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ON THE MERKULOV CONSTRUCTION OF A_∞-(CO)ALGEBRAS*
ПРО МЕРКУЛОВСЬКУ КОНСТРУКЦІЮ A_∞-(КО)АЛГЕБР

The aim of this short note is to complete some aspects of a theorem proved by S. Merkulov in [Int. Math. Res. Not. IMRN. – 1999. – **3**. – P. 153 – 167] (Theorem 3.4), as well as to provide a complete proof of the dual result for dg coalgebras.

В цьому короткому повідомленні ми доповнюємо деякі аспекти теореми, що була доведена Меркуловим в [Int. Math. Res. Not. IMRN. – 1999. – **3**. – P. 153 – 167] (теорема 3.4), а також наводимо повне доведення дуального результату для dg-коалгебр.

1. Introduction. The objective of this short note is twofold:

(i) Complete some aspects of a theorem of S. Merkulov – which in principle produces an A_∞-algebra from a certain dg submodule of a dg algebra –, showing that the construction also gives a morphism of A_∞-algebras and that both are strictly unitary under some further assumptions (see Theorem 3.1). These last extra components were not considered in the original statement by Merkulov, but the existence of the morphism of A_∞-algebras appears in a particular case in [6] (Proposition 2.3).

(ii) Provide a precise statement together with a complete proof of a dual version of the previous theorem for dg coalgebras. This is done in Theorem 4.1.

We are mainly interested in (ii), because we need such a result in [1] for our study of the A_∞-coalgebra structure on the group $\operatorname{Tor}^A_\bullet(K,K)$ of a nonnegatively graded algebra A and some of their associated A_∞-comodules (e.g., in Theorem 2.11 and Proposition 2.16 of that article). It was used in particular to compute the A_∞-module structure of $\operatorname{Ext}^\bullet_A(M,k)$ over the Yoneda algebra of a generalized Koszul algebra A, where M is a generalized Koszul module over A. Incidentally, we find more convenient to work with the formulation in [7], for in [1] we need to deal with the slightly greater generality of (nonsymmetric) bimodules over a (noncommutative) algebra.

The proof of the statements added to the result of Merkulov follows the usual philosophy of specific manipulations of equations. However, these new results, which appear in Theorem 3.1, cannot be directly deduced from [7] (Theorem 3.4). In particular, the construction of the morphism of A_∞-algebras added to the result by Merkulov allows to compare the dg algebra one starts with and the constructed A_∞-algebra, which seems relevant to us, and it was indeed needed in [6]. Finally, let us add that the proof of Theorem 4.1 is parallel to the one for dg algebras, and, as in the case of dg algebras, the result obtained for dg coalgebras is slightly more general than those obtained in homological perturbation theory, since our assumptions are in general weaker than those of a SDR (cf. [3], or the nice exposition in [4], Section 6).

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2. Preliminaries on basic algebraic structures. In what follows, k will denote a field and K will be a noncommutative unitary k-algebra. By *module* (sometimes decorated by adjectives such as graded, or dg) we mean a (not necessarily symmetric) bimodule over K (correspondingly decorated), such that the induced bimodule structure over k is symmetric. For dg algebras, dg coalgebras and A_∞-algebras, we follow the sign conventions of [5], whereas for A_∞-coalgebras we shall use the ones given in [2] (Subsection 2.1). We also recall that, if $V = \oplus_{n\in\mathbb{Z}} V^n$ is a (cohomological) graded K-module, $V[m]$ is the graded module over K whose nth homogeneous component $V[m]^n$ is given by V^{m+n}, for all $n, m \in \mathbb{Z}$, and it is called the *shift* of V. We are not going to consider any shift on other gradings, such as the Adams grading. All morphisms between modules will be K-linear on both sides (satisfying further requirements if the modules are decorated as before). One trivially sees that all the standard definitions of graded or dg (co)algebra, or even A_∞-(co)algebra, eventually provided with an Adams grading, and (co)modules over them make perfect sense in the monoidal category of graded K-bimodules, correspondingly provided with an Adams grading. All unadorned tensor products $\otimes$ would be over K.

Finally, $\mathbb{N}$ will denote the set of (strictly) positive integers, whereas $\mathbb{N}_0$ will be the set of nonnegative integers. Similarly, for $N \in \mathbb{N}$, we denote by $\mathbb{N}_{\geq N}$ the set of positive integers greater than or equal to N.

3. On the theorem of Merkulov. Let (A, μ_A, d_A) be a dg algebra provided with an Adams grading and let $W \subseteq A$ be a dg submodule of A respecting the Adams degree. We assume that there is a linear map $Q\colon A \to A[-1]$ of total degree zero, where $A[-1]$ denotes the shift of the cohomological degree, satisfying that the image of $\mathrm{id}_A - [d_A, Q]$ lies in W, where $[d_A, Q] = d_A \circ Q + Q \circ d_A$ is the graded commutator. For all $n \geq 2$, construct $\lambda_n : A^{\otimes n} \to A$ as follows. Setting formally $\lambda_1 = -Q^{-1}$, define

$$\lambda_n = \sum_{i=1}^{n-1} (-1)^{i+1} \mu_A \circ \big((Q \circ \lambda_i) \otimes (Q \circ \lambda_{n-i})\big) \tag{3.1}$$

for $n \geq 2$. We have the following result, whose first part is [7] (Theorem 3.4), whereas the rest is a slightly more general version of [6] (Proposition 2.3 and Lemma 2.5).

Theorem 3.1. *Let (A, μ_A, d_A) be a dg algebra provided with an Adams grading and let $\iota\colon (W, d_W) \to (A, d_A)$ be a dg submodule of A (respecting the Adams degree). Suppose there is a linear map $Q\colon A \to A[-1]$ of total degree zero, where $A[-1]$ denotes the shift of the cohomological degree, satisfying that the image of $\mathrm{id}_A - [d_A, Q]$ lies in $\iota(W)$. For all $n \in \mathbb{N}$, define $m_n\colon W^{\otimes n} \to W$ as follows. Set $m_1 = d_W = d_A \circ \iota$ and $m_n = \big(\mathrm{id}_A - [d_A, Q]\big) \circ \lambda_n \circ \iota^{\otimes n}$, for $n \geq 2$. Then $(W, m_\bullet)$ is an Adams graded A_∞-algebra.*

Define the collection $f_\bullet\colon W \to A$, where $f_n\colon W^{\otimes n} \to A$ is the linear map of total degree $(1-n, 0)$ given by $f_n = -Q \circ \lambda_n \circ \iota^{\otimes n}$ for $n \in \mathbb{N}$. Then $f_\bullet$ is a morphism of Adams graded A_∞-algebras, and it is a quasiisomorphism if and only if ι is so.

Furthermore, assume A has a unit 1_A, there is an element $1_W \in W$ such that $\iota(1_W) = 1_A$, $Q \circ Q = 0$ and $Q \circ \iota = 0$. Then 1_W is a strict unit of the Adams graded A_∞-algebra $(W, m_\bullet)$, and $f_\bullet\colon W \to A$ is a morphism of strictly unitary Adams graded A_∞-algebras.

Proof. For the first part, the proof given in [7], based on that of the corresponding Lemmas 3.2 and 3.3, applies *verbatim* in this context. We also note that the sign convention of that article agrees with the one we follow here.

To prove the second assertion we proceed as follows. It suffices to prove that $f_\bullet$ is a morphism of A_∞-algebras, for the quasiisomorphism property is immediate. We have thus to show the following reduced form of the Stasheff identities on morphisms $\mathrm{MI}(n)$ (see [5], Definition 4.1):

$$\sum_{(r,s,t)\in\mathcal{I}_n}(-1)^{r+st}f_{r+1+t}\circ\big(\mathrm{id}_W^{\otimes r}\otimes m_s\otimes\mathrm{id}_W^{\otimes t}\big)=$$

$$=d_A\circ f_n+\sum_{p=1}^{n-1}(-1)^{p-1}\mu_A\circ\Big((Q\circ\lambda_p\circ\iota^{\otimes p})\otimes\Big(Q\circ\lambda_{n-p}\circ\iota^{\otimes(n-p)}\Big)\Big) \tag{3.2}$$

for all $n\in\mathbb{N}$, where $\mathcal{I}_n=\big\{(r,s,t)\in\mathbb{N}_0\times\mathbb{N}\times\mathbb{N}_0: r+s+t=n\big\}$. The case $n=1$ is trivial since ι is a morphism of complexes. Moreover, the case $n=2$ is also clear, since the left member of (3.2) gives

$$f_1\circ m_2-f_2\circ(\mathrm{id}_W\otimes m_1+m_1\otimes\mathrm{id}_W)=$$
$$=\big(\mathrm{id}_A-[d_A,Q]\big)\circ\mu_A\circ\iota^{\otimes 2}+Q\circ\mu_A\circ\iota^{\otimes 2}\circ(\mathrm{id}_W\otimes d_W+d_W\otimes\mathrm{id}_W)=$$
$$=\mu_A\circ\iota^{\otimes 2}-d_A\circ Q\circ\mu_A\circ\iota^{\otimes 2}, \tag{3.3}$$

where we have used the Leibniz property for the derivation d_A. The right member of (3.2) gives

$$d_A\circ f_2+\mu_A\circ\big((Q\circ\lambda_1\circ\iota)\otimes(Q\circ\lambda_1\circ\iota)\big)=-d_A\circ Q\circ\mu_A\circ\iota^{\otimes 2}+\mu_A\circ\iota^{\otimes 2},$$

which clearly coincides with (3.3). We shall now consider $n>2$. Using the same notation as in [7], by the definition of the tensors Φ_n and Θ_n given in Lemmas 3.2 and 3.3, respectively, we see that

$$Q\circ(\Phi_n+\Theta_n)\circ\iota^{\otimes n}=Q\circ d_A\circ\lambda_n\circ\iota^{\otimes n}-\sum_{(r,s,t)\in\mathcal{I}_n^*}(-1)^{r+st}f_{r+1+t}\circ\big(\mathrm{id}_W^{\otimes r}\otimes m_s\otimes\mathrm{id}_W^{\otimes t}\big)$$

for all $n\in\mathbb{N}$, where $\mathcal{I}_n^*=\big\{(r,s,t)\in\mathbb{N}_0\times\mathbb{N}\times\mathbb{N}_0: r+s+t=n, r+t>0\big\}$. Moreover, by the previously mentioned lemmas, the tensor Φ_n and Θ_n vanish, which implies that

$$\sum_{(r,s,t)\in\mathcal{I}_n}(-1)^{r+st}f_{r+1+t}\circ\big(\mathrm{id}_W^{\otimes r}\otimes m_s\otimes\mathrm{id}_W^{\otimes t}\big)=f_1\circ m_n+Q\circ d_A\circ\lambda_n\circ\iota^{\otimes n}.$$

On the other hand,

$$f_1\circ m_n+Q\circ d_A\circ\lambda_n\circ\iota^{\otimes n}=\lambda_n\circ\iota^{\otimes n}-d_A\circ Q\circ\lambda_n\circ\iota^{\otimes n}=$$
$$=\sum_{i=1}^{n-1}(-1)^{i+1}\mu_A\circ\big((Q\circ\lambda_i\circ\iota^{\otimes i})\otimes(Q\circ\lambda_{n-i}\circ\iota^{\otimes(n-i)})\big)-d_A\circ Q\circ\lambda_n\circ\iota^{\otimes n},$$

where we have used the definition of m_n in the first equality and equation (3.1) in the last one. It is clear that the last member of the previous chain of identities coincides with the right member of (3.2), as was to be shown.

The proof of the third assertion follows the same pattern as the one given in [6] (Lemma 2.5), but since we are assuming a weaker assumption on Q (called G in that article), we describe roughly how it is done. By the definition of f_2, we see that the condition $Q\circ\iota=0$ implies that $f_2(1_W\otimes w)=f_2(w\otimes 1_W)=0$. The fact ι is a morphism of dg modules and $d_A(1_A)=0$

imply that $m_1(1_W) = 0$. Suppose now that we have proved that, for $2 \leq i \leq n-1$, $f_i(w_1, \dots, w_i)$ vanishes if there exists $j \in \{1, \dots, i\}$ such that $w_j = 1_W$. By (3.1) and the inductive hypothesis, we see that $\lambda_n(w_1, \dots, w_n)$ vanishes if there exists $j \in \{2, \dots, n-1\}$ such that $w_j = 1_W$, $\lambda_n(1_W, w_2, \dots, w_n) = f_{n-1}(w_2, \dots, w_n)$, and $\lambda_n(w_1, \dots, w_{n-1}, 1_W) = (-1)^n f_{n-1}(w_1, \dots, w_{n-1})$ for all $w_1, \dots, w_n \in W$. From the definition of f_n and the assumption that $Q \circ Q = 0$ we conclude that $f_n(w_1, \dots, w_n)$ vanishes if there exists $j \in \{1, \dots, n\}$ such that $w_j = 1_W$. Moreover, since the image of $\big(\mathrm{id}_A - [d_A, Q]\big)$ lies in $\iota(W)$ and $Q \circ \iota = 0$, we see that

$$0 = Q \circ \big(\mathrm{id}_A - [d_A, Q]\big) = Q - Q \circ d_A \circ Q = \big(\mathrm{id}_A - [d_A, Q]\big) \circ Q, \tag{3.4}$$

where we have used in the last two equalities that $Q \circ Q$ vanishes. Using our previous description of λ_n in terms of $f_{n-1} = -Q \circ \lambda_{n-1} \circ \iota^{\otimes(n-1)}$ for $n \geq 3$ (if it does not vanish already) and (3.4), we get that $m_n(w_1, \dots, w_n)$ vanishes if there exists $j \in \{1, \dots, n\}$ such that $w_j = 1_W$.

4. The dual result. We shall briefly present the dual procedure to the one introduced by S. Merkulov in [7] to produce an A_∞-algebra structure from a particular data on a dg submodule of a dg algebra. In our case, we produce an A_∞-coalgebra structure on a quotient dg module of a dg coalgebra. We also note that, even though the results of the article of Merkulov are stated for vector spaces, they are clearly seen to be true (by exactly the same arguments) in our more general situation of bimodules over the k-algebra K.

Let (C, Δ_C, d_C) be a dg coalgebra provided with an Adams grading and let $(C, d_C) \twoheadrightarrow (W, d_W)$ be a dg module quotient of C respecting the Adams degree. Denote by $\mathcal{K}$ the kernel of the previous quotient and assume that there is a linear map $Q\colon C \to C[-1]$ of total degree zero, where $C[-1]$ denotes the shift of the cohomological degree whereas the Adams degree remains unchanged, satisfying that $\mathrm{id}_C - [d_C, Q]$ vanishes on $\mathcal{K}$, where $[d_C, Q] = d_C \circ Q + Q \circ d_C$ is the graded commutator. For all $n \geq 2$, define $\gamma_n : C \to C^{\otimes n}$ as follows. Setting formally $\gamma_1 = -Q^{-1}$, define

$$\gamma_n = \sum_{i=1}^{n-1} (-1)^{i+1} \big((\gamma_{n-i} \circ Q) \otimes (\gamma_i \circ Q)\big) \circ \Delta_C \tag{4.1}$$

for $n \geq 2$. We shall say that Q is *admissible* if the family $\{\gamma_n\}_{n \in \mathbb{N}_{\geq 2}}$ is locally finite, i.e., it satisfies that the induced map $C \to \prod_{n\geq 2} C^{\otimes n}$ factors through the canonical inclusion $\oplus_{n \geq 2} C^{\otimes n} \to \prod_{n \geq 2} C^{\otimes n}$.

Fact 4.1. *The identity*

$$-\sum_{p=2}^{n-1} \big(\gamma_p \otimes (\gamma_{n-p} \circ Q)\big) \circ \gamma_2 + \sum_{p=2}^{n-1} (-1)^p \big((\gamma_{n-p} \circ Q) \otimes \gamma_p\big) \circ \gamma_2 = 0$$

holds.

Proof. The identity just follows by replacing the two occurrences of γ_p by the recurrent expression given by (4.1) and simplifying the corresponding terms.

Fact 4.2. *Let e be an endomorphism of C of degree zero and define*

$$E_n(e) = \sum_{(r,s,t) \in \mathcal{I}_n^*} (-1)^{rs+t} (\mathrm{id}_C^{\otimes r} \otimes (\gamma_s \circ e) \otimes \mathrm{id}_C^{\otimes t}) \circ \gamma_{r+1+t},$$

where $\mathcal{I}_n^* = \big\{(r, s, t) \in \mathbb{N}_0 \times \mathbb{N}_{\geq 2} \times \mathbb{N}_0 : r + s + t = n$ and $r + t > 0\big\}$. Then

$$E_n(e) = \sum_{t=1}^{n-2}\sum_{i=1}^{t}(-1)^i(\gamma_{n-t}e\otimes \mathrm{id}_C^{\otimes t})(\gamma_i Q\otimes \gamma_{t-i+1}Q)\Delta_C+$$
$$+\sum_{r=1}^{n-2}\sum_{i=1}^{r}(-1)^{r(n-r)+r-i}(\mathrm{id}_C^{\otimes r}\otimes \gamma_{n-r}e)(\gamma_i Q\otimes \gamma_{1+r-i}Q)\Delta_C+$$
$$+\sum_{(r,s,t)\in\hat{\mathcal{I}}_n}\sum_{i=r+1}^{r+t}(-1)^{rs+r-i}\Big(\big((\mathrm{id}_C^{\otimes r}\otimes\gamma_s e\otimes \mathrm{id}_C^{\otimes(i-r-1)})\gamma_i Q\big)\otimes\gamma_{r+1+t-i}Q\Big)\Delta_C+$$
$$+\sum_{(r,s,t)\in\hat{\mathcal{I}}_n}\sum_{i=1}^{r}(-1)^{rs+r+s-i(s+1)}\Big(\gamma_i Q\otimes\big((\mathrm{id}_C^{\otimes(r-i)}\otimes\gamma_s e\otimes \mathrm{id}_C^{\otimes t})\gamma_{r+1+t-i}Q\big)\Big)\Delta_C, \tag{4.2}$$

where $\hat{\mathcal{I}}_n = \big\{(r,s,t)\in\mathbb{N}\times\mathbb{N}_{\geq 2}\times\mathbb{N}\colon r+s+t=n\big\}$, *and we have omitted the composition symbol* $\circ$ *to economize space.*

Proof. The statement follows from the next chain of identities, which uses the definition (4.1):

$$E_n(e) = \sum_{t=1}^{n-2}(-1)^t(\gamma_{n-t}e\otimes\mathrm{id}_C^{\otimes t})\gamma_{1+t} + \sum_{r=1}^{n-2}(-1)^{r(n-r)}(\mathrm{id}_C^{\otimes r}\otimes\gamma_{n-r}e)\gamma_{r+1}+$$
$$+\sum_{(r,s,t)\in\hat{\mathcal{I}}_n}(-1)^{rs+t}\big(\mathrm{id}_C^{\otimes r}\otimes\gamma_s e\otimes\mathrm{id}_C^{\otimes t}\big)\gamma_{r+1+t} =$$
$$=\sum_{t=1}^{n-2}\sum_{i=1}^{t}(-1)^i(\gamma_{n-t}e\otimes \mathrm{id}_C^{\otimes t})(\gamma_i Q\otimes \gamma_{t-i+1}Q)\Delta_C+$$
$$+\sum_{r=1}^{n-2}\sum_{i=1}^{r}(-1)^{r(n-r)+r-i}\big(\mathrm{id}_C^{\otimes r}\otimes \gamma_{n-r}e\big)(\gamma_i Q\otimes \gamma_{1+r-i}Q)\Delta_C+$$
$$+\sum_{(r,s,t)\in\hat{\mathcal{I}}_n}\sum_{i=r+1}^{r+t}(-1)^{rs+r-i}\Big(\big((\mathrm{id}_C^{\otimes r}\otimes\gamma_s e\otimes \mathrm{id}_C^{\otimes(i-r-1)})\gamma_i Q\big)\otimes\gamma_{r+1+t-i}Q\Big)\Delta_C+$$
$$+\sum_{(r,s,t)\in\hat{\mathcal{I}}_n}\sum_{i=1}^{r}(-1)^{rs+r+s-i(s+1)}\Big(\gamma_i Q\otimes\big((\mathrm{id}_C^{\otimes(r-i)}\otimes\gamma_s e\otimes \mathrm{id}_C^{\otimes t})\gamma_{r+1+t-i}Q\big)\Big)\Delta_C,$$

where we have omitted the composition symbol $\circ$ to reduce space.

Lemma 4.1. *For* $n\in\mathbb{N}_{\geq 3}$, *define*

$$\Gamma_n = \sum_{(r,s,t)\in\mathcal{I}_n^*}(-1)^{rs+t}(\mathrm{id}_C^{\otimes r}\otimes\gamma_s\otimes\mathrm{id}_C^{\otimes t})\circ\gamma_{r+1+t},$$

where $\mathcal{I}_n^* = \big\{(r,s,t)\in\mathbb{N}_0\times\mathbb{N}_{\geq 2}\times\mathbb{N}_0\colon r+s+t=n\,\mathrm{and}\,r+t>0\big\}$. *Then* $\Gamma_n\equiv 0$ *for all* $n\geq 3$.

Proof. First note that $\Gamma_3 = (\mathrm{id}_C\otimes\Delta_C - \Delta_C\otimes\mathrm{id}_C)\circ\Delta_C$, so the coassociativity of C implies that Γ_3 vanishes.

Let us now consider $n>3$. We note that $\Gamma_n = E_n(\mathrm{id}_C)$, so it can be written as indicated in equation (4.2). Moreover, the terms corresponding to $i=1$ in the first sum and to $i=r$ in the second sum in that latter expression of Γ_n cancel due to Fact 4.1. Using the definition of Γ_m for $3 \leq m \leq n-1$ in the remaining terms of the new expression of Γ_n we get

$$\Gamma_n = \sum_{i=1}^{n-3} (-1)^{n-i} \big((\gamma_i \circ Q) \otimes (\Gamma_{n-i} \circ Q) \big) \circ \Delta_C - \sum_{j=1}^{n-3} \big((\Gamma_{n-j} \circ Q) \otimes (\gamma_j \circ Q) \big) \circ \Delta_C.$$

The lemma thus follows from an inductive argument.

Lemma 4.2. *For $n \in \mathbb{N}_{\geq 2}$, define*

$$H_n = \gamma_n \circ d_C + (-1)^{n-1} \sum_{r=0}^{n-1} \Big(\mathrm{id}_C^{\otimes r} \otimes d_C \otimes \mathrm{id}_C^{\otimes (n-r-1)} \Big) \circ \gamma_n -$$
$$- \sum_{(r,s,t) \in \mathcal{I}_n^*} (-1)^{rs+t} \Big(\mathrm{id}_C^{\otimes r} \otimes (\gamma_s \circ [d_C, Q]) \otimes \mathrm{id}_C^{\otimes t} \Big) \circ \gamma_{r+1+t},$$

where $\mathcal{I}_n^* = \big\{ (r,s,t) \in \mathbb{N}_0 \times \mathbb{N}_{\geq 2} \times \mathbb{N}_0 : r+s+t = n$ and $r+t > 0 \big\}$. Then $H_n \equiv 0$ for all $n \geq 2$.

Proof. Note that $H_2 = \Delta_C \circ d_C - (\mathrm{id}_C \otimes d_C + d_C \otimes \mathrm{id}_C) \circ \Delta_C$, so it vanishes by the Leibniz identity for C.

Assume now that $n > 2$. Note that

$$H_n = \gamma_n \circ d_C - (-1)^n \sum_{r=0}^{n-1} \Big(\mathrm{id}_C^{\otimes r} \otimes d_C \otimes \mathrm{id}_C^{\otimes (n-r-1)} \Big) \circ \gamma_n - E_n\big([d_C, Q]\big).$$

Using the definition (4.1), we can write

$$\begin{aligned} H_n = & \sum_{i=1}^{n-1} \big((\gamma_i \circ Q \circ d_C) \otimes (\gamma_{n-i} \circ Q) \big) \circ \Delta_C - \\ & - \sum_{i=1}^{n-1} (-1)^{n-i} \big((\gamma_i \circ Q) \otimes (\gamma_{n-i} \circ Q \circ d_C) \big) \circ \Delta_C + \\ & + \sum_{i=0}^{n-1} \sum_{r=0}^{i-1} (-1)^i \Big(\Big(\Big(\mathrm{id}_C^{\otimes r} \otimes d_C \otimes \mathrm{id}_C^{\otimes (i-r-1)} \Big) \circ (\gamma_i \circ Q) \Big) \otimes (\gamma_{n-i} \circ Q) \Big) \circ \Delta_C - \\ & - \sum_{i=0}^{n-1} \sum_{r=0}^{n-i-1} \Big((\gamma_i \circ Q) \otimes \Big(\Big(\mathrm{id}_C^{\otimes r} \otimes d_C \otimes \mathrm{id}_C^{\otimes (n-i-r-1)} \Big) \circ (\gamma_{n-i} \circ Q) \Big) \Big) \circ \Delta_C - \\ & - E_n(d_C \circ Q) - E_n(Q \circ d_C). \end{aligned} \tag{4.3}$$

Fact 4.2 expresses $E_n(Q \circ d_C)$ as a linear combination of sums satisfying that the terms corresponding to $i = 1$ of its first sum and the terms corresponding to $i = r$ of its second sum cancel the first two sums in (4.3). Combining the remaining terms of (4.3) and using the definition of H_m, for $3 \leq m \leq n-1$, we get

$$H_n = \sum_{i=1}^{n-2} (-1)^{n-i} \big((\gamma_i \circ Q) \otimes (H_{n-i} \circ Q) \big) \otimes \Delta_C - \sum_{j=1}^{n-2} \big((H_{n-j} \circ Q) \otimes (\gamma_j \circ Q) \big) \circ \Delta_C.$$

The lemma thus follows from an inductive argument.

Theorem 4.1. *Let* (C, Δ_C, d_C) *be a dg coalgebra provided with an Adams grading and let* $\rho\colon (C, d_C) \to (W, d_W)$ *be a quotient dg module of* C (*respecting the Adams degree*) *with kernel* $\mathcal{K}$. *Suppose there is an admissible linear map* $Q\colon C \to C[-1]$ *of total degree zero, where* $C[-1]$ *denotes the shift of the cohomological degree, satisfying that* $\mathrm{id}_C - [d_C, Q]$ *vanishes on* $\mathcal{K}$. *For all* $n \in \mathbb{N}$, *define* $\Delta_n\colon W \to W^{\otimes n}$ *as follows. Set* $\Delta_1 = d_W$ *and* Δ_n *to be the unique map satisfying that* $\Delta_n \circ \rho = \rho^{\otimes n} \circ \gamma_n \circ \big(\mathrm{id}_C - [d_C, Q]\big)$ *for* $n \geq 2$. *Then* $(W, \Delta_\bullet)$ *is an Adams graded* A_∞*-coalgebra.*

Define the collection $f_\bullet\colon C \to W$, *where* $f_n\colon C \to W^{\otimes n}$ *is the linear map of homological degree* $n-1$ *and zero Adams degree given by* $f_n = -\rho^{\otimes n} \circ \gamma_n \circ Q$ *for* $n \in \mathbb{N}$. *Then* $f_\bullet$ *is a morphism of Adams graded* A_∞*-coalgebras.*

Furthermore, assume C *has a counit* ϵ_C, *there a linear map* $\epsilon_W\colon W \to \mathcal{K}$ *such that* $\epsilon_W \circ \rho = \epsilon_C$, $Q \circ Q = 0$ *and* $\rho \circ Q = 0$. *Then,* ϵ_W *is a strict counit of the Adams graded* A_∞*-coalgebra* $(W, \Delta_\bullet)$, *and* $f_\bullet\colon C \to W$ *is a morphism of strictly counitary Adams graded* A_∞*-coalgebras.*

Proof. The first part follows from the fact that the Stasheff identity $\mathrm{SI}(n)$ for $n \geq 3$ for the operations $\Delta_\bullet$ is by definition given by $\rho^{\otimes n} \circ (\Gamma_n + H_n) \circ \big(\mathrm{id} - [d_C, Q]\big)$, so it vanishes. The two first Stasheff identities $\mathrm{SI}(1)$ and $\mathrm{SI}(2)$ are trivial, for W is a quotient of C.

To prove the second assertion we proceed as follows. We have thus to prove the following reduced form of the Stasheff identities on morphisms $\mathrm{MI}(n)$ (see [2], eq. (2.2)):

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{rs+t}\big(\mathrm{id}_W^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}_W^{\otimes t}\big) \circ f_{r+1+t} =$$

$$= f_n \circ d_C + \sum_{p=1}^{n-1} (-1)^{n-p-1}\Big((\rho^{\otimes p} \circ \gamma_p \circ Q) \otimes (\rho^{\otimes (n-p)} \circ \gamma_{n-p} \circ Q)\Big) \circ \Delta_C \tag{4.4}$$

for all $n \in \mathbb{N}$, where $\mathcal{I}_n = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 \colon r+s+t = n\}$. The case $n=1$ is trivial since ρ is a morphism of complexes. Moreover, the case $n=2$ is also clear, since the left member of (4.4) gives

$$\Delta_2 \circ f_1 - \big(\mathrm{id}_W \otimes \Delta_1 + \Delta_1 \otimes \mathrm{id}_W\big) \circ f_2 =$$
$$= \rho^{\otimes 2} \circ \Delta_C \circ \big(\mathrm{id}_A - [d_C, Q]\big) + \rho^{\otimes 2} \circ (\mathrm{id}_C \otimes d_C + d_C \otimes \mathrm{id}_C) \circ \Delta_C \circ Q =$$
$$= \rho^{\otimes 2} \circ \Delta_C - \rho^{\otimes 2} \circ \Delta_C \circ Q \circ d_C, \tag{4.5}$$

where we have used the Leibniz property for the coderivation d_C. The right member of (4.4) gives

$$f_2 \circ d_C + \big((\rho \circ \gamma_1 \circ Q) \otimes (\rho \circ \gamma_1 \circ Q)\big) \circ \Delta_C = -\rho^{\otimes 2} \circ \Delta_C \circ Q \circ d_C + \rho^{\otimes 2} \circ \Delta_C,$$

which clearly coincides with (4.5). We shall now consider $n > 2$. By the definition of the tensors Γ_n and H_n given in Lemmas 4.1 and 4.2, respectively, we see that

$$\rho^{\otimes n} \circ (\Gamma_n + H_n) \circ Q = \rho^{\otimes n} \circ \gamma_n \circ d_C \circ Q + \sum_{(r,s,t)\in\mathcal{I}_n^*} (-1)^{rs+t}(\mathrm{id}_W^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}_W^{\otimes t}) \circ f_{r+1+t}$$

for all $n \in \mathbb{N}$, where we recall that $\mathcal{I}_n^* = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 \colon r+s+t = n, r+t > 0\}$. Moreover, by the previously mentioned lemmas, the tensor Γ_n and H_n vanish, which implies that

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{rs+t}(\mathrm{id}_W^{\otimes r} \otimes \Delta_s \otimes \mathrm{id}_W^{\otimes t}) \circ f_{r+1+t} = \Delta_n \circ f_1 - \rho^{\otimes n} \circ \gamma_n \circ d_C \circ Q.$$

On the other hand,

$$\Delta_n \circ f_1 - \rho^{\otimes n} \circ \gamma_n \circ d_C \circ Q = \rho^{\otimes n} \circ \gamma_n - \rho^{\otimes n} \circ \gamma_n \circ Q \circ d_C =$$

$$= \sum_{i=1}^{n-1} (-1)^{n-i-1} \big((\rho^{\otimes i} \circ \gamma_i \circ Q) \otimes (\rho^{\otimes (n-i)} \circ \gamma_{n-i} \circ Q) \big) \circ \Delta_C - \rho^{\otimes n} \circ \gamma_n \circ Q \circ d_C,$$

where we have used the definition of Δ_n in the first equality, and equation (4.1) in the last one. It is clear that the last member of the previous chain of identities coincides with the right member of (4.4), as was to be shown.

The proof of the third assertion is parallel to the one given to Theorem 3.1. Using the definition of f_2, we see that the condition $\rho \circ Q = 0$ implies that $(\epsilon_W \otimes \mathrm{id}_W) \circ f_2 = (\mathrm{id}_W \otimes \epsilon_W) \circ f_2 = 0$. The fact ρ is a morphism of dg modules and $\epsilon_C \circ d_C = 0$ imply that $\epsilon_W \circ \Delta_1 = 0$. Suppose now that we have proved that, for $2 \leq i \leq n-1$, $\left(\mathrm{id}_W^{\otimes j} \otimes \epsilon_W \otimes \mathrm{id}_W^{\otimes (i-j-1)} \right) \circ f_i$ vanishes for all $j \in \{0, \ldots, i-1\}$. By (4.1) and the inductive hypothesis, we see that $\left(\mathrm{id}_W^{\otimes j} \otimes \epsilon_W \otimes \mathrm{id}_W^{\otimes (n-j-1)} \right) \circ$ $\circ \rho^{\otimes n} \circ \gamma_n$ vanishes for all $j \in \{1, \ldots, n-2\}$, $\left(\epsilon_W \otimes \mathrm{id}_W^{\otimes (n-1)} \right) \circ \rho^{\otimes n} \circ \gamma_n = (-1)^{n-1} f_{n-1}$, and $\left(\mathrm{id}_W^{\otimes (n-1)} \otimes \epsilon_W \right) \circ \rho^{\otimes n} \circ \gamma_n = f_{n-1}$. By the definition of f_n and the assumption that $Q \circ Q = 0$ we conclude that $(\mathrm{id}_W^{\otimes j} \otimes \epsilon_W \otimes \mathrm{id}_W^{\otimes (n-j)}) \circ f_n$ vanishes for all $j \in \{0, \ldots, n-1\}$. Moreover, since the image of $\big(\mathrm{id}_C - [d_C, Q]\big)$ vanishes on the kernel of ρ and $\rho \circ Q = 0$, we see that

$$0 = \big(\mathrm{id}_C - [d_C, Q]\big) \circ Q = Q - Q \circ d_C \circ Q = Q \circ \big(\mathrm{id}_C - [d_C, Q]\big), \tag{4.6}$$

where we have used in the last two equalities that $Q \circ Q$ vanishes. Using our previous description of $\left(\mathrm{id}_W^{\otimes j} \otimes \epsilon_W \otimes \mathrm{id}_W^{\otimes (n-j-1)} \right) \circ \rho^{\otimes n} \circ \gamma_n$ in terms of $f_{n-1} = -\rho^{\otimes (n-1)} \circ \gamma_{n-1} \circ Q$ for $n \geq 3$ (if it does not vanish already) and (4.6), we get that $\left(\mathrm{id}_W^{\otimes j} \otimes \epsilon_W \otimes \mathrm{id}_W^{\otimes (n-j)} \right) \circ \Delta_n$ vanishes for all $j \in \{0, \ldots, n-1\}$.

The structure of A_∞-coalgebra on W given by the previous theorem is called a *Merkulov model* on W, or simply a *model*. As in the case of dg algebras, note that the result stated in the first two paragraphs of the previous theorem is slightly more general than those obtained in homological perturbation theory, since the conditions of a SDR are not necessarily satisfied (cf. [3] or [4], Section 6).

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