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ON SPLICED SEQUENCES AND THE DENSITY OF POINTS WITH RESPECT TO A MATRIX CONSTRUCTED BY USING A WEIGHT FUNCTION*

ПРО СПЛЕТЕНІ ПОСЛІДОВНОСТІ ТА ГУСТИНУ ТОЧОК ВІДНОСНО МАТРИЦІ, ЩО СКОНСТРУЙОВАНА ЗА ДОПОМОГОЮ ВАГОВОЇ ФУНКЦІЇ

Following the line of investigation in [Linear Algebra and Appl. – 2015. – **487**. – P. 22 – 42], for $y \in \mathbb{R}$ and a sequence $x = (x_n) \in \ell^\infty$ we define a new notion of density δ_g with respect to a weight function g of indices of the elements x_n close to y, where $g : \mathbb{N} \to [0, \infty)$ is such that $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$. We present the relationships between the densities δ_g of indices of (x_n) and the variation of the Cesàro-limit of (x_n). Our main result states that if the set of limit points of (x_n) is countable and $\delta_g(y)$ exists for any $y \in \mathbb{R}$, then $\lim_{n\to\infty} \dfrac{1}{g(n)} \sum_{i=1}^{n} x_i = \sum_{y\in\mathbb{R}} \delta_g(y) \cdot y$, which is an extended and much more general form of the "natural density version of the Osikiewicz theorem". Note that in [Linear Algebra and Appl. – 2015. – **487**. – P. 22 – 42], the regularity of the matrix was used in the entire investigation, whereas in the present paper the investigation is actually performed with respect to a special type of matrix, which is not necessarily regular.

У цьому викладі ми слідуємо роботі [Linear Algebra and Appl. – 2015. – **487**. – P. 22–42]. Так, для $y \in \mathbb{R}$ і послідовності $x = (x_n) \in \ell^\infty$ ми вводимо нове поняття густини δ_g відносно вагової функції g від індексів елементів x_n, близьких до y, де функція $g : \mathbb{N} \to [0, \infty)$ така, що $g(n) \to \infty$ і $n/g(n) \nrightarrow 0$. Наведено співвідношення між густинами δ_g індексів елементів (x_n) і варіаціями границі Чезаро для (x_n). В основному результаті стверджується, що у випадку, коли множина граничних значень для (x_n) є зліченною, а $\delta_g(y)$ існує для всіх $y \in \mathbb{R}$, $\lim_{n\to\infty} \dfrac{1}{g(n)} \sum_{i=1}^{n} x_i = \sum_{y\in\mathbb{R}} \delta_g(y) \cdot y$, що є розширеною та набагато більш загальною формою „природної густинної версії теореми Осікевича". Відмітимо, що в [Linear Algebra and Appl. – 2015. – **487**. – P. 22 – 42] регулярність матриці використовувалась протягом усього дослідження. Водночас у нашій роботі дослідження насправді виконується для спеціального типу матриці, що необов'язково є регулярною.

1. Introduction. For $n, m \in \mathbb{N}$ with $n < m$, let $[n, m]$ denote the set $\{n, n+1, n+2, \ldots, m\}$. Let $A \subset \mathbb{N}$. Define

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{n} \qquad \text{and} \qquad \underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n}.$$

The numbers $\overline{d}(A)$ and $\underline{d}(A)$ are called the upper natural density and the lower natural density of A, respectively. If $\overline{d}(A) = \underline{d}(A)$, then this common value is called the natural density of A and we denote it by $d(A)$. Let $\mathcal{I}_d$ be the family of all subsets of $\mathbb{N}$ which have natural density 0. Then $\mathcal{I}_d$ is a proper nontrivial admissible ideal of subsets of $\mathbb{N}$. The notion of natural density was used by Fast [8] and Scoenberg [23] to define the notion of statistical convergence.

In [4] a natural extension of the notions of natural density and statistical convergence were introduced, by replacing n with a non linear term n^α, $0 < \alpha < 1$, in the definition of asymptotic density. The motivation came from the urge to investigation different kinds of densities and the problem of comparing them with the natural density. Very recently in has been shown in [2] that one can Further, extend the concept of natural density by considering natural density of weight g where

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$g\colon \mathbb{N} \to [0,\infty)$ is a function with $\lim_{n\to\infty} g\left(n\right) = \infty$ and $\dfrac{n}{g\left(n\right)} \nrightarrow 0$. It has been observed in [2] that one can construct uncountably many noncomparable P-ideals corresponding to different choices of the weight function $g,$ all different from the ideal $\mathcal{I}_d$.

In another direction Osikiewicz had developed the ideas of finite and infinite splices in [20]. Let $E_1, E_2, E_3, \ldots, E_k, \ldots$ be a partition of $\mathbb{N}$ into countable number of sequences. Let $y_1,\, y_2,\, y_3,\, \ldots$ $\ldots,\, y_k,\, \ldots$ be distinct real numbers. Let (x_n) be such that

$$\lim_{n\to\infty, n\in E_i} x_n = y_i.$$

Then (x_n) is called an infinite-splice. (In the same way Osikiewicz defined a finite splice taking finite number of sequences and finite number of distinct real numbers.) He proved the following.

Theorem 1 (Natural density (or Cesàro) version of Osikiewicz theorem [20]). *Assume that* (x_n) *is a splice over a partition* $\{E_i\}$. *Let* $y_i = \lim_{n\to\infty, n\in E_i} x_n$. *Assume that* $d(E_i)$ *exists for each* i *and*

$$\sum_i d(E_i) = 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = \sum_i y_i \cdot d(E_i).$$

In fact Osikiewicz considered a more general case, namely matrix summability method and A-density with the use of regular infinite matrices A the details of which are presented in the preliminaries. Very recently in [3] a new approach was made to study the general version of Osikiewicz theorem by defining the notion of the A-density of a point and an alternative version of the same result was established. In fact it was shown that the assumptions of Osikiewicz theorem imply those of the following theorem.

Theorem 2. *Suppose that* $x = (x_n)$ *is a bounded sequence,* $\delta_A(y)$ *exists for every* $y \in \mathbb{R}$ *and* $\sum_{y\in D} \delta_A(y) = 1$. *Then*

$$\lim_{n\to\infty} (Ax)_n = \sum_{y\in D} \delta_A(y) \cdot y.$$

Consequently, the Osikiewicz result follows from Theorem 2.

As a natural consequence in this paper we extend the ”natural density version of Osikiewicz theorem” with the help of a weighted density function. But instead of considering the original approach of Osikiewicz, we follow the more natural line of investigation of [3]. In order to do that we define the notion of the density of a point with respect to a weight function and prove some of its consequences. Note that the corresponding results do not follow from the results of [3] as the redefined matrix with respect to a weight function is not necessarily a regular matrix. This shows that results similar to [20] or [3] can be obtained for special kinds of nonregular matrices also. For simplicity we do not use the matrix notation inside the body of the paper.

2. Preliminaries. We first present the necessary definitions and notations which will form the background of this article.

If $x = (x_n)$ is a sequence and $A = (a_{n,k})$ is a summability matrix, then by Ax we denote the sequence $((Ax)_1, (Ax)_2, (Ax)_3, \dots)$ where $(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k$. The matrix A is called regular if $\lim_{n\to\infty} x_n = L$ implies $\lim_{n\to\infty}(Ax)_n = L$. The well-known Silverman – Toeplitz theorem characterizes regular matrices in the following way. A matrix A is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$,

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$,

(iii) $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{n,k}| < \infty$.

We say that a nonnegative matrix $A = (a_{ij})$ is nonregular if it fails to satisfy any of the three conditions (i), (ii) and (iii) prescribed above.

For a nonnegative regular matrix A and $E \subset \mathbb{N}$, following Freedman and Sember [12], the A-density of E, denoted by $\delta_A(E)$, is defined as follows:

$$\underline{\delta_A}(E) = \liminf_{n\to\infty} \sum_{k\in E} a_{n,k} = \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \mathbb{1}_E(k) = \liminf_{n\to\infty} (A\mathbb{1}_E)_n,$$

where $\mathbb{1}_E$ is a 0-1 sequence such that $\mathbb{1}_E(k) = 1 \iff k \in E$. If $\overline{\delta_A}(E) = \underline{\delta_A}(E)$ then we say that the A-density of E exists and it is denoted by $\delta_A(E)$. Clearly, if A is the Cesàro matrix, i.e.,

$$a_{n,k} = \begin{cases} \dfrac{1}{n} & \text{if } n \geq k, \\ 0 & \text{otherwise}, \end{cases}$$

then δ_A coincides with the natural density.

Throughout by ℓ^∞ we denote the set of all bounded sequences of reals.

We first recall the original Osikiewicz theorem.

Theorem 3 (Osikiewicz [20]). *Assume that A is a nonnegative regular summability matrix. Assume that $(x_n) \in \ell^\infty$ is a splice over a partition $\{E_i\}$. Let $y_i = \lim_{n\to\infty, n\in E_i} x_n$. Assume that $\delta_A(E_i)$ exists for each i and*

$$\sum_i \delta_A(E_i) = 1.$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k = \sum_i y_i \cdot \delta_A(E_i).$$

In [3] in a new approach, the authors had defined for a sequence (x_n) the density $\delta_A(y)$ of indices of those x_n which are close to y which was not dealt with till then in the literature. This was a more general approach than that of Osikiewicz.

Fix $(x_n) \in \ell^\infty$. For $y \in \mathbb{R}$ let

$$\overline{\delta_A}(y) = \lim_{\varepsilon\to 0^+} \overline{\delta_A}(\{n\colon |x_n - y| \leq \varepsilon\})$$

and

$$\underline{\delta_A}(y) = \lim_{\varepsilon\to 0^+} \underline{\delta_A}(\{n\colon |x_n - y| \leq \varepsilon\}).$$

If $\overline{\delta_A}(y) = \underline{\delta_A}(y)$, then the common value is denoted by $\delta_A(y)$.

The main result of [3] was the following.

Theorem 4. *Let $x = (x_n) \in \ell^\infty$. Suppose that the set of limit points of (x_n) is countable and $\delta_A(y)$ exists for any $y \in \mathbb{R}$ where A is a nonnegative regular matrix. Then*

$$\lim_{n\to\infty}(Ax)_n = \sum_{y\in\mathbb{R}} \delta_A(y) \cdot y.$$

Now recall that a nonempty family $\mathcal{I}$ of subsets of $\mathbb{N}$ is an ideal in $\mathbb{N}$ if for $A,\ B \subset \mathbb{N}$: (i) $A,\ B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (ii) $A \in \mathcal{I},\ B \subset A$ imply $B \in \mathcal{I}$. Further, if $\bigcup_{A\in\mathcal{I}} A = \mathbb{N}$, i.e., $\{n\} \in \mathcal{I}\ \ \forall\, n \in \mathbb{N}$, then $\mathcal{I}$ is called admissible or free. An ideal $\mathcal{I}$ is called a P-ideal if for any sequence of sets (D_n) from $\mathcal{I}$, there is another sequence of sets (C_n) in $\mathcal{I}$ such that $D_n \vartriangle C_n$ is finite for all n and $\bigcup_n C_n \in \mathcal{I}$. Equivalently, if for each sequence (A_n) of sets from $\mathcal{I}$ there exists $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$, then $\mathcal{I}$ becomes a P-ideal.

For a bounded sequence (x_n), we now recall the following definitions (see [17]):

(i) (x_n) is $\mathcal{I}$-convergent to y if for any $\varepsilon > 0$, $\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}$.

(ii) A point y is called an $\mathcal{I}$-cluster point of (x_n) if $\{n : |x_n - y| \leq \varepsilon\} \notin \mathcal{I}$ for any $\varepsilon > 0$.

(iii) y is called an $\mathcal{I}$-limit point of (x_n) if there is a set $B \subset \mathbb{N}$, $B \notin \mathcal{I}$, such that $\lim_{n\in B} x_n = y$.

We now start our main discussions. Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g\,(n) = \infty$. The upper density of weight g was defined in [2] by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{g\,(n)}$$

for $A \subset \mathbb{N}$. The lower density of weight g, $\underline{d}_g(A)$ is defined in a similar way. Then the family

$$\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d}_g(A) = 0\}$$

forms an ideal. It has been observed in [2] that $\mathbb{N} \in \mathcal{I}_g$ if and only if $\dfrac{n}{g\,(n)} \to 0$. Therefore, we additionally assume that $n/g\,(n) \nrightarrow 0$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ becomes a proper admissible P-ideal of $\mathbb{N}$ (see [2]). As a natural consequence we can consider the following definition.

Definition 1. *A sequence (x_n) of real numbers is said to converge d_g-statistically to x if, for any given $\epsilon > 0$, $\overline{d}_g(A_\epsilon) = 0$ where*

$$A_\epsilon = \{n \in \mathbb{N} :\ |x_n - x| \geq \epsilon\}.$$

Further, one should observe that if we define $A = (a_{ij}),\ i, j = 1, 2, \ldots, \infty$, such that

$$a_{ij} = \begin{cases} \dfrac{1}{g(i)} & \text{if} \quad i \leq j, \\ 0 & \text{otherwise,} \end{cases}$$

where $g : \mathbb{N} \to (0, \infty)$ is a weight function defined above then clearly A is not necessarily a regular matrix (though for certain choices of g, for example, if $g(n) = n + 1$, the generated matrix would be regular). In fact for appropriately chosen functions g the corresponding matrices may not satisfy all the three conditions of a regular matrix. For example, if we take $g(n) = \sqrt{n}$, then for the corresponding matrix

(i) $\lim_{n\to\infty} a_{n,k} = 0$,

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \infty$.

Our main result, Theorem 5, reveals that we can actually obtain Osikiewicz like theorems for matrices which are not necessarily regular.

We now define the main concepts, namely, the notions of g-densities at a point where the upper density of weight g is defined by

$$\overline{\delta_g}(y) = \lim_{\varepsilon \to 0^+} \overline{\delta_g}\{n \colon |x_n - y| \le \varepsilon\}$$

and the lower density of weight g is defined by

$$\underline{\delta_g}(y) = \lim_{\varepsilon \to 0^+} \underline{\delta_g}\{n \colon |x_n - y| \le \varepsilon\}.$$

If $\overline{\delta_g}(y) = \underline{\delta_g}(y)$, then the common value is denoted by $\delta_g(y)$.

3. Main results. The main result which we are going to establish in this paper is the following.

Theorem 5. *Let $x = (x_n) \in \ell^\infty$ and the set of limit points of (x_n) is countable. Let A be a matrix generated by a weight function g for wich $\lim_{n\to\infty} \dfrac{n}{g(n)} = M$ (say) finitely exists. Suppose $\delta_g(y)$ exists for all $y \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} (Ax)_n = \sum_{y\in\mathbb{R}} \delta_g(y)\cdot y$$

or, equivalently,

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{i=1}^{n} x_i = \sum_{y\in\mathbb{R}} \delta_g(y)\cdot y.$$

We start with the following observation.

Lemma 1. *Let $(x_n) \in \ell^\infty$ and $\delta_g(y)$ exists for all $y \in \mathbb{R}$. Then $D := \{y \in \mathbb{R} \colon \delta_g(y) > 0\}$ is countable and*

$$\sum_{y\in D} \delta_g(y) \le \limsup_n \frac{n}{g(n)}.$$

Proof. If $\limsup_n \dfrac{n}{g(n)} = \infty$, then there is nothing to prove. So let $\limsup_n \dfrac{n}{g(n)} = M < \infty$ and let (r_n) be a strictly decreasing sequence converging to M. For $m \in \mathbb{N}$ let

$$D_m := \left\{ y \in \mathbb{R} \,\middle|\, \delta_g(y) \ge \frac{1}{m} \right\}.$$

Note that $D_1 \subset D_2 \subset \ldots \subset D_m \subset \ldots$ and $D = \bigcup_m D_m$. Now if $y_1, y_2, \ldots, y_l \in D_m$ be distinct, let us choose $\varepsilon = \min_{i\neq j} \dfrac{|y_i - y_j|}{3} > 0$. Consequently, the sets $E_i = \{n \colon x_n \in (y_i - \varepsilon, y_i + \varepsilon)\}$ are pairwise disjoint. Moreover,

$$\underline{\delta_g}(E_i) \ge \delta_g(y_i) \ge \frac{1}{m} \Rightarrow \liminf_n \frac{|E_i \cap [1,n]|}{g(n)} \ge \frac{1}{m}.$$

Then, for any $\tau > 0$, there exists $n_1 \in \mathbb{N}$ such that $\dfrac{|E_i \cap [1,n]|}{g(n)} > \dfrac{1}{m} - \tau$ for all $n \ge n_1$. Again $\limsup_n \dfrac{n}{g(n)} < r_n$ for every $n \in \mathbb{N}$. So, for any fixed r_p, we get $n_2 \in \mathbb{N}$ such that

$\dfrac{n}{g(n)} < M + \delta < r_p \quad \forall\, n \geq n_2$ and for a suitably chosen δ. Let $n_0 = \max\{n_1, n_2\}$. As E_i's are disjoint, we have

$$\frac{|(\bigcup_{i=1}^{l} E_i) \cap [1,n]|}{g(n)} = \sum_{i=1}^{l} \frac{|E_i \cap [1,n]|}{g(n)} \geq \frac{l}{m} - l\tau \qquad \forall\, n \geq n_0.$$

But

$$\frac{\left|(\bigcup_{i=1}^{l} E_i) \cap [1,n]\right|}{g(n)} \leq \frac{n}{g(n)} < r_p.$$

Now note that $\dfrac{l}{m} - l\tau \leq r_p$ evidently implies $l \leq \dfrac{mr_p}{1-\tau m}$. Hence, choosing τ so that $1 - \tau m > 0$ we observe that l must be finite. Thus, D_m is finite for each m which implies that $D = \bigcup_m D_m$ can be at most countable.

Again

$$\sum_{y \in D_m} \delta_g(y) \leq \sum_{i=1}^{l} \underline{\delta_g}(E_i) = \sum_{i=1}^{l} \liminf_{n} \frac{|E_i \cap [1,n]|}{g(n)} \leq$$

$$\leq \sum_{i=1}^{l} \left(\frac{|E_i \cap [1,n]|}{g(n)} + \frac{\varepsilon_0}{l} \right) \quad \text{for all} \quad n \geq N \quad \text{(say)}$$

where ε_0 is arbitrary. So

$$\sum_{y \in D_m} \delta_g(y) \leq \frac{\left|(\bigcup_{i=1}^{l} E_i) \cap [1,n]\right|}{g(n)} + \varepsilon_0 \leq \frac{n}{g(n)} + \varepsilon_0 \leq r_p$$

for suitably chosen ε_0. Finally, in view of the fact that $D = \bigcup_m D_m$ we get

$$\sum_{y \in D} \delta_g(y) = \lim_{m \to \infty} \sum_{y \in D_m} \delta_g(y) \leq r_p.$$

Since this is true for any r_p, letting $p \to \infty$ we get $\sum_{y \in D} \delta_g(y) \leq M$.

Lemma 1 is proved.

Note that in general, one cannot prove that $D = \{y \in \mathbb{R} \colon \delta_g(y) > 0\}$ is nonempty. Also the above lemma would not remain true if one would change $\delta_g(y)$ to $\overline{\delta_g}(y)$, that is $D_1 := \{y \in \mathbb{R} \colon \overline{\delta_g}(y) > 0\}$ need not be countable. An example in this respect is given in [3] for $g(n) = n$.

The next result extends the natural density version of the Osikiewicz theorem. We will later show that the condition $\sum_{y \in D} \delta_A(y) = M$ implies that the set of indices of (x_n) can be divided into appropriate splices. The method which we use in our proof is similar to that of Osikiewicz, but not analogous as we use essentially new arguments.

Theorem 6. *Let (x_n) be a bounded sequence and g be a weight function such that $\delta_g(y)$ exists for every $y \in \mathbb{R}$ and moreover $\sum_{y \in D} \delta_g(y) = M$. Then*

$$\lim_{n \to \infty} \frac{1}{g(n)} \sum_{i=1}^{n} x_i = \sum_{y \in D} \delta_g(y) \cdot y.$$

Proof. Since (x_n) is bounded, there exists a $K > 0$ such that $|x_n| \le K$ for every $n \in \mathbb{N}$. Let $D = \{y_i : i = 1, 2, \ldots\}$ where y_i's are distinct. Let $\varepsilon > 0$ be given and let $r \in \mathbb{N}$ be such that

$$\sum_{i=1}^{r} \delta_g(y_i) > M - \varepsilon \quad \text{and} \quad \left| \sum_{i=r+1}^{\infty} y_i \cdot \delta_g(y_i) \right| < \varepsilon.$$

Let $N \in \mathbb{N}$ be such that

$$\frac{1}{3} \min_{i \neq j} |y_i - y_j| > \frac{1}{N} \qquad \text{for all} \quad i,\ j \in 1, 2, \ldots, r$$

and such that the sets $E_i = \left\{ j : |x_j - y_i| < \dfrac{1}{N} \right\}$ have the following property:

$$\delta_g(y_i) - \frac{\varepsilon}{r(k+1)} \le \underline{\delta_g}(E_i) \le \overline{\delta_g}(E_i) \le \delta_g(y_i) + \frac{\varepsilon}{r(k+1)}$$

for $i = 1, 2, \ldots, r$. Obviously $E_1, \ldots, E_r$ are pairwise disjoint. Now we can choose an m_0 $(\in \mathbb{N})$ such that

$$\underline{\delta_g}(E_i) - \frac{1}{N} < \frac{|E_i \cap [1, n]|}{g(n)} < \overline{\delta_g}(E_i) + \frac{1}{N}$$

for every $n \ge m_0$ and $i = 1, 2, \ldots, r$. Therefore,

$$\delta_g(y_i) - \frac{1}{N} - \frac{\varepsilon}{r(k+1)} < \frac{|E_i \cap [1, n]|}{g(n)} < \delta_g(y_i) + \frac{1}{N} + \frac{\varepsilon}{r(k+1)}$$

and, consequently,

$$\left| \frac{|E_i \cap [1, n]|}{g(n)} - \delta_g(y_i) \right| < \frac{1}{N} + \frac{\varepsilon}{r(k+1)} \tag{1}$$

for every $n \ge m_0$ and $i = 1, 2, \ldots, r$. Then, for $n \ge m_0$, we have

$$\frac{1}{g(n)} \sum_{i=1}^{n} x_i \le \frac{|E_1 \cap [1, n]|}{g(n)} \left(y_1 + \frac{1}{N} \right) + \frac{|E_2 \cap [1, n]|}{g(n)} \left(y_2 + \frac{1}{N} \right) + \ldots$$

$$\ldots + \frac{|E_r \cap [1, n]|}{g(n)} \left(y_r + \frac{1}{N} \right) + K \frac{|(E_1 \cup \ldots \cup E_r)^c \cap [1, n]|}{g(n)}.$$

Now we can choose $m_1 > m_0$ such that, for all $n \ge m_1$,

$$\frac{n}{g(n)} < M + \varepsilon.$$

Then

$$M + \varepsilon > \frac{n}{g(n)} = \frac{|\bigcup_{i=1}^{r} E_i \cap [1, n]|}{g(n)} + \frac{|(\bigcup_{i=1}^{r} E_i)^c \cap [1, n]|}{g(n)}$$

and, consequently,

$$\frac{|\bigcup_{i=1}^{r} E_i \cap [1, n]|}{g(n)} = \sum_{i=1}^{r} \frac{|E_i \cap [1, n]|}{g(n)} > \sum_{i=1}^{r} \delta_g(y_i) - \frac{r}{N} - \frac{\varepsilon}{K+1}.$$

Therefore, for $n \geq m_1$, we obtain

$$\frac{|(\bigcup_{i=1}^{r} E_i)^c \cap [1,n]|}{g(n)} < (M+\varepsilon) - \left(M - \frac{r}{N} - \left(1 + \frac{1}{K+1}\right)\varepsilon\right) = \frac{r}{N} + \left(2 + \frac{1}{K+1}\right)\varepsilon.$$

Subsequently we get, for $n \geq m_1$,

$$\frac{1}{g(n)}\sum_{i=1}^{n} x_i \leq \frac{|E_1 \cap [1,n]|}{g(n)}\left(y_1 + \frac{1}{N}\right) + \frac{|E_2 \cap [1,n]|}{g(n)}\left(y_2 + \frac{1}{N}\right) + \ldots$$

$$\ldots + \frac{|E_r \cap [1,n]|}{g(n)}\left(y_r + \frac{1}{N}\right) + \frac{Kr}{n} + \left(2 + \frac{1}{K+1}\right)\varepsilon K.$$

Analogously,

$$\frac{1}{g(n)}\sum_{i=1}^{n} x_i \geq \frac{|E_1 \cap [1,n]|}{g(n)}\left(y_1 - \frac{1}{N}\right) - \frac{|E_2 \cap [1,n]|}{g(n)}\left(y_2 - \frac{1}{N}\right) + \ldots$$

$$\ldots + \frac{|E_r \cap [1,n]|}{g(n)}\left(y_r - \frac{1}{N}\right) - \frac{Kr}{n} - \left(2 + \frac{1}{K+1}\right)\varepsilon K.$$

Thus,

$$\frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{r}\frac{|E_i \cap [1,n]|}{g(n)}\left(y_i + \frac{1}{N}\right) \leq \frac{Kr}{n} + \left(2 + \frac{1}{K+1}\right)\varepsilon K \tag{2}$$

and

$$\frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{r}\frac{|E_i \cap [1,n]|}{g(n)}\left(y_i - \frac{1}{N}\right) \geq -\frac{Kr}{n} - \left(2 + \frac{1}{K+1}\right)\varepsilon K. \tag{3}$$

Hence, by using (1) and (2), we have

$$\frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{\infty}\delta_g(y_i)\cdot y_i = \frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{r}\delta_g(y_i).y_i - \sum_{i=r+1}^{\infty}\delta_g(y_i)\cdot y_i \leq$$

$$\leq \frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{r}\delta_g(y_i).y_i + \left|\sum_{i=r+1}^{\infty}\delta_g(y_i)\cdot y_i\right| \leq$$

$$\leq \frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{r}\delta_g(y_i\cdot y_i + \varepsilon =$$

$$= \left[\frac{1}{g(n)}\sum_{i=1}^{n} x_i - \sum_{i=1}^{r}\frac{|E_i \cap [1,n]|}{g(n)}\left(y_i + \frac{1}{N}\right)\right] + \sum_{i=1}^{r}\left[\frac{|E_i \cap [1,n]|}{g(n)}\left(y_i + \frac{1}{N}\right) - \delta_g(y_i)\cdot y_i\right] + \varepsilon \leq$$

$$\leq \sum_{i=1}^{r}\left[\left(\frac{|E_i \cap [1,n]|}{g(n)} - \delta_g(y_i)\right)\left(y_i + \frac{1}{N}\right)\right] + \frac{1}{N}\sum_{i+1}^{r}\delta_g(y_i) + \frac{Kr}{N} + \left(2K + \frac{K}{K+1} + 1\right)\varepsilon \leq$$

$$\leq \sum_{i=1}^{r}\left[\left|\left(\frac{|E_i \cap [1,n]|}{g(n)} - \delta_g(y_i)\right)\right|\left(|y_i| + \frac{1}{N}\right)\right] + \frac{M}{n} + \frac{Kr}{N} + \left(2K + \frac{K}{K+1} + 1\right)\varepsilon \leq$$

$$\leq r\left(\frac{1}{N}+\frac{\varepsilon}{r(K+1)}\right)\left(K+1+\frac{1}{N}\right)+\frac{M}{n}+\frac{Kr}{N}+\left(2K+\frac{K}{K+1}+1\right)\varepsilon.$$

Analogously, from (1) and (3), we get

$$\frac{1}{g(n)}\sum_{i=1}^{n}x_i-\sum_{i=1}^{\infty}\delta_g(y_i)\cdot y_i\geq$$

$$\geq -r\left(\frac{1}{N}+\frac{\varepsilon}{r(K+1)}\right)\left(K+1+\frac{1}{N}\right)-\frac{M}{n}-\frac{Kr}{N}-\left(2K+\frac{K}{K+1}+1\right)\varepsilon.$$

Since N can be chosen arbitrarily large, we obtain

$$\left|\frac{1}{g(n)}\sum_{i=1}^{n}x_i-\sum_{i=1}^{\infty}\delta_g(y_i)\cdot y_i\right|\leq\left(2K+\frac{K}{K+1}+1\right)\varepsilon$$

for every $\varepsilon>0$. Therefore, we can conclude that

$$\lim_{n\to\infty}\frac{1}{g(n)}\sum_{i=1}^{n}x_i=\sum_{i=1}^{\infty}\delta_g(y_i)\cdot y_i.$$

Theorem 6 is proved.

Next we establish the following result.

Proposition 1. *Let (x_n) be a splice over a partition $\{E_i\}$, $y_i=\lim_{\substack{n\to\infty\\ n\in E_i}}x_n$, $\delta_g(E_i)$ exists for each i and $\sum_{i=1}^{\infty}\delta_g(E_i)=M$. Then $\delta_g(y)$ exists for every $y\in\mathbb{R}$ and $\delta_g(y_i)=\delta_g(E_i)$.*

Proof. Let $\varepsilon>0$ be given. We choose $N\in\mathbb{N}$ such that $\sum_{i=1}^{N}\delta_g(E_i)>M-\varepsilon$. Let $\delta>0$ be such that the intervals $(y_i-\delta,y_i+\delta)$ are pairwise disjoint for $i=1,2,\ldots,N$. It is noted that the sets of indices $E_i\setminus\{k\colon x_k\in(y_i-\delta,y_i+\delta)\}$ are finite. So $\underline{\delta_g}(y_i)\geq\delta_g(E_i)$. On the other hand, we have

$$\overline{\delta g}(y_i)=\lim_{\eta\to 0^+}\overline{\delta_g}\left\{k\colon x_k\in(y_i-\eta,y_i+\eta)\right\}=$$

$$=M-\lim_{\eta\to 0^+}\underline{\delta_g}\left\{k\colon x_k\notin(y_i-\eta,y_i+\eta)\right\}\leq$$

$$\leq M-\underline{\delta_g}\left\{k\colon x_k\notin(y_i-\delta,y_i+\delta)\right\}\leq$$

$$\leq M-\sum_{\substack{m=1\\ m\neq i}}^{N}\delta_g(E_m)<\delta_g(E_i)+\varepsilon.$$

Thus,

$$\delta_g(y_i)=\delta_g(E_i).$$

Finally, let y be not in the set of limits $\{y_i\}$. As before for any $\varepsilon>0$ we can find N such that $\sum_{i=1}^{N}\delta_g(E_i)>M-\varepsilon$. Let η be the distance from y to the set $\{y_1,\ldots,y_N\}$. Then $\overline{\delta_g}\left(\left\{m\colon|x_m-y|<\dfrac{\eta}{2}\right\}\right)<\varepsilon$ and, consequently, $\delta_g(y)=0$.

Proposition 1 is proved.

Next we establish the following result which not only forms the basis of a necessary condition for the existence of the limit $\lim_{n\to\infty} \dfrac{1}{g(n)} \sum_{i=1}^{n} x_i$, but at the same time is an interesting observation.

Proposition 2. *Suppose* $x = (x_n) \in l^\infty$. *If* $\delta_g(y) = M$, *then* My *is a limit point of the sequence* $\lim_{n\to\infty} \dfrac{1}{g(n)} \sum_{i=1}^{n} x_i$.

Proof. Since (x_n) is bounded, there is $K > 0$ such that $|x_n| \le K$ for all $N \in \mathbb{N}$. Let y be such that $\overline{\delta_g}(y) = M$. Also let $N \in \mathbb{N}$ and let $E_N = \left\{ j \in \mathbb{N} \colon |x_j - y| < \dfrac{1}{N} \right\}$. Then there exists $k_N \ge N$ such that

$$\frac{|E_N \cap [1, k_N]|}{g(k_N)} > \overline{\delta_g}(E_N) - \frac{1}{N} = M - \frac{1}{N}.$$

Again as we have $\lim_{n\to\infty} \dfrac{n}{g(n)} = M$, we get

$$\frac{k_N}{g(k_N)} < M + \frac{1}{N}.$$

Since $y - \dfrac{1}{N} < x_k < y + \dfrac{1}{N}$ for all $x_k \in E_N$ and also $-K \le x_k \le K$ for each $x_k \notin E_N$, so we have

$$\frac{|E_N \cap [1, k_N]|}{g(k_N)} \left(y - \frac{1}{N} \right) - \frac{|E_N^c \cap [1, k_N]|}{g(k_N)} K \le \frac{1}{g(k_N)} \sum_{i=1}^{k_N} x_i \le$$

$$\le \frac{|E_N \cap [1, k_N]|}{g(k_N)} \left(y + \frac{1}{N} \right) + \frac{|E_N^c \cap [1, k_N]|}{g(k_N)} K.$$

Thus,

$$y \left(\frac{k_N}{g(k_N)} - M \right) - \frac{1}{N} \frac{|E_N \cap [1, k_N]|}{g(k_N)} - \frac{|E_N^c \cap [1, k_N]|}{g(k_N)} (K + y) \le$$

$$\le \frac{1}{g(k_N)} \sum_{i=1}^{k_N} x_i - My \le y \left(\frac{k_N}{g(k_N)} - M \right) + \frac{1}{N} \frac{|E_N \cap [1, k_N]|}{g(k_N)} + \frac{|E_N^c \cap [1, k_N]|}{g(k_N)} (K - y)$$

and, consequently,

$$\left| \frac{1}{g(k_N)} \sum_{i=1}^{k_N} x_i - My \right| \le \left| \frac{|E_N \cap [1, k_N]|}{g(k_N)} \frac{1}{N} \right| + \left| \frac{|E_N^c \cap [1, k_N]|}{g(k_N)} (K + |y|) \right| + \frac{1}{N} |y|.$$

Since

$$\frac{|E_N^c \cap [1, k_N]|}{g(k_N)} = \frac{k_N}{g(k_N)} - \frac{|E_N \cap [1, k_N]|}{g(k_N)} < M + \frac{1}{N} - \left(M - \frac{1}{N} \right) = \frac{2}{N},$$

we obtain

$$\left| \frac{1}{g(k_N)} \sum_{i=1}^{k_N} x_i - My \right| \le \left(\frac{M}{N} + \frac{1}{N^2} \right) + \frac{|E_N^c \cap [1, k_N]|}{g(k_N)} (K + |y|) + \frac{1}{N} |y| \le$$

$$\leq \left(\frac{M}{N}+\frac{1}{N^2}\right)+\frac{2}{N}(K+|y|)+\frac{1}{N}|y|.$$

Therefore,

$$\lim_{N\to\infty}\frac{1}{g(k_N)}\sum_{i=1}^{k_N}x_i = My.$$

Proposition 2 is proved.

***Corollary* 1.** *Let (x_n) be a bounded sequence. Suppose that there are y and z $(y\neq z)$ with $\delta_g(y)=\delta_g(z)=M$. Then the limit $\lim_{n\to\infty}\dfrac{1}{g(n)}\sum_{i=1}^{n}x_i$ does not exist.*

One should note that Corollary 1 cannot be weakened by assuming $\overline{\delta_g}(y),\ \overline{\delta_g}(z)>r$ for some $r\in(0,M)$. A counter example is given in Proposition 9 in [3] for $g(n)=n$.

Now we recall some important results from [3] which will be useful in the sequel.

Lemma 2 [3]**.** *Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$. Assume that $X:=\{n\colon x_n\in[a,b]\}\notin\mathcal{I}$. Suppose that*

$$\{n\colon a\leq x_n\leq t-\varepsilon\}\in\mathcal{I} \text{ or } \{n\colon t+\varepsilon\leq x_n\leq b\}\in\mathcal{I}$$

for any $t\in(a,b)$ and any $\varepsilon>0$ such that $\varepsilon<\min\{t-a,b-t\}$. Then there is $y\in[a,b]$ such that $\{n\colon |x_n-y|\geq\varepsilon\}\in\mathcal{I}$ for every $\varepsilon>0$.

Proposition 3 [3]**.** *Let $\mathcal{I}$ be a P-ideal. Assume that $(x_n)\in\ell^\infty$ does not have any $\mathcal{I}$-limit points. Then the set of limit points of (x_n), i.e., the set*

$$\{y\in\mathbb{R}\colon x_{n_k}\to y \text{ for some increasing sequence } (n_k) \text{ of natural numbers}\},$$

is uncountable and closed.

***Corollary* 2** [3]**.** *Let $[a,b]$ be a fixed interval and $\mathcal{I}$ be a P-ideal. Assume that $\{n\colon x_n\in\in[a,b]\}\notin\mathcal{I}$ and any point $y\in(a,b)$ is not an $\mathcal{I}$-limit point of (x_n). Then the set of limit points of (x_n) in $[a,b]$, i.e., the set*

$$\{y\in(a,b)\colon x_{n_k}\to y \text{ for some increasing sequence } (n_k) \text{ of natural numbers}\},$$

is uncountable and closed.

***Corollary* 3** [3]**.** *Let $(x_n)\in\ell^\infty$. Assume that the set of limit points of (x_n) is countable. Then the sequence (x_n) has at least one $\mathcal{I}$-limit point for every P-ideal $\mathcal{I}$.*

Now we prove certain results analogous to the results of [3] which will help us to reach our final goal.

Lemma 3. *Let $r\in(0,1)$, $r_1\geq r_2\geq\ldots$, $\lim_{n\to\infty}r_n=r$ and (E_n) be a decreasing sequence of subsets of $\mathbb{N}$.*

(i) *If $\underline{\delta_g}(E_n)=r_n, n\in\mathbb{N}$, then there is a subset E of $\mathbb{N}$ with $\underline{\delta_g}(E)=r$ and such that $E_n\setminus E$ is finite for all n. Moreover, if $\overline{\delta_g}(E_n)\to r$, then $\delta_g(E)=r$.*

(ii) *If $\overline{\delta_g}(E_n)=r_n, n\in\mathbb{N}$, then there is a subset E of $\mathbb{N}$ with $\overline{\delta_g}(E)=r$ and such that $E_n\setminus E$ is finite for all n.*

Proof. (i) Let (p_n) be an increasing sequence of natural numbers such that $\dfrac{|E_n \cap [1,j]|}{g(j)} > > r_n - \dfrac{1}{3n}$ for every $j \geq p_n$. For each $n \in \mathbb{N}$ now choose $m_n > p_n$ such that

$$\frac{\big|(E_n \cap [1,m_n]) \cap [1,j]\big|}{g(j)} > r_n - \frac{1}{3n} - \frac{1}{3n} > r_n - \frac{1}{n}$$

for all j, $p_n \leq j \leq p_{n+1}$. Thus, we have two increasing sequences of natural numbers (p_n) and (m_n) such that, for all $j \in [p_n, p_{n+1}]$,

$$\frac{\big|(E_n \cap [1,m_n]) \cap [1,j]\big|}{g(j)} > r_n - \frac{1}{n}.$$

Put $E = \bigcup_{n=1}^{\infty} E_n \cap [1, m_{n+1}]$. Take $p_n \leq j < p_{n+1}$. Then

$$\frac{|E \cap [1,j]|}{g(j)} \geq \frac{\big|(E_n \cap [1,m_n]) \cap [1,j]\big|}{g(j)} > r_n - \frac{1}{n}.$$

Thus, $\liminf_{n\to\infty} \dfrac{|E \cap [1,n]|}{g(n)} \geq r$, which means that $\underline{\delta_g}(E) \geq r$. Since $E_1 \supset E_2 \supset \ldots$, so $\bigcup_{n=1}^{\infty} E_n \cap [1, m_{n+1}] \subset E_j$ and $E_j \setminus E \subset \bigcup_{n=1}^{j-1} E_n \cap [1, m_{n+1}]$. Therefore, $E_j \setminus E$ is finite, and, consequently, $\underline{\delta_g}(E) \leq \underline{\delta_g}(E_j)$ and $\overline{\delta_g}(E) \leq \overline{\delta_g}(E_j)$. Hence, $\underline{\delta_g}(E) = r$ and if $\overline{\delta_g}(E_n) \to r$, then $\delta_g(E) = r$.

(ii) As before we can choose two increasing sequences of natural numbers (p_n) and (m_n) such that

$$\frac{\big|(E_n \cap [1,m_n]) \cap [1,p_n]\big|}{g(p_n)} > r_n - \frac{1}{n}$$

for every n. Put $E = \bigcup_{n=1}^{\infty} E_n \cap [1, m_{n+1}]$. Then

$$\frac{|E \cap [1,p_{n+1}]|}{g(p_{n+1})} \geq \frac{\big|(E_n \cap [1,m_{n+1}]) \cap [1,p_{n+1}]\big|}{g(p_{n+1})} \geq r_{n+1} - \frac{1}{n+1}.$$

Thus, $\overline{\delta_g}(E) \geq r$. Since $E_n \setminus E$ is finite for all n, it now readily follows that $\overline{\delta_g}(E) = r$.

Lemma 3 is proved.

Theorem 7. *As before let $\mathcal{I}_g = \{A \subset \mathbb{N} : \delta_g(A) = 0\}$. Let $(x_n) \in \ell^\infty$. A point $y \in \mathbb{R}$ is an $\mathcal{I}_g$-limit point of (x_n) if and only if $\overline{\delta_g}(y) > 0$. Moreover, if $\delta_g(y) > 0$, then there is $E \subset \mathbb{N}$ with $\delta_g(E) = \delta_g(y)$ and $\lim_{n\in E} x_n = y$.*

Proof. Assume that $\overline{\delta_g}(y) = 0$ and suppose y is an $\mathcal{I}_g$-limit point of (x_n). Then there is $E \subset \mathbb{N}$ such that $\overline{\delta_g}(E) > 0$ and $\lim_{n\in E} x_n = y$. Note that $\big\{j : |x_j - y| \leq \varepsilon\big\} \setminus E$ is finite for all $\varepsilon > 0$. Hence, $\overline{\delta_g}(E) \leq \overline{\delta_g}\big(\{j : |x_j - y| \leq \varepsilon\}\big)$ for every $\varepsilon > 0$. Therefore, $\overline{\delta_g}(E) = 0$ which is a contradiction.

Conversely, let $\overline{\delta_g}(y) > 0$. Let $E_n = \left\{j : |x_j - y| \leq \dfrac{1}{n}\right\}$. Then (E_n) is a decreasing sequence with $\overline{\delta_g}(E_n) \to \overline{\delta_g}(y)$. By Lemma 3 there is E such that $E_n \setminus E$ is finite for all n with $\overline{\delta_g}(E) = = \overline{\delta_g}(y)$. Since almost all elements of E are contained in E_n, clearly $\lim_{\substack{j\to\infty\\ j\in E}} x_j = y$. Hence, y is an $\mathcal{I}_g$-limit point of (x_n).

The last part of the assertion follows in a similar way.

Theorem 7 is proved.

***Corollary* 4.** *Let $(x_n) \in \ell^\infty$. A point $y \in \mathbb{R}$ is an $\mathcal{I}_g$-cluster point of (x_n) and it is not an $\mathcal{I}_g$-limit point if and only if*:

(i) $\overline{\delta_g}\left(\left\{j : |x_j - y| \le \dfrac{1}{n}\right\}\right) > 0$ *for every* n,

(ii) $\delta_g(y) = \lim\limits_{n\to\infty} \overline{\delta_g}\left(\left\{j : |x_j - y| \le \dfrac{1}{n}\right\}\right) = 0.$

Proposition 4. *Let (x_n) be a bounded sequence. Assume that $y_1, y_2, \ldots$ are the only distinct real numbers such that $\delta_g(y_i) > 0$ for all i. Then there exists a partition $E_1, E_2, \ldots$ such that $\delta_g(E_i) = \delta_g(y_i)$ for all i and $\lim_{n\in E_i} x_n = y_i$.*

Proof. By Theorem 7 there are $E_1', E_2', \ldots$ with $\lim_{n\in E_i'} x_n = y_i$. Note that $E_i' \cap E_j'$ is finite if $i \ne j$. Define $E_1, E_2, \ldots$ in the following way. Let $E_1'' = E_1', E_m'' = E_m' \setminus \bigcup_{i=1}^{m-1} E_i'$ for $m \ge 2$. Since $E_m' \cap \bigcup_{i=1}^{m-1} E_i'$ is finite, so $\delta_g(E_m'') = \delta_g(E_m') = \delta_g(y_m)$ for $m \in \mathbb{N}$. Let $E = \mathbb{N} \setminus \bigcup_{m=1}^{\infty} E_m'$. If E is finite, then put $E_1 = E \cup E_1''$ and $E_m = E \cup E_m''$ for $m \ge 2$. If the set E is infinite, then enumerate it as $\{n_1, n_2, \ldots\}$ and put $E_m = E_m'' \cup \{n_m\}$. Clearly $\lim_{n\in E_m} x_n = y_m$.

Proposition 4 is proved.

Proposition 5. *Let $\{E_n : n = 1, 2, \ldots\}$ be a partition on $\mathbb{N}$ such that $\sum_{n=1}^{\infty} \delta_g(E_n) <$ $< \limsup\limits_n \dfrac{n}{g(n)} = M$ (say). Then there is a partition $\{F_n : n = 0, 1, 2, \ldots\}$ of $\mathbb{N}$ such that*

(i) $F_n \subset E_n$,

(ii) $\delta_g(F_n) = \delta_g(E_n)$ *for all* n,

(iii) $\delta_g(F_0) = M - \sum_{n=1}^{\infty} \delta_g(E_n)$.

Proof. Let (ε_n) be a strictly decreasing sequence of positive real numbers converging to 0. We have $\delta_g(E_1) = \limsup\limits_n \left[\dfrac{|E_1 \cap [1, n]|}{g(n)}\right]$. Furthermore, $\delta_g(E_1^c) = M - \delta_g(E_1)$. So

$$\limsup_n \frac{|E_1^c \cap [1, n]|}{g(n)} = M - \delta_g(E_1) \Rightarrow \frac{|E_1^c \cap [1, n]|}{g(n)} \ge M - \delta_g(E_1) + \varepsilon_1$$

for all $n \ge N_1$(say). Since $\dfrac{1}{g(n)} \to 0$ as $n \to \infty$, we can choose $N_2 \in \mathbb{N}$ large enough such that $\dfrac{|E_1^c \cap \{1\}|}{g(n)} < 2\varepsilon_1$ for all $n \ge N_2$. Let $m_1 = \max\{N_1, N_2\}$, and we set $m_0 = 1$. Then

$$\frac{|E_1^c \cap [1, n]|}{g(n)} - \frac{|E_1^c \cap \{1\}|}{g(n)} \ge M - \delta_g(E_1) - \varepsilon_1$$

for each $n \ge m_1$ and

$$\frac{|[m_0, j] \setminus E_1|}{g(j)} \ge M - \delta_g(E_1) - \varepsilon_1$$

for all $j \ge m_1$. Similarly, we have

$$\delta_g\big[(E_1 \cup E_2)^c\big] = M - \delta_g(E_1 \cup E_2) = M - \delta_g(E_1) - \delta_g(E_2),$$

i.e.,

$$\limsup_n \left[\frac{\big|(E_1 \cup E_2)^c \cap [1, n]\big|}{g(n)}\right] = M - \delta_g(E_1) - \delta_g(E_2) \Rightarrow$$

$$\Rightarrow \frac{\big|(E_1 \cup E_2)^c \cap [1,n]\big|}{g(n)} \geq M - (\delta_g(E_1) + \delta_g(E_2)) + \varepsilon_2$$

for all $n \geq K_1$(say). Now we choose a large $K_2 > N_2$ such that

$$\frac{\big|(E_1 \cup E_2)^c \cap [1,m_1]\big|}{g(n)} < 2\varepsilon_2$$

for all $n \geq K_2$. Let $m_2 = \max\{K_1, K_2\}$. Then

$$\frac{\big|(E_1 \cup E_2)^c \cap [1,n]\big|}{g(n)} - \frac{\big|(E_1 \cup E_2)^c \cap [1,m_1]\big|}{g(n)} \geq M - (\delta_g(E_1) + \delta_g(E_2)) - \varepsilon_2$$

whenever $n \geq m_2$ and this implies that

$$\frac{\big|[m_1, j] \setminus (E_1 \cup E_2)\big|}{g(j)} \geq M - (\delta_g(E_1) + \delta_g(E_2)) - \varepsilon_2$$

for all $j \geq m_2$.

Inductively we can define an increasing sequence $(m_n : n = 0, 1, 2, \ldots)$ of natural numbers such that, for all $n = 1, 2, \ldots,$

$$\frac{\big|[m_{n-1}, j] \setminus (E_1 \cup E_2 \cup \ldots \cup E_n)\big|}{g(j)} \geq M - \sum_{i=1}^{n} \delta_g(E_i) - \varepsilon_n$$

whenever $j \geq m_n$. Now let $F_0 = \bigcup_{n=1}^{\infty}([m_{n-1}, m_{n+1}] \setminus \bigcup_{i=1}^{n} E_i)$. Then, for $m_n \leq j \leq m_{n+1}$, we have

$$\frac{|F_0 \cap [1,j]|}{g(j)} \geq \frac{\big|[m_{n-1}, j] \setminus (E_1 \cup E_2 \cup \ldots \cup E_n)\big|}{g(j)} \geq M - \sum_{i=1}^{n} \delta_g(E_i) - \varepsilon_n$$

for every n. So

$$\underline{\delta_g}(F_0) = \liminf_{j} \frac{|(F_0 \cap [1,j]|}{g(j)} \geq M - \sum_{i=1}^{\infty} \delta_g(E_i) \tag{4}$$

as $\varepsilon_n \to 0$.

Now let $F_n = E_n \setminus F_0$, $n = 1, 2, \ldots$. Then $F_0 \cap E_n$ is finite for all n (note that $F_0 \cap E_1 = = \varphi,\ F_0 \cap E_2 \subset [1, m_1],\ F_0 \cap E_3 \subset [1, m_2], \ldots$). So $\delta_g(F_n) = \delta_g(E_n)$ for all n and we obtain

$$\bigcup\nolimits_{n=1}^{\infty} F_n = \bigcup\nolimits_{n=1}^{\infty} (E_n \setminus F_0) = \bigcup\nolimits_{n=1}^{\infty} E_n \setminus F_0 = \mathbb{N} \setminus F_0 \Rightarrow F_0 = \mathbb{N} \setminus \bigcup\nolimits_{n=1}^{\infty} F_n.$$

Hence,

$$\overline{\delta_g}(F_0) = M - \underline{\delta_g}(F_0^c) = M - \underline{\delta_g}\left(\bigcup\nolimits_{n=1}^{\infty} F_n\right) \leq M - \sum_{n=1}^{\infty} \underline{\delta_g}(F_n) = M - \sum_{n=1}^{\infty} \delta_g(E_n). \tag{5}$$

Combining (4) and (5), we finally observe that

$$\delta_g(F_0) = M - \sum_{n=1}^{\infty} \delta_g(E_n).$$

Proposition 5 is proved.

Finally, we prove a sufficient condition for a bounded sequence (x_n) to have the property that $\sum_{y \in \mathbb{R}} \delta_g(y) = M$.

Theorem 8. *Let (x_n) be a bounded sequence. Suppose that the set of limit points of (x_n) is countable and $\delta_g(y)$ exists for all $y \in \mathbb{R}$. Then* $\sum_{y\in\mathbb{R}} \delta_g(y) = M$.

Proof. Let $D := \{y \in \mathbb{R} \colon \delta_g(y) > 0\}$. If possible let $\sum_{y\in D} \delta_g(y) < M$. As the number of limit points is countable, we are in a position to use Corollary 16 [3].

Let y be an $\mathcal{I}_g$-limit point of (x_n). Then there exists $B \subset \mathbb{N}$, $\delta_g(B) > 0$ such that $\lim_{n\in B} x_n = y$. So for any $\varepsilon > 0$, $\{n \colon |x_n - y| \le \varepsilon\} \supseteq B \setminus B_0$ where $B_0 \subset \mathbb{N}$ is finite. Observe that

$$\overline{\delta_g}\{n \colon |x_n - y| \le \varepsilon\} \ge \delta_g(B) \Rightarrow \lim_{\varepsilon\to 0^+} \left[\overline{\delta_g}\{n \colon |x_n - y| \le \varepsilon\}\right] \ge \delta_g(B) \Rightarrow \delta_g(y) \ge \delta_g(B) > 0.$$

So $D \neq \varnothing$. Now from Lemma 1 it follows that that D is countable. We enumerate D as $\{y_1, y_2, \ldots\}$. By Proposition 4 there is a partition $\{E_1, E_2, \ldots\}$ of $\mathbb{N}$ such that $\delta_g(E_k) = \delta_g(y_k)$ and $\lim_{\substack{n\to\infty \\ n\in E_k}} = y_k$. Again applying Proposition 5 we get that there is a partition $\{F_0, F_1, F_2, \ldots\}$ of $\mathbb{N}$ such that $F_k \subset E_k$, $\delta_g(F_k) = \delta_g(E_k)$ for $k = 1, 2, \ldots$; $F_0 = \mathbb{N}\setminus\bigcup_{i=1}^{\infty} F_i$ so that $\delta_g(F_0) = M - \sum_{k=1}^{\infty} \delta_g(F_k)$. Obviously, $\delta_g(F_0) > 0$.

Now we consider the sequence $(x_n)_{n\in F_0}$ and the ideal $\mathcal{I}_{g|F_0} = \{E \subset F_0 \colon E \in \mathcal{I}_g\}$. Since $\delta_g(y) = 0$ for all $y \notin D$, so by Theorem 7 y cannot be an $\mathcal{I}_g$-limit of $(x_n)_{n\in F_0}$. Consequently, y cannot be an $\mathcal{I}_{g|F_0}$-limit point of $(x_n)_{n\in F_0}$. Now if any y_i is an $\mathcal{I}_{g|F_0}$-limit point of $(x_n)_{n\in F_0}$, then there would be a set $B \subset \mathbb{N}$, $B \subset F_0$ such that $B \notin \mathcal{I}_{g|F_0}$ and $\lim_{n\in B} x_n = y_i$. Now $B \subset F_0$ and $B \notin \mathcal{I}_{g|F_0}$ implies $B \notin \mathcal{I}_g$. Again $B \subset F_0$ implies $B \cap F_i = \varnothing$ for all $i = 1, 2, \ldots$. So $\lim_{n\to\infty} x_n \in B \cup F_i = y_i$ for all i. Consequently,

$$\delta_g(y_i) = \overline{\delta_g}(y_i) = \lim_{\varepsilon\to 0^+} \left[\overline{\delta_g}\{n \colon |x_n - y_i| \le \varepsilon\}\right] \ge \overline{\delta_g}(B \cup F_i) > \overline{\delta_g}(F_i) = \delta_g(y_i)$$

which is a contradiction. So no y_i is an $\mathcal{I}_{g|F_0}$-limit point of $(x_n)_{n\in F_0}$, i.e.,$(x_n)_{n\in F_0}$ has no $\mathcal{I}_{g|F_0}$-limit point.

Now to verify that the ideal $\mathcal{I}_{g|F_0}$ is a P-ideal let $A_1, A_2, \ldots \in \mathcal{I}_{g|F_0}$. Then $A_1, A_2, \ldots \in \mathcal{I}_g$. As $\mathcal{I}_g$ is a P-ideal, we can get $A_\infty \in \mathcal{I}_g$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$. Now $A_\infty \cap F_0 \in \mathcal{I}_{g|F_0}$ and $A_n \subset F_0$ for all n implies $A_n \setminus (A_\infty \cap F_0)$ is finite for all n. So $\mathcal{I}_{g|F_0}$ is a P-ideal such that $(x_n)_{n\in F_0}$ has no $\mathcal{I}_{g|F_0}$-limit point. Hence, the set of limit points of $(x_n)_{n\in F_0}$ must be uncountable (see Proposition 14 [3]), i.e., (x_n) will have uncountably many limit points which contradicts the assumption of the statement. Hence, it follows that $\sum_{y\in\mathbb{R}} \delta_g(y) = M$.

Theorem 8 is proved.

Finally, combining Theorem 6 with Theorem 8, we get the desired proof of our main result.

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