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UNIQUENESS OF DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS*

ПРО ЄДИНІСТЬ РІЗНИЦЕВО-ДИФЕРЕНЦІАЛЬНИХ ПОЛІНОМІВ МЕРОМОРФНИХ ФУНКЦІЙ

We investigate the problems of uniqueness of difference-differential polynomials of finite-order meromorphic functions sharing a small function ignoring multiplicity and obtain some results that extend the results of K. Liu, X. L. Liu, and T. B. Cao.

Вивчаються проблеми єдиності різницево-диференціальних поліномів мероморфних функцій скінченного порядку, що поділяють малу функцію (нехтуючи кратністю). Отримано деякі результати, що узагальнюють результати K. Liu, X. L. Liu i T. B. Cao.

1. Introduction and results. In this paper, a meromorphic function always means it is meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with standard notations of the Nevanlinna theory of entire and meromorphic functions as explained in [5, 6, 14].

Let $f(z)$ and $\alpha(z)$ be two meromorphic functions. We say that $\alpha(z)$ is a small function with respect to $f(z)$ if $T(r,\alpha(z)) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o\big(T(r,f)\big)$ as $r \to \infty$, outside an exceptional set E of finite logarithmic measure, i.e., $\lim_{r\to\infty} \int_{(1,r]\cap E} \dfrac{dt}{t} < \infty$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. If, for $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the quantities $f(z)-a$ and $g(z)-a$ have the same set of zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities). At the same time, if we do not consider the multiplicities, then $f(z)$ and $g(z)$ are said to share the value a IM (ignoring multiplicities). Let $f(z)$ and $g(z)$ share the value 1 IM and let z_0 be a 1-point of $f(z)$ of order p and a 1-point of $g(z)$ of order q. We denote the counting function of the 1-points of both $f(z)$ and $g(z)$ with $p > q$ by $\overline{N}_L\left(r, \dfrac{1}{f-1}\right)$. In the same way, we can define $\overline{N}_L\left(r, \dfrac{1}{g-1}\right)$.

Let $f(z)$ be a non-constant meromorphic function. Let a be a finite complex number, and let k be a positive integer. By $N_{k)}\left(r, \dfrac{1}{f-a}\right)$ $\left(\text{or } \overline{N}_{k)}\left(r, \dfrac{1}{f-a}\right)\right)$, we denote the counting function of the roots of $f(z)-a$ with multiplicity $\le k$(IM) and by $N_{(k}\left(r, \dfrac{1}{f-a}\right)$ $\left(\text{or } \overline{N}_{(k}\left(r, \dfrac{1}{f-a}\right)\right)$, we denote the counting function of the roots of $f(z)-a$ with multiplicity $\ge k$ (IM). We set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \ldots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Further, we define the order $\rho(f)$ of a meromorphic function $f(z)$ by

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$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r},$$

and the hyper order $\rho_2(f)$ of a meromorphic function $f(z)$ by

$$\rho_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

Let m be a non-negative integer, $a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0)$ be complex constants. Define

$$P(w) = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_1 w + a_0. \tag{1.1}$$

In 2010, X. G. Qi, L. Z. Yang and K. Liu [12] considered the problems of uniqueness regarding the difference polynomials of entire functions and obtained the following result.

Theorem A. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order and c be a non-zero complex constant. If $n \geq 6$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share 1 CM, then $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 that satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In 2011, X. M. Li, W. L. Li, H. X. Yi, Z. T. Wen [7] have improved the above result and obtained the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order and $\alpha(z)$ be a meromorphic function such that $\rho(\alpha) < \rho(f)$, let c be a non-zero complex constant and let $n \geq 7$ be an integer. If $f(z)^n(f(z)-1)f(z+c) - \alpha(z)$ and $g(z)^n(g(z)-1)g(z+c) - \alpha(z)$ share 0 CM, then $f(z) \equiv g(z)$.*

Next, K. Liu, X. L. Liu, T. B. Cao [8 – 10] proved the following results.

Theorem C. *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions of finite order. Suppose that c is a non-zero constant and $n \in \mathbb{N}$. If $n \geq 26$, $f(z)^n f(z+c)$ and $g(z)^n g(z+c)$ share 1 IM, then $f = tg$ or $fg = t$, where $t^{n+1} = 1$.*

Theorem D. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, $n \geq 5k+12$. If $[f(z)^n f(z+c)]^{(k)}$ and $\left[g(z)^n g(z+c)\right]^{(k)}$ share the value 1 IM, then either $f(z) = c_1 e^{Cz}$, $g(z) = c_2 e^{-Cz}$, where c_1, c_2 and C are constants satisfying $(-1)^k (c_1c_2)^{n+1} \left[(n+1)C\right]^{2k} = 1$ or $f = tg$, where $t^{n+1} = 1$.*

Theorem E. *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) > 1$, $n \geq 5k + 4m + 12$. If $\left[f^n(f^m-1)f(z+c)\right]^{(k)}$ and $\left[g^n(g^m-1)g(z+c)\right]^{(k)}$ share the value 1 IM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

In this paper, we shall extend these results to meromorphic functions and obtain the following two theorems.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant finite order meromorphic functions. Suppose that $a(z)(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$. Let $k(>0)$ and $m(>0)$ be two integers satisfying $n > 4m + 13k + 19$, $P(w)$ be as defined in* (1.1) *and c be a non-zero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c, poles of $f(z)$ are not zeros of $f(z+c)$ and poles of $g(z)$ are not zeros of $g(z+c)$. If $\left[f^nP(f)f(z+c)\right]^{(k)}$ and $\left[g^nP(g)g(z+c)\right]^{(k)}$ share $a(z)$ IM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two cases holds*:

(1) *$f \equiv tg$, for a constant t such that $t^d = 1$, where $d = GCD(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$;*

(2) *$f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f,g)\equiv 0$, where* $R(w_1,w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1} + \ldots + a_0)w_1(z+c) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1} + \ldots + a_0)w_2(z+c)$.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non-constant finite order meromorphic functions. Suppose that $a(z)(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$. Let $k(>0)$ be integer satisfying $n > 13k+19$, $P(w) = a_0$, where $a_0 \neq 0$ is a complex constant and c be a non-zero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c, poles of $f(z)$ are not zeros of $f(z+c)$ and poles of $g(z)$ are not zeros of $g(z+c)$. If $\big[f^nP(f)f(z+c)\big]^{(k)}$ and $\big[g^nP(g)g(z+c)\big]^{(k)}$ share $a(z)$ IM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two cases holds:*

(1) *$f(z)\equiv tg(z)$ for a constant t such that $t^{n+1}=1$;*

(2) $a_0^2\big[f^nf(z+c)\big]^{(k)}\big[g^ng(z+c)\big]^{(k)} = a^2(z)$.

2. Some lemmas. We need the following lemmas to prove our results.

Lemma 2.1 [2]**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed non-zero complex constant. Then, for each $\epsilon > 0$, we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right) + m\left(r,\frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2.2 [3]**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed non-zero complex constant. Then, for each $\epsilon > 0$, we have*

$$T\big(r,f(z+c)\big) = T(r,f) + O(r^{\rho-1+\epsilon}).$$

It is evident that $S\big(r,f(z+c)\big) = S(r,f)$.

Lemma 2.3 [11]**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed non-zero complex constant. Then*

(i) $N\left(r,\dfrac{1}{f(z+c)}\right) \le N\left(r,\dfrac{1}{f}\right) + S(r,f)$,

(ii) $N\big(r,f(z+c)\big) \le N(r,f) + S(r,f)$,

(iii) $\overline{N}\left(r,\dfrac{1}{f(z+c)}\right) \le \overline{N}\left(r,\dfrac{1}{f}\right) + S(r,f)$,

(iv) $\overline{N}\big(r,f(z+c)\big) \le \overline{N}(r,f) + S(r,f)$,

outside an exceptional set with finite logarithmic measure.

Lemma 2.4 [15]**.** *Let $f(z)$ be a non-constant meromorphic function and p, k be two positive integers. Then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f),$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f).$$

Lemma 2.5 ([13], Lemma 3)**.** *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. If $f(z)$ and $g(z)$ share 1 CM, then one of the following three cases holds*:

(1) $T(r,f) \le N_2\left(r,\dfrac{1}{f}\right) + N_2\left(r,\dfrac{1}{g}\right) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$;

the same inequality holds for $T(r,g)$;

(2) $fg = 1$;

(3) $f \equiv g$.

Lemma 2.6 [15]**.** *Let $f_1(z)$ and $f_2(z)$ be two non-constant meromorphic functions. If $c_1 f_1 + c_2 f_2 = c_3$, where c_1, c_2 and c_3 are non-zero constants, then*

$$T(r, f_1) \leq \overline{N}(r, f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + S(r, f_1).$$

Define

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \tag{2.1}$$

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right), \tag{2.2}$$

where $F = \dfrac{\big[f^n P(f) f(z+c)\big]^{(k)}}{a(z)}$ and $G = \dfrac{\big[g^n P(g) g(z+c)\big]^{(k)}}{a(z)}$, both $f(z)$ and $g(z)$ are meromorphic functions of finite order, c is a non-zero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c, $a(z) (\not\equiv 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$, which has no common zeros or poles with $f(z)$ and $g(z)$.

Using the similar method as in Lemma 2.14 of Banerjee [1], we obtain the following lemma.

Lemma 2.7. *Let F, G and H be defined as in* (2.1)*. If F and G share 1 IM and ∞ IM, and $H \not\equiv 0$, then $F \not\equiv G$ and*

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + 7\overline{N}(r, F) + S(r, F) + S(r, G),$$

the same inequality holds for $T(r, G)$.

Lemma 2.8 [16]**.** *Let F, G and V be defined as in* (2.2)*. If F and G share ∞ IM and $V \equiv 0$, then $F \equiv G$.*

Lemma 2.9 [16]**.** *If F and G share 1 IM, then*

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, F) + S(r, F) + S(r, G).$$

Lemma 2.10. *Let $f(z)$, $g(z)$ be two non-constant finite order meromorphic functions such that poles of $f(z)$ are not zeros of $f(z+c)$ and poles of $g(z)$ are not zeros of $g(z+c)$, F, G and V be defined as in* (2.2)*, $P(w)$ be defined as in* (1.1) *and $n (> 3)$, $k (> 0)$, $m (\geq 0)$ be three integers. Let c be a non-zero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c. If $V \not\equiv 0$, F and G share 1 and ∞ IM, then*

$$(n + m + k - 5)\overline{N}(r, f) \leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g)$$

and

$$(n + m + k - 5)\overline{N}(r, g) \leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + S(r, f) + S(r, g).$$

Proof. Let z_0 be a pole of $f(z)$ and $g(z)$ with multiplicities p and q, respectively. By using hypotheses $V \not\equiv 0$, F and G share ∞ IM, pole of $f(z)$ is not a zero of $f(z+c)$ and a pole of $g(z)$ is not a zero of $g(z+c)$, we get z_0 is pole of F with multiplicity $(n+m)p+k$ and pole of G with multiplicity $(n+m)q+k$.

Thus z_0 is zero of $\dfrac{F'}{F-1} - \dfrac{F'}{F}$ with multiplicity $(n+m)p+k-1 \geq n+m+k-1$ and also z_0 is zero of $\dfrac{G'}{G-1} - \dfrac{G'}{G}$ with multiplicity $(n+m)q+k-1 \geq n+m+k-1$, hence z_0 is zero of V with multiplicity at least $n+m+k-1$. Thus

$$(n+m+k-1)\overline{N}(r,f) \leq N\left(r, \frac{1}{V}\right) \tag{2.3}$$

and

$$(n+m+k-1)\overline{N}(r,g) \leq N\left(r, \frac{1}{V}\right). \tag{2.4}$$

By the lemma of the logarithmic derivative, we have

$$m(r,V) = S(r,f) + S(r,g).$$

Now consider

$$N\left(r, \frac{1}{V}\right) \leq T(r,V) \leq m(r,V) + N(r,V) \leq N(r,V) + S(r,f) + S(r,g). \tag{2.5}$$

Since $F(z)$ and $G(z)$ share the value 1 IM, zeros of $F(z)-1$ and zeros of $G(z)-1$ of different multiplicities contribute to poles of V and also since $F(z)$ and $G(z)$ share the value ∞ IM, the poles of $F(z)$ and $G(z)$ of different multiplicities contributes to zeros of V. Thus from (2.2) and (2.5), we deduce

$$N\left(r, \frac{1}{V}\right) \leq \overline{N}\left(r, \frac{1}{F}\right) +$$

$$+\overline{N}\left(r, \frac{1}{G}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r,f) + S(r,g). \tag{2.6}$$

Since F and G share 1 IM, by Lemma 2.9 and (2.6), we get

$$N\left(r, \frac{1}{V}\right) \leq 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + S(r,f) + S(r,g). \tag{2.7}$$

By Lemma 2.3, we obtain

$$\overline{N}(r,F) = \overline{N}\left(r, \frac{\left[f^n P(f) f(z+c)\right]^{(k)}}{a(z)}\right) \leq$$

$$\leq \overline{N}(r,f) + \overline{N}(r, f(z+c)) + S(r,f) \leq 2\overline{N}(r,f) + S(r,f). \tag{2.8}$$

Similarly,

$$\overline{N}(r,G) \leq 2\overline{N}(r,g) + S(r,g). \tag{2.9}$$

From (2.7)–(2.9) and using that $f(z)$ and $g(z)$ share ∞ IM, we have

$$N\left(r,\frac{1}{V}\right)\le 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+2\overline{N}(r,f)+2\overline{N}(r,g)+S(r,f)+S(r,g)\le$$

$$\le 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+4\overline{N}(r,f)+S(r,f)+S(r,g). \qquad (2.10)$$

It follows from (2.3) and (2.10) that

$$(n+m+k-1)\overline{N}(r,f)\le 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+4\overline{N}(r,f)+S(r,f)+S(r,g),$$

i.e.,

$$(n+m+k-5)\overline{N}(r,f)\le 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g).$$

Similarly,

$$(n+m+k-5)\overline{N}(r,g)\le 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g).$$

Lemma 2.11 [4]. *Let $f(z)$ be a non-constant finite order meromorphic function. Let $P(f)$ be as defined in* (1.1) *and c be a non-zero complex constant such that $f(z)$ is not periodic function of period c. Then*

$$(n+m-1)T(r,f)+S(r,f)\le T(r,f^nP(f)f(z+c))\le (n+m+1)T(r,f)+S(r,f).$$

Lemma 2.12 [4]. *Let $f(z)$ be a transcendental finite order meromorphic function. Let $k(>0)$ be integer satisfying $n>k+5$, c be a non-zero complex constant such that $f(z)$ is not periodic function of period c and let $P(w)$ be as defined in* (1.1). *Suppose that $a(z)(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$. Then $(f^nP(f)f(z+c))^{(k)}-a(z)$ has infinitely many zeros.*

Lemma 2.13 [4]. *Let $f(z)$ and $g(z)$ be two non-constant finite order meromorphic functions. Let $P(w)$ be as defined in* (1.1). *Let $k(>0)$, $m(\ge 0)$ be integers satisfying $n>2k+m+5$ and c be a non-zero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c. If $\left[f^nP(f)f(z+c)\right]^{(k)}\equiv\left[g^nP(g)g(z+c)\right]^{(k)}$, then $f^nP(f)f(z+c)\equiv g^nP(g)g(z+c)$.*

Lemma 2.14 [4]. *Let $f(z)$ and $g(z)$ be two non-constant finite order meromorphic functions. Let c be a non-zero complex constant such that $f(z)$ and $g(z)$ are not periodic functions of period c and $k(>0)$ be integer satisfying $n>k+5$. Let $P(w)$ be as defined in* (1.1). *Suppose that $a(z)(\not\equiv 0,\infty)$ is a small function with respect to $f(z)$ with finitely many zeros and poles. If $\left(f^nP(f)f(z+c)\right)^{(k)}\left(g^nP(g)g(z+c)\right)^{(k)}=a^2(z)$, $f(z)$ and $g(z)$ share ∞ IM, then $P(w)$ reduces to a non-zero monomial, namely, $P(w)=a_iw^i\not\equiv 0$ for some $i\in\{0,1,\ldots,m\}$.*

3. Proof of the theorems. ***3.1. Proof of Theorem* 1.1.** Let F, G, H and V be as defined in (2.1) and (2.2). If $F_1=f^nP(f)f(z+c)$ and $G_1=g^nP(g)g(z+c)$, then F and G share 1 and ∞ IM. Suppose that $H\not\equiv 0$. Then according to Lemmas 2.7 and 2.8, $F\not\equiv G$ and $V\not\equiv 0$ and it follows that

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + 2\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + 7\overline{N}(r,F) +$$

$$+ S(r,F) + S(r,G). \tag{3.1}$$

By Lemma 2.4 with $p = 2$, Lemma 2.3 and (3.1), we obtain

$$T(r,F_1) \le N_2\left(r,\frac{1}{G}\right) + 2\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + N_{k+2}\left(r,\frac{1}{F_1}\right) + 7\overline{N}(r,F) +$$

$$+ S(r,F) + S(r,G) \le$$

$$\le N_{k+2}\left(r,\frac{1}{G_1}\right) + k\overline{N}(r,G_1) + 2N_{k+1}\left(r,\frac{1}{F_1}\right) + 2k\overline{N}(r,F_1) + N_{k+1}\left(r,\frac{1}{G_1}\right) +$$

$$+ k\overline{N}(r,G_1) + N_{k+2}\left(r,\frac{1}{F_1}\right) + 7\overline{N}(r,F) + S(r,F) + S(r,G) \le$$

$$\le (k+2)\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{P(g)}\right) + N\left(r,\frac{1}{g(z+c)}\right) + 2k\overline{N}(r,g) +$$

$$+ 2(k+1)\overline{N}\left(r,\frac{1}{f}\right) + 2N\left(r,\frac{1}{P(f)}\right) + 2N\left(r,\frac{1}{f(z+c)}\right) + 4k\overline{N}(r,f) +$$

$$+ (k+1)\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{P(g)}\right) + N\left(r,\frac{1}{g(z+c)}\right) + 2k\overline{N}(r,g) +$$

$$+ (k+2)\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P(f)}\right) + N\left(r,\frac{1}{f(z+c)}\right) +$$

$$+ 14\overline{N}(r,f) + S(r,f) + S(r,g),$$

i.e.,

$$T(r,F_1) \le (3k+4)\overline{N}\left(r,\frac{1}{f}\right) + (2k+3)\overline{N}\left(r,\frac{1}{g}\right) + 3N\left(r,\frac{1}{P(f)}\right) + 2N\left(r,\frac{1}{P(g)}\right) +$$

$$+ 3N\left(r,\frac{1}{f}\right) + 2N\left(r,\frac{1}{g}\right) + (8k+14)\overline{N}(r,f) + S(r,f) + S(r,g).$$

By Lemma 2.11, the above inequality reduces

$$(n+m-1)T(r,f) \le (3k+3m+7)T(r,f) + (2k+2m+5)T(r,g) + (8k+14)\overline{N}(r,f) +$$

$$+ S(r,f) + S(r,g). \tag{3.2}$$

Similarly,

$$(n+m-1)T(r,g) \le (3k+3m+7)T(r,g) + (2k+2m+5)T(r,f) + (8k+14)\overline{N}(r,f) +$$

$$+ S(r,f) + S(r,g). \tag{3.3}$$

From (3.2) and (3.3), we get

$$(n+m-1)(T(r,f)+T(r,g)) \le (5k+5m+12)(T(r,f)+T(r,g)) + 2(8k+14)\overline{N}(r,f)+$$
$$+S(r,f)+S(r,g),$$

i.e.,

$$(n-4m-5k-13)\big(T(r,f)+T(r,g)\big) \le 2(8k+14)\overline{N}(r,f)+S(r,f)+S(r,g). \tag{3.4}$$

Since $V \not\equiv 0,$ F and G share 1 and ∞ IM, by Lemma 2.10, we have

$$(n+m+k-5)\overline{N}(r,f) \le 2\overline{N}\left(r,\frac{1}{F}\right)+2\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g). \tag{3.5}$$

By Lemma 2.4 with $p=1,$ (3.5) reduces

$$(n+m+k-5)\overline{N}(r,f) \le 2(k+1)\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{P(f)}\right)+2N\left(r,\frac{1}{f(z+c)}\right)+$$
$$+2k\overline{N}(r,f)+2k\overline{N}(r,f(z+c))+2(k+1)\overline{N}\left(r,\frac{1}{g}\right)+$$
$$+2N\left(r,\frac{1}{P(g)}\right)+2N\left(r,\frac{1}{g(z+c)}\right)+2k\overline{N}(r,g)+$$
$$+2k\overline{N}(r,g(z+c))+S(r,f)+S(r,g) \le$$
$$\le 2(k+m+2)T(r,f)+2(k+m+2)T(r,g)+8k\overline{N}(r,f)+S(r,f)+S(r,g),$$

i.e.,

$$(n+m-7k-5)\overline{N}(r,f) \le 2(k+m+2)\left[T(r,f)+T(r,g)\right]+S(r,f)+S(r,g). \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$[(n-4m-5k-13)(n+m-7k-5)-4(8k+14)(k+m+2)][T(r,f)+T(r,g)] \le$$
$$\le S(r,f)+S(r,g),$$

which is a contradiction because $n>4m+13k+19.$ Thus, $H\equiv 0.$
Similar to the proof of Lemma 2.5 applied to the functions F and G, we obtain the following cases:

(i) $T(r,F) \le N_2\left(r,\dfrac{1}{F}\right)+N_2\left(r,\dfrac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G),$

(ii) $FG\equiv 1,$

(iii) $F\equiv G.$

By the condition on $n,$ the case (i) is impossible.

By Lemma 2.14, the case (ii) is impossible.

Hence, we get only the case (iii), i.e., $\left[f^nP(f)f(z+c)\right]^{(k)} \equiv \left[g^nP(g)g(z+c)\right]^{(k)}$, then, by Lemma 2.13, we obtain $f^nP(f)f(z+c)\equiv g^nP(g)g(z+c)$, i.e.,

$$f^n\left(a_mf^m+a_{m-1}f^{m-1}+\ldots+a_1f+a_0\right)f(z+c)\equiv$$
$$\equiv g^n\left(a_mg^m+a_{m-1}g^{m-1}+\ldots+a_1g+a_0\right)g(z+c). \tag{3.7}$$

Let $h = \dfrac{f}{g}$. If h is a constant then substituting $f = gh$ and $f(z+c) = g(z+c)h(z+c)$ in (3.7), we deduce $a_m g^{n+m}\big(h^{n+m}h(z+c)-1\big)g(z+c)+a_{m-1}g^{n+m-1}\big(h^{n+m-1}h(z+c)-1\big)g(z+c)+\ldots$ $\ldots+a_1 g^{n+1}\big(h^{n+1}h(z+c)-1\big)g(z+c)+a_0 g^n\big(h^n h(z+c)-1\big)g(z+c)\equiv 0$, which implies $h^d = 1$, where $d = GCD(n+m+1,\ldots,n+m+1-i,\ldots,n+1), a_{m-i}\neq 0$ for $i = 0,1,\ldots,m$. Thus $f(z)\equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m+1,\ldots,n+m+1-i,\ldots$ $\ldots,n+1)$, $a_{m-i}\neq 0$ for $i = 0,1,\ldots,m$, which is the conclusion (1) of Theorem 1.1. If h is not a constant then $f(z)$ and $g(z)$ satisfy the algebraic difference equation $R(f,g)\equiv 0$, where $R(w_1,w_2) = w_1^n(a_m w_1^m + a_{m-1}w_1^{m-1}+\ldots+a_0)w_1(z+c) - w_2^n(a_m w_2^m + a_{m-1}w_2^{m-1}+\ldots$ $\ldots+a_0)w_2(z+c)$, which is the conclusion (2) of Theorem 1.1.

***3.2. Proof of Theorem* 1.2.** Substituting $a_1 = a_2 = \ldots = a_m = 0$ in $P(w)$ and proceeding as in the proof of Theorem 1.1, we complete the proof of Theorem 1.2.

References

1. *Banerjee A.* Meromorphic functions sharing one value // Int. J. Math. Sci. – 2005. – **22**. – P. 3587 – 3598.
2. *Bergweiler W., Langley J. K.* Zeros of differences of meromorphic functions // Math. Proc. Cambridge Phil. Soc. – 2007. – **142**. – P. 133 – 147.
3. *Chiang Y. M., Feng S. J.* On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane // Ramanujan J. – 2008. – **16**. – P. 105 – 129.
4. *Dyavanal R. S., Mathai M. M.* Uniqueness of difference-differential polynomials of meromorphic functions and its applications // Indian J. Math. and Math. Sci. – 2016. – **12**, № 1. – P. 11 – 30.
5. *Hayman W. K.* Meromorphic functions. – Oxford: Clarendon Press, 1964.
6. *Laine I.* Nevanlinna theory and complex differential equations. – Berlin: De Gruyter, 1993.
7. *Li X. M., Li W. L., Yi H. X., Wen Z. T.* Uniqueness theorems of entire functions whose difference polynomials share a meromorphic function of a smaller order // Ann. Polon. Math. – 2011. – **102**, № 2. – P. 111 – 127.
8. *Liu K., Liu X. L., Cao T. B.* Value distribution and uniqueness of difference polynomials // Appl. Math. J. Chinese Univ. – 2011. – Article ID 234215. – 12 p.
9. *Liu K., Liu X. L., Cao T. B.* Some results on zeros and uniqueness of difference-differential polynomials // Appl. Math. J. Chinese Univ. – 2012. – **27**, № 1. – P. 94 – 104.
10. *Liu K., Liu X. L., Cao T. B.* Some results on zeros distributions and uniqueness of derivatives of difference polynomials // arXiv:1107.0773 [math.CV] (2011).
11. *Luo X., Lin W. C.* Value sharing results for shifts of meromorphic functions // J. Math. Anal. and Appl. – 2011. – **377**. – P. 441 – 449.
12. *Qi X. G., Yang L. Z., Liu K.* Uniqueness and periodicity of meromorphic functions concerning the difference operator // Comput. Math. Appl. – 2010. – **60**, № 6. – P. 1739 – 1746.
13. *Yang C. C., Hua X. H.* Uniqueness and value-sharing of meromorphic functions // Ann. Acad. Sci. Fenn. Math. – 1997. – **22**. – P. 395 – 406.
14. *Yang C. C., Yi H. X.* Uniqueness theory of meromorphic functions. – Kluwer Acad. Publ., 2003.
15. *Yi H. X.* Uniqueness of meromorphic functions and a question of C. C. Yang // Complex Var. – 1990. – **14**. – P. 169 – 176.
16. *Yi H. X.* Meromorphic functions that share three sets // Kodai Math. J. – 1997. – **20**. – P. 22 – 32.

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