-one if and only if* Q *has a root in the unit disc.*

(iii) $T \in (c, c)$ *is one-to-one if and only if* Q *has a root in* $D \setminus \{1\}$ *or* 1 *is a double root of* Q.

(iv) $T \in (\ell_\infty, \ell_\infty)$ *is one-to-one if and only if* Q *has a root in* D° *or a double root on* ∂D.

Proof. (i) $T \in (c_0, c_0)$ is not one-to-one if and only if there exists $x = (x_0, x_1, x_2, \ldots) \neq \theta$ in c_0 such that $Tx = \theta$. $Tx = \theta$ for nonzero $x = (x_n) \in c_0$ if and only if x satisfies system of equations (7). Hence, by Lemma 1.1, $Tx = \theta$ for nonzero $x = (x_n) \in c_0$ if and only if (8) holds for $\theta \neq x \in c_0$.

For the case $\alpha_1 \neq \alpha_2$, (8) holds for $\theta \neq x \in c_0$ if and only if $|\alpha_1| > 1$ and $|\alpha_2| > 1$. Similarly, for the case $\alpha_1 = \alpha_2$, (8) holds for $\theta \neq x \in c_0$ if and only if $|\alpha_1| > 1$.

Hence, $Tx = \theta$ with $x \neq \theta$ if and only if roots of Q are outside the unit disc. Equivalently, $T \in (c_0, c_0)$ is one-to-one if and only if Q has a root in the unit disc.

(ii) $T \in (\ell_1, \ell_1)$ is not one-to-one if and only if there exists $x = (x_0, x_1, x_2, \ldots) \neq \theta$ in ℓ_1 such that $Tx = \theta$. $Tx = \theta$ for nonzero $x = (x_n) \in \ell_1$ if and only if x satisfies system of equations (7). Hence, by Lemma 1.1, $Tx = \theta$ for nonzero $x = (x_n) \in \ell_1$ if and only if (8) holds for $\theta \neq x \in \ell_1$.

For the case $\alpha_1 \neq \alpha_2$, (8) holds for $\theta \neq x \in \ell_1$ if and only if $|\alpha_1| > 1$ and $|\alpha_2| > 1$. Similarly, for the case $\alpha_1 = \alpha_2$, (8) holds for $\theta \neq x \in \ell_1$ if and only if $|\alpha_1| > 1$.

So, $Tx = \theta$ with $\theta \neq x \in \ell_1$ if and only if roots of Q are outside the unit disc. Equivalently, $T \in (\ell_1, \ell_1)$ is one-to-one if and only if Q has a root in the unit disc.

(iii) Before we begin the proof, we remind that; if $z_n = 1/z^n$ is a complex sequence we have

$$z_n = \frac{1}{z^n} \begin{cases} \text{converges to } 0, & \text{if } |z| > 1, \\ \text{converges to } 1, & \text{if } z = 1, \\ \text{diverges}, & \text{otherwise}. \end{cases}$$

In particular, if $|z| = 1$ and $z \neq 1$, the sequence $1/z^n$ "spins" around the unit circle; i.e., diverges.

$T \in (c, c)$ is not one-to-one if and only if there exists $x = (x_0, x_1, x_2, \ldots) \neq \theta$ in c such that $Tx = \theta$. $Tx = \theta$ for nonzero $x = (x_n) \in c$ if and only if x satisfies system of equations (7). Hence, by Lemma 1.1, $Tx = \theta$ for nonzero $x = (x_n) \in c$ if and only if (8) holds for $\theta \neq x \in c$.

For the case $\alpha_1 \neq \alpha_2$, (8) holds for $\theta \neq x \in c$ if and only if the three cases: $|\alpha_1| > 1$ and $|\alpha_2| > 1$ or $|\alpha_1| > 1$ and $\alpha_2 = 1$ or $\alpha_1 = 1$ and $|\alpha_2| > 1$. For the case $\alpha_1 = \alpha_2$, (8) holds for $\theta \neq x \in c$ if and only if $|\alpha_1| > 1$.

So, $Tx = \theta$ with $\theta \neq x \in c$ if and only if roots of Q are outside the unit disc or one of the roots is outside the unit disc and the other one is 1. Equivalently, $T \in (c, c)$ is one-to-one if and only if Q has a root in $D \setminus \{1\}$ or 1 is a double root of Q.

(iv) $T \in (\ell_\infty, \ell_\infty)$ is not one-to-one if and only if there exists $x = (x_0, x_1, x_2, \ldots) \neq \theta$ in ℓ_∞ such that $Tx = \theta$. $Tx = \theta$ for nonzero $x = (x_n) \in \ell_\infty$ if and only if x satisfies system of equations (7). Hence, by Lemma 1.1, $Tx = \theta$ for nonzero $x = (x_n) \in \ell_\infty$ if and only if (8) holds for $\theta \neq x \in \ell_\infty$.

For the case $\alpha_1 \neq \alpha_2$, (8) holds for $\theta \neq x \in \ell_\infty$ if and only if $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. For the case $\alpha_1 = \alpha_2$, (8) holds for $\theta \neq x \in \ell_\infty$ if and only if $|\alpha_1| > 1$.

So, $Tx = \theta$ with $\theta \neq x \in \ell_\infty$ if and only if $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$ with $\alpha_1 \neq \alpha_2$ or $\alpha_1 = \alpha_2$ with $|\alpha_1| > 1$. Equivalently, $T \in (\ell_\infty, \ell_\infty)$ is one-to-one if and only if Q has a root in D° or Q has a double root on ∂D.

We have the following two lemmas as a consequence of the corresponding results in [13] and [4], respectively.

Lemma 1.2. $(I - \alpha L) \in (c_0, c_0)$ *is onto if and only if* α *is not on the unit circle.*

Lemma 1.3. $(R - \alpha I) \in (c_0, c_0)$ *is onto if and only if* α *is outside the unit disc.*

If $U : \mu \to \mu$ (μ is ℓ_1 or c_0) is a bounded linear operator represented by the matrix A, then it is known that the adjoint operator $U^* : \mu^* \to \mu^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 and the dual space ℓ_1^* of ℓ_1 is isometrically isomorphic to the Banach space ℓ_∞.

Lemma 1.4 [11, p. 59]**.** *U has a dense range if and only if U^* is one-to-one.*

***Corollary* 1.2.** *If $U \in (\mu, \mu)$, then $\sigma_r(U, \mu) = \sigma_p(U^*, \mu^*) \setminus \sigma_p(U, \mu)$.*

Theorem 1.3. $T \in (c_0, c_0)$ *is onto if and only if roots of Q are not on the unit circle and at least one root of Q is outside the unit disc.*

Proof. We will use the representation (2) of T:

$$T = s(I - \alpha_1 L)(R - \alpha_2 I) = s(I - \alpha_2 L)(R - \alpha_1 I).$$

Suppose $T \in (c_0, c_0)$ is onto. An operator of the form $I - \alpha L$ or $R - \alpha I$ is in (c_0, c_0) for any $\alpha \in \mathbb{C}$. So the operators $I - \alpha_1 L$ and $I - \alpha_2 L$ are both onto. Therefore, by Lemma 1.2, α_1 and α_2 are not on the unit circle. Let us assume here that both α_1 and α_2 are in D°. Then the associated function of the adjoint operator $T^* \in (\ell_1, \ell_1)$, which is represented by the transpose T^t, is P. Both roots of P are outside the unit disc. This means, by Theorem 1.2, T^* is not one-to-one and, by Lemma 1.4, T does not have a dense range and so T is not onto. Then, our assumption is not true, so at least one root of Q is outside the unit disc.

For the inverse implication, suppose the roots α_1 and α_2 of Q are not on the unit circle and at least one root, say α_2, is outside the unit disc. Then, by Lemma 1.2, $I - \alpha_1 L$ is onto and, by Lemma 1.3, $R - \alpha_2 I$ is onto. So, $T = s(I - \alpha_1 L)(R - \alpha_2 I)$ is onto.

***Corollary* 1.3.** *$T \in (c, c)$ is onto if and only if roots of Q are not on the unit circle and at least one root of Q is outside the unit disc.*

Proof. We prove by showing that ontoness of T in (c_0, c_0) and (c, c) are equivalent. Suppose T is onto over (c_0, c_0). Then, by Theorem 1.3, $\gamma := Q(1) = s + q + r \neq 0$. Then

$$\begin{bmatrix} q & r & 0 & 0 & \cdots \\ s & q & r & 0 & \cdots \\ 0 & s & q & r & \cdots \\ 0 & 0 & s & q & \cdots \\ 0 & 0 & 0 & s & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} q+r \\ s+q+r \\ s+q+r \\ s+q+r \\ s+q+r \\ s+q+r \\ \vdots \end{bmatrix} = (q+r+s) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} - s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Now, by letting $e = (1, 1, \ldots)$ and $e_1 = (1, 0, 0, \ldots)$, we have the equation

$$Te = \gamma e - s e_1.$$

Since T is onto over (c_0, c_0) there exists $x \in c_0$ such that $Tx = s e_1$. Then, by linearity of T, we get

$$T(e + x) = Te + Tx = \gamma e - s e_1 + s e_1 = \gamma e.$$

Then, for $e' := (e + x)/\gamma$, we obtain $Te' = e$. Let $y = (y_k)$ be an arbitrary element of c with $\delta = \lim y_k$. Then clearly $y - \delta e \in c_0$ and so there exists $x'' \in c_0$ such that $Tx'' = y - \delta e$. Hence,

$$T(x'' + \delta e') = Tx'' + \delta Te' = y - \delta e + \delta e = y.$$

Now $x'' + \delta e' \in c$, since

$$x'' + \delta e' = x'' + \frac{\delta}{\gamma}(e + x) \to 0 + \frac{\delta}{\gamma}(1 + 0) = \frac{\delta}{\gamma} \in \mathbb{C}.$$

So, T is onto over (c, c).

For the inverse implication, suppose T is onto over (c, c). Then $Te = \gamma e - s e_1 \notin c_0$, because if $Te \in c_0$, then we have the contradiction $T \in (c, c_0)$. So, $\gamma \neq 0$. Let $y = (y_k)$ be an arbitrary element of c_0. Since $c_0 \subset c$ we have $y \in c$ and since T is onto over (c, c), there exists $x = (x_k) \in c$ such that $Tx = y$. Let $\delta = \lim x_k$. Then $x - \delta e \in c_0$. By Theorem 1.1 (ii) $T \in (c_0, c_0)$, and so we must have $T(x - \delta e) \in c_0$. By linearity of T we get

$$T(x - \delta e) = Tx - \delta Te = y - \delta(\gamma e - s e_1) \in c_0.$$

Now since $y \in c_0$ and $(\gamma e - s e_1) \notin c_0$ we must have $\delta = 0$ and so $x \in c_0$. Therefore, T is onto over (c_0, c_0).

The following theorem gives a general result about the resolvent set of a bounded operator over a Banach space. (For a proof see, e.g., [7].)

Theorem 1.4. *Let X be a Banach space and $U \in B(X)$. Then $\lambda \in \rho(U, X)$ if and only if U_λ is bijective.*

2. The spectra and fine spectra.

Theorem 2.1.

$$\sigma(T, c_0) = \begin{cases} Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big), & \text{if } |r| \le |s|, \\ Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big), & \text{if } |r| > |s|. \end{cases}$$

Proof. Suppose $|r| \le |s|$ and $\lambda \in \sigma(T, c_0)$. By Theorem 1.4, $T - \lambda I$ is not onto or is not one-to-one. The associated function of $T - \lambda I$ is $(Q - \lambda)(z) = sz + q - \lambda + rz^{-1}$. The product of the roots of $Q - \lambda$ is r/s. Since $|r/s| \le 1$, at least one root is in D, which means, by Theorem 1.2, $T - \lambda I$ is one-to-one. So $T - \lambda I$ is not onto. Now, by Theorem 1.3, $Q - \lambda$ has a root on the unit circle or both roots are in D°. Suppose β is a root of $Q - \lambda$. If both roots are in D°, then $|r/s| < |\beta| < 1$ and $\lambda = Q(\beta)$, which means $\lambda \in Q\Big(D^\circ \setminus \dfrac{r}{s} D\Big)$. If $Q - \lambda$ has a root on the circle, then $|\beta| = 1$ or $|\beta| = |r/s|$ with $\lambda = Q(\beta)$, which means $\lambda \in Q\Big(\partial D \cup \dfrac{r}{s}\partial D\Big)$. So, we have $\lambda \in Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big)$. Therefore, $\sigma(T, c_0) \subseteq Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big)$.

For the reverse inclusion, suppose $|r| \le |s|$ and $\lambda \in Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big)$. Then $\lambda = Q(\beta)$ with $|r/s| \le |\beta| \le 1$. Then both roots of $Q - \lambda$ are in D° or $Q - \lambda$ has a root on the unit circle. Hence, by Theorem 1.3, $T - \lambda I$ is not onto and, by Theorem 1.4, $\lambda \in \sigma(T, c_0)$. So, we have $Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big) \subseteq \sigma(T, c_0)$. Hence, for $|r| \le |s|$ we have $\sigma(T, c_0) = Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big)$.

Now, suppose $|r| > |s|$ and $\lambda \in \sigma(T, c_0)$. By Theorem 1.4, $T - \lambda I$ is not onto or is not one-to-one. If $T - \lambda I$ is not onto, by Theorem 1.3, $Q - \lambda$ has a root on the unit circle or both roots are in D°. But, both roots cannot be in D°, since the product of the roots, r/s, is absolutely greater than 1. So, $Q - \lambda$ has a root on the unit circle, which means $\lambda = Q(\beta)$ for some β with $|\beta| = 1$ or $|\beta| = |r/s|$. Then $\lambda \in Q\Big(\partial D \cup \dfrac{r}{s}\partial D\Big)$. If $T - \lambda I$ is not one-to-one, by Theorem 1.2, both roots of $Q - \lambda$ are outside D. Let β be a root of $Q - \lambda$. If both roots are outside D, then $1 < |\beta| < |r/s|$ and $\lambda = Q(\beta)$, which means $\lambda \in Q\Big(\dfrac{r}{s} D^\circ \setminus D\Big)$. So, we have $\lambda \in Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big)$. Therefore, $\sigma(T, c_0) \subseteq Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big)$.

For the reverse inclusion, suppose $|r| > |s|$ and $\lambda \in Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big)$. Then $\lambda = Q(\beta)$ with $1 \le |\beta| \le |r/s|$. Then, both roots of $Q - \lambda$ are outside D or $Q - \lambda$ has a root on the unit circle. Hence, by Theorems 1.2 and 1.3, $T - \lambda I$ is not one-to-one or not onto, and, by Theorem 1.4, $\lambda \in \sigma(T, c_0)$. So, we have $Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big) \subseteq \sigma(T, c_0)$. Hence, for $|r| > |s|$ we get $\sigma(T, c_0) = Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big)$.

Theorem 2.2. *For $\mu \in \{\ell_1, c, \ell_\infty\}$,*

$$\sigma(T, \mu) = \begin{cases} Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big), & \text{if } |r| \le |s|, \\ Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big), & \text{if } |r| > |s|. \end{cases}$$

Proof. We will use the fact that the spectrum of a bounded operator over a Banach space is equal to the spectrum of the adjoint operator. The adjoint operator is the transpose of the matrix for c_0. So $\sigma(T, \ell_1) = \sigma(T^*, c_0^*) = \sigma(T^t, c_0)$. The associated function of T^t is $P(z) = rz + q + sz^{-1}$.

So, by Theorem 2.1, we get

$$\sigma(T,\ell_1)=\begin{cases}P\Big(D\setminus \dfrac{s}{r}D^\circ\Big), & \text{if } |s|\le|r|,\\ P\Big(\dfrac{s}{r}D\setminus D^\circ\Big), & \text{if } |s|>|r|.\end{cases}$$

We can see that, for $|s|\le|r|$, we have $P\Big(D\setminus \frac{s}{r}D^\circ\Big)=Q\Big(\frac{r}{s}D\setminus D^\circ\Big)$, and, for $|s|>|r|$, we get $P\Big(\frac{s}{r}D\setminus D^\circ\Big)=Q\Big(D\setminus \frac{r}{s}D^\circ\Big)$. Hence, $\sigma(T,\ell_1)=\sigma(T,c_0)$.

We know by Cartlidge [9] that if a matrix operator U is bounded on c, then $\sigma(U,c)=\sigma(U,\ell_\infty)$. Hence, we have $\sigma(T,c)=\sigma(T,\ell_\infty)=\sigma(T^{**},c_0^{**})=\sigma(T,c_0)$.

Theorem 2.3. *For* $\mu\in\{\ell_1,c_0\}$,

$$\sigma_p(T,\mu)=\begin{cases}Q\Big(\dfrac{r}{s}D^\circ\setminus D\Big), & \textit{if } |r|>|s|,\\ \varnothing, & \textit{if } |r|\le|s|.\end{cases}$$

Proof. Suppose $\lambda\in\sigma_p(T,\mu)$ for $\mu\in\{\ell_1,c_0\}$. Then $T-\lambda I$ is not one-to-one. By Theorem 1.2, $T-\lambda I$ is not one-to-one if and only if roots of $Q-\lambda$ are outside D. The product of the roots of $Q-\lambda$ is r/s. So, if β is a root of $Q-\lambda$, then $1<|\beta|<|r/s|$ and $Q(\beta)=\lambda$. Hence, $\lambda\in Q\Big(\frac{r}{s}D^\circ\setminus D\Big)$. So, we have $\sigma_p(T,\mu)\subseteq Q\Big(\frac{r}{s}D^\circ\setminus D\Big)$.

For the reverse inclusion, suppose $\lambda\in Q\Big(\frac{r}{s}D^\circ\setminus D\Big)$. Then there exists $\beta\in\mathbb{C}$ with $1<|\beta|<|r/s|$ such that $Q(\beta)=\lambda$. Now, β is a root of $Q-\lambda$ and is outside D. The other root is $r/(s\beta)$ which is also outside D, since $|r/(s\beta)|>1$. So both roots of $Q-\lambda$ are outside D, which means, by Theorem 1.2, that $T-\lambda I$ is not one-to-one. Hence, $\lambda\in\sigma_p(T,\mu)$. So, we have $Q\Big(\frac{r}{s}D^\circ\setminus D\Big)\subseteq\sigma_p(T,\mu)$.

The following two theorems can be proved by using similar arguments, so we give them without proof.

Theorem 2.4. $$\sigma_p(T,c)=\begin{cases}Q\Big(\dfrac{r}{s}D^\circ\setminus D\Big)\cup Q(\{1\}), & \textit{if } |r|>|s|,\\ \varnothing, & \textit{if } |r|\le|s|.\end{cases}$$

Theorem 2.5. $$\sigma_p(T,\ell_\infty)=\begin{cases}Q\Big(\dfrac{r}{s}D\setminus D^\circ\Big), & \textit{if } |r|>|s|,\\ Q\left(\partial D\setminus\left\{\pm\sqrt{\dfrac{r}{s}}\right\}\right), & \textit{if } |r|=|s|,\\ \varnothing, & \textit{if } |r|<|s|.\end{cases}$$

Theorem 2.6. $$\sigma_r(T,c_0)=\begin{cases}\varnothing, & \textit{if } |r|\ge|s|,\\ Q\Big(D^\circ\setminus\dfrac{r}{s}D\Big), & \textit{if } |r|<|s|.\end{cases}$$

Proof. Let us do the proof only for the case $|r|<|s|$. When $|r|<|s|$, $\sigma_p(T^t,\ell_1)=P\Big(\frac{s}{r}D^\circ\setminus D\Big)=Q\Big(D^\circ\setminus\frac{r}{s}D\Big)$ by Theorem 2.3. Now, using Corollary 1.2, we have $\sigma_r(T,c_0)=\sigma_p(T^*,c_0^*)\setminus\sigma_p(T,c_0)=\sigma_p(T^t,\ell_1)\setminus\sigma_p(T,c_0)=Q\Big(D^\circ\setminus\frac{r}{s}D\Big)$.

If $U: c\to c$ is a bounded matrix operator represented by the matrix A, then $U^*: c^*\to c^*$ acting on $\mathbb{C}\oplus\ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix},$$

where χ is the limit of the sequence of row sums of A minus the sum of the limits of the columns of A, and b is the column vector whose kth entry is the limit of the kth column of A for each $k \in \mathbb{N}$. Then, for $T: c \to c$, the matrix T^* is of the form

$$\begin{bmatrix} s+q+r & 0 \\ 0 & T^t \end{bmatrix} = \begin{bmatrix} Q(1) & 0 \\ 0 & T^t \end{bmatrix}.$$

Theorem 2.7. $\sigma_r(T, c) = \begin{cases} \varnothing, & \text{if } |r| > |s|, \\ Q(\{1\}), & \text{if } |r| = |s|, \\ Q\left(D^\circ \setminus \dfrac{r}{s} D\right) \cup Q(\{1\}), & \text{if } |r| < |s|. \end{cases}$

Proof. Let us do the proof only for the case $|r| < |s|$. The other cases can be proved similarly, so we omit them. Let $x = (x_0, x_1, \ldots) \in \mathbb{C} \oplus \ell_1$ be an eigenvector of T^* corresponding to the eigenvalue λ. Then we have $(s+q+r)x_0 = \lambda x_0$ and $Tx' = \lambda x'$, where $x' = (x_1, x_2, \ldots)$. If $x_0 \neq 0$, then $\lambda = s+q+r$, and $s+q+r$ is an eigenvalue, since $T^*(1,0,0,\ldots) = (s+q+r)(1,0,0,\ldots)$. If $x_0 = 0$, then x' is an eigenvector of T^t over ℓ_1 and $T^t x' = \lambda x'$. By Theorem 2.3, $\lambda \in$ $\in \sigma_p(T^t, \ell_1) = P\Big(\dfrac{s}{r} D^\circ \setminus D\Big) = Q\Big(D^\circ \setminus \dfrac{r}{s} D\Big)$. Hence, $\sigma_p(T^*, c^*) = Q\Big(D^\circ \setminus \dfrac{r}{s} D\Big) \cup Q\big(\{1\}\big)$. Then $\sigma_r(T, c) = \sigma_p(T^*, c^*) \setminus \sigma_p(T, c) = Q\Big(D^\circ \setminus \dfrac{r}{s} D\Big) \cup Q(\{1\})$.

As a consequence of Theorems 2.3 and 2.5, we have the following result.

Theorem 2.8. $\sigma_r(T, \ell_1) = \begin{cases} \varnothing, & \text{if } |r| > |s|, \\ Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big) \setminus Q\left(\left\{\pm\sqrt{\dfrac{r}{s}}\right\}\right), & \text{if } |r| \leq |s|. \end{cases}$

The spectrum σ is the disjoint union of σ_p, σ_r, and σ_c, so we obtain the following theorem as a consequence of Theorems 2.3, 2.6, 2.7 and 2.4.

Theorem 2.9. *We have*

$$\sigma_c(T, c_0) = Q(\partial D) \cup Q\left(\frac{r}{s} \partial D\right),$$

$$\sigma_c(T, c) = Q(\partial D) \setminus Q(\{1\}),$$

$$\sigma_c(T, \ell_1) = \begin{cases} Q(\partial D), & \text{if } |r| > |s|, \\ Q\left(\left\{\pm\sqrt{\dfrac{r}{s}}\right\}\right), & \text{if } |r| \leq |s|. \end{cases}$$

3. The resolvent operator and some applications.

Theorem 3.1. *Let $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$. The resolvent operator T^{-1} over μ exists, is continuous and the domain of T^{-1} is the whole space μ if and only if one of the roots of the function Q is in D°, and the other root is outside D. In this case, say $|\alpha_1| < 1 < |\alpha_2|$, the matrix representation of T^{-1} is $S = (s_{nk})$ defined by*

$$s_{nk} = \frac{1}{s(\alpha_1 - \alpha_2)} \begin{cases} \left(\alpha_2^{k+1} - \alpha_1^{k+1}\right) \alpha_2^{-n-1}, & \text{if } n \geq k, \\ \left(\alpha_1^{-n-1} - \alpha_2^{-n-1}\right) \alpha_1^{k+1}, & \text{if } n < k. \end{cases}$$

Proof. By Theorem 1.4, the resolvent operator T^{-1} over μ exists, is continuous and the domain of T^{-1} is the whole space μ if and only if 0 is in the resolvent set $\rho(T,\mu)$. Hence, for $|r| \leq |s|$, $0 \notin Q\Big(D \setminus \dfrac{r}{s}D^\circ\Big)$ and for $|r| > |s|$, $0 \notin Q\Big(\dfrac{r}{s}D \setminus D^\circ\Big)$. In both cases, this is equivalent to saying, one of the roots of Q is absolutely less than 1, and the other root is absolutely greater than 1.

Now, let us show that $S(Tx) = x$ for all $x = (x_0, x_1, \ldots) \in \ell_\infty$. By definition we have $(Tx)_k = sx_{k-1} + qx_k + rx_{k+1}$, where $x_{-1} = 0$. Then, for $n \in \mathbb{N}$,

$$(S(Tx))_n = \sum_{k=0}^{\infty} s_{nk}(Tx)_k = \sum_{k=0}^{\infty} s_{nk}(sx_{k-1} + qx_k + rx_{k+1}).$$

This sum is absolutely convergent since rows of S are in ℓ_1 and $x \in \ell_\infty$. So, we can change the order of summation and get

$$(S(Tx))_n = \sum_{k=0}^{\infty} (ss_{n(k+1)} + qs_{nk} + rs_{n(k-1)})x_k,$$

where $s_{n(-1)} = 0$. Now, it is not difficult to check that $ss_{n(k+1)} + qs_{nk} + rs_{n(k-1)} = \delta_{nk}$ for all $n, k \in \mathbb{N}$, where δ_{nk} is the Kronecker delta. Hence, we have $(S(Tx))_n = x_n$ for all $n \in \mathbb{N}$. So, we get $S(Tx) = x$ for all $x \in \ell_\infty$.

***Corollary* 3.1.** *Let $\mu \in \{c_0, c, \ell_1, \ell_\infty\}$. For $\lambda \notin \sigma(T,\mu)$ the matrix representation of $(T-\lambda)^{-1}$ is $V = (v_{nk})$ defined by*

$$v_{nk} = \frac{1}{s(\beta_1 - \beta_2)} \begin{cases} \left(\beta_2^{k+1} - \beta_1^{k+1}\right)\beta_2^{-n-1}, & \text{if } n \geq k, \\ \left(\beta_1^{-n-1} - \beta_2^{-n-1}\right)\beta_1^{k+1}, & \text{if } n < k, \end{cases}$$

where β_1 and β_2 are the roots of $Q - \lambda$ satisfying $|\beta_1| < 1 < |\beta_2|$.

***Example* 3.1.** When T is a symmetric tridiagonal matrix we have $s = r$ and

$$T = \begin{bmatrix} q & r & 0 & 0 & 0 & 0 & \cdots \\ r & q & r & 0 & 0 & 0 & \cdots \\ 0 & r & q & r & 0 & 0 & \cdots \\ 0 & 0 & r & q & r & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

then $\sigma(T,\mu) = Q\Big(D \setminus \dfrac{r}{r}D^\circ\Big) = Q(D \setminus D^\circ) = Q(\partial D)$ for $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$, where $Q(z) = q + r(z + z^{-1})$. Therefore,

$$\sigma(T,\mu) = \{q + 2r\cos\theta : \theta \in [0,\pi]\} = [q - 2r, q + 2r]$$

which is one of the main results of [5]; $[q-2r, q+2r]$ is the closed line segment in the complex plane with endpoints $q - 2r$ and $q + 2r$.

***Example* 3.2.** When $|s| = |r|$, it can be proved that the spectrum is always a closed line segment. For example, let

$$T = \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & i & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & i & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & i & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where i is the complex number $\sqrt{-1}$. Then

$$\sigma(T,\mu) = Q\Big(D \setminus \frac{i}{1} D^\circ\Big) = Q(D \setminus D^\circ) = Q(\partial D) \quad \text{for} \quad \mu \in \{\ell_1, c_0, c, \ell_\infty\},$$

where $Q(z) = z + 1 + iz^{-1}$. Therefore,

$$\sigma(T,\mu) = \{1 + (\cos\theta + i\sin\theta) + i(\cos\theta - i\sin\theta) : \theta \in [0, 2\pi]\} =$$
$$= \{1 + (1+i)(\cos\theta + \sin\theta) : \theta \in [0, 2\pi]\} =$$
$$= \Big[1 - \sqrt{2}(1+i), 1 + \sqrt{2}(1+i)\Big].$$

***Example* 3.3.** Let

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $\sigma(T,\mu) = Q\Big(D \setminus \frac{1}{2} D^\circ\Big)$ for $\mu \in \{\ell_1, c_0, c, \ell_\infty\}$, where $Q(z) = 2z + z^{-1}$. For the boundaries we have

$$Q(\partial D) = Q\left(\frac{1}{2}\partial D\right) = \{2(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) : \theta \in [0, 2\pi]\} =$$
$$= \{(3\cos\theta + i\sin\theta : \theta \in [0, 2\pi]\} = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{3^2} + y^2 = 1\right\}.$$

Hence, $\sigma(T,\mu)$ is the elliptical region $\left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{3^2} + y^2 \leq 1\right\}$.

Now, let us give an application of Theorem 2.2, related to the system of equations

$$y_k = s x_{k-1} + q x_k + r x_{k+1}, \qquad k = 0, 1, 2, \ldots, \tag{9}$$

where $x_{-1} = 0$.

Theorem 3.2. *Let r, q and s be complex numbers with $r, s \neq 0$, and $Q(z) = sz + q + rz^{-1}$ with roots α_1, α_2 satisfying $|\alpha_1| \leq |\alpha_2|$. Let the complex sequences $x = (x_n)$ and $y = (y_n)$ be solutions of the system of equations* (9). *Then the following are equivalent*:

(i) *boundedness of (y_n) always implies a unique bounded solution (x_n),*

(ii) *convergence of (y_n) always implies a unique convergent solution (x_n),*

(iii) *$y_n \to 0$ always implies a unique solution (x_n) with $x_n \to 0$,*

(iv) *$\sum |y_n| < \infty$ always implies a unique solution (x_n) with $\sum |x_n| < \infty$,*

(v) $|\alpha_1| < 1 < |\alpha_2|$.

Proof. Let $|r| \leq |s|$. The system of equations (9) hold, so we have $Tx = y$. Then Q is the function associated with T. Let us prove only (i)⇔(v) and omit the proofs of (ii)⇔(v), (iii)⇔(v), (iv)⇔(v) since they are similarly proved. Suppose boundedness of (y_n) always implies a unique bounded solution (x_n). Then the operator $T - 0I = T \in (\ell_\infty, \ell_\infty)$ is bijective. This means, by Theorems 1.4 and 2.2, $0 \notin \sigma(T, \ell_\infty) = Q\Big(D \setminus \dfrac{r}{s} D^\circ\Big)$. This is equivalent to $|\alpha_1| < 1 < |\alpha_2|$. If $|r| > |s|$, similarly we get $0 \notin Q\Big(\dfrac{r}{s} D \setminus D^\circ\Big)$, which is also equivalent to $|\alpha_1| < 1 < |\alpha_2|$.

For the reverse implication, suppose $|\alpha_1| < 1 < |\alpha_2|$. So, $\lambda = 0 \notin \sigma(T, \ell_\infty)$. Now, by Theorem 1.4, $T = T - 0I$ is bijective on ℓ_∞, which means that the boundedness of (y_n) implies a bounded unique solution (x_n).

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