UDC 517.9

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ON A FRANKL-TYPE BOUNDARY-VALUE PROBLEM FOR A MIXED-TYPE DEGENERATING EQUATION

ПРО ГРАНИЧНУ ЗАДАЧУ ТИПУ ФРАНКЛЯ ДЛЯ РІВНЯННЯ МІШАНОГО ТИПУ, ЩО ВИРОДЖУЄТЬСЯ

We investigate the existence and uniqueness of solutions for an analog of the Frankl-type boundary-value problem for a parabolic-hyperbolic-type equation. The uniqueness of solution is proved by using the extreme principle and the existence is proved by the method of integral equations.

Вивчається задача про існування та єдиність розв'язків для аналога граничної задачі типу Франкля для рівняння параболічно-гіперболічного типу. Єдиність розв'язку доведено за допомогою принципу екстремуму, а його існування — за допомогою методу інтегральних рівнянь.

1. Introduction. The first time by I. M. Gelfand [6] was offered to study of boundary-value problems for a parabolic-hyperbolic-type equations. In the sequel it appeared a number of works devoted to several local and nonlocal boundary-value problems for the degenerating hyperbolic (see [8, 23]) equation $L\,u \equiv (-y)^m u_{xx} - x^n u_{yy} = 0$, $y < 0$, m, $n > 0$, and degenerating parabolic-hyperbolic-type equation (see [2, 10]):

$$0 = \begin{cases} y^{m_1} u_{xx} - x^{n_1} u_y, & x > 0, \\ y^{m_2} u_{xx} - (-x)^{n_2} u_{yy}, & x < 0, \quad m_k, n_k = \text{const}, \quad k = 1, 2. \end{cases}$$

Since A. V. Bitsadze's works, in the theory of partial differential equations there was a new direction, in which the Frankl problem for the first time was formulated and investigated for the modeling equations of the mixed type. In works [4, 5], the Frankl problem was discussed for the special mixed-type equation of second order $u_{xx} + \operatorname{sign} y\, u_{yy} = 0$. The Frankl problem for the mixed equation with parabolic degeneracy $\operatorname{sign}(y)|y|^m u_{xx} + u_{yy} = 0$ is a mathematical model of problem of gas dynamic, was discussed by M. M. Smirnov [25]. Existence of solution of Frankl problem for general Lavrent'ev – Bitsadze equations was proved in work of Guo-chun Wen and H. Begehr [7]. The basic review of boundary-value problems for the mixed-type equations with Frankl condition it is possible to find in the work J. M. Rassias [22]. Later, in works [1, 19, 20, 23, 24], for the mixed equations with one and two lines of degenerating was investigated several local and nonlocal problems with Frankl-type conditions. In a series of papers (see [12 – 14]) the authors considered some classes of boundary-value problems for the degenerating parabolic-hyperbolic-type equations involving Riemann – Liouville fractional derivatives.

2. Initial necessary dates. ***2.1. Riemann – Liouville integral-differential operator.***

Definition 1. *Let $f(x)$ be an absolutely continuous function over (a, b). Then the left and right Riemann – Liouville fractional integrals of order α $(\alpha \in R^+)$ are, respectively* (*see* [15, *p.* 69]),

$$\left(I^{\alpha}_{a+}f\right)x=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{x}f(t)(x-t)^{\alpha-1}dt,\quad x>a, \tag{1}$$

$$\left(I^{\alpha}_{-b}f\right)x=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{b}f(t)(x-t)^{\alpha-1}dt,\quad x<b. \tag{2}$$

The Riemann–Liouville fractional derivatives ${}_{RL}D^{\alpha}_{ax}f$ and ${}_{RL}D^{\alpha}_{xb}f$ of order $\alpha,\ \alpha\in R^{+}$, are defined by (see [15, p. 26])

$$\left({}_{RL}D^{\alpha}_{ax}f\right)x=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int\limits_{a}^{x}\frac{f(t)}{(x-t)^{\alpha-n+1}}dt,\qquad n=[\alpha]+1,\quad x>a, \tag{3}$$

$$\left({}_{RL}D^{\alpha}_{xb}f\right)x=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int\limits_{x}^{b}\frac{f(t)}{(t-x)^{\alpha-n+1}}dt,\qquad n=[\alpha]+1,\quad x<b, \tag{4}$$

respectively, where $[\alpha]$ is the integer part of α. In particular, for $\alpha=N\cup\{0\}$, we have

$$\left({}_{RL}D^{0}_{ax}f\right)x=f(x),\quad \left({}_{RL}D^{0}_{xb}f\right)x=f(x),\quad \left({}_{RL}D^{n}_{ax}f\right)x=f^{(n)}(x),$$

$$\left({}_{RL}D^{n}_{xb}f\right)x=(-1)^{n}f^{(n)}(x),\quad n\in N,$$

where $f^{(n)}(x)$ is the usual derivative of $f(x)$ of order n.

Lemma 1 (An extreme principle for the fractional derivative operations)**.** *Let positive not decreasing function $w(t)$ and a function $f(t)$ be continuous in $[a,b]$. If the function $f(t)$ reaches the positive maximum (a negative minimum) in the segment $[a,b]$ on the point $t=x,\ a<x<b$, and in as much as small vicinity of this point derivative of function $w(t)f(t)$ satisfy Gölder condition with an indicator $\gamma>\alpha$, then ${}_{RL}D^{\alpha}_{ax}wf>0$ $({}_{RL}D^{\alpha}_{xb}wf<0)$. The similar remark takes place for the operator ${}_{RL}D^{\alpha}_{xb}$ if $w(t)$ is a positive not increasing function on the $[a,b]$.*

2.2. Gauss hypergeometric function. Gauss hypergeometric function $F(a,b,c,z)$ is defined in the unit desk as the sum of the hypergeometric series (see [15, p. 27])

$$F(a,b,c,z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{5}$$

where $|z|<1,\ a,\ b\in\mathbb{C}\ c\in\mathbb{C}\backslash Z^{-}_{0}$ and $(a)_0=1,\ (a)_n=a(a+1)\ldots(a+n-1)=\dfrac{\Gamma(a)}{\Gamma(a+n)}$, $n=1,2,\ldots.$

One such analytic continuation is given by Euler integral representation:

$$F(a,b,c,z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int\limits_{0}^{1}x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx, \tag{6}$$

$$0<\operatorname{Re}b<\operatorname{Re}c,\quad |\arg(1-z)|<\pi.$$

The Gauss hypergeometric function $F(a,b,c,z)$ allows the following estimation:

$$F(a,b,c,z) \leq \begin{cases} c_1, & \text{if} \quad c-a-b>0, \quad 0 \leq z \leq 1, \\ c_2(1-z)^{c-a-b}, & \text{if} \quad c-a-b<0, \quad 0<z<1, \\ (1+|\ln(1-z)|)\, c_3, & \text{if} \quad c-a-b=0, \end{cases} \tag{7}$$

$$F(a,1-a,c,z) = (1-z)^{c-1} F\left(\frac{c-a}{2}, \frac{c+a-1}{2}, c, 4z(1-z)\right). \tag{8}$$

Generalized fractional integro-differential operators with Gauss hypergeometric function $F(a,b,c,z)$ defined for real $a,\ b,\ c,$ and $x>0$ be given by formula

$$F_{ox}\begin{bmatrix} a, & b \\ c, & x^k \end{bmatrix} f(x) = \frac{k}{\Gamma(c)} \int\limits_0^x \frac{f(t)t^{k-1}}{\left(x^k-t^k\right)^{1-c}} F\left(a,b,c,\frac{x^k-t^k}{x^k}\right) dt, \tag{9}$$

where $c>0,\ k>0$.

3. Formulation of the problem $\boldsymbol{TF}$. In the present research work we consider the parabolic-hyperbolic-type equation

$$0 = \begin{cases} y^{m_0} u_{xx} - x^{n_0} u_y & \text{in} \quad \Omega_1\,, \\ (-y)^{m_1} u_{xx} - x^{n_1} u_{yy} & \text{in} \quad \Omega_2, \end{cases} \tag{10}$$

where $m_i, n_i = \text{const} > 0,\ i = 0,\ 1,$ in the rectangular domain $\Omega_1 = \{(x,y):\ 0<x<h_1,\ 0< y<h_2\}$ and Ω_2 is characteristic triangle in the lower half plane $y<0$ restricted with segment AB (on the line $y=0$) and with two characteristics $AC:\ \dfrac{1}{q_1}x^{q_1} - \dfrac{1}{p_1}(-y)^{p_1} = 0$ and $BC:\ \dfrac{1}{q_1}x^{q_1} + \dfrac{1}{p_1}(-y)^{p_1} = 1$ of Eq. (10), and by the segment $(0,h_1)$ of the line $y=0$ with $A\,(0,0)$, $B\,(h_1,0)$ and $C\left(\left(\dfrac{q_1}{2}\right)^{1/q_1}, -\left(\dfrac{p_1}{2}\right)^{1/p_1}\right)$. Here $2p_1 = m_1+2$, $2q_1 = n_1+2$, $h_1 = q_1^{\,1/q_1}$, $h_2>0$, and $m_1>n_1$.

Let Ω be the union of Ω_1, the segment $J=(0,h_1)$ and $\Omega_2:\ \Omega = \Omega_1 \cup \Omega_2 \cup J$. Introduce designations $\alpha_0 = (n_0+1)/(n_1+2)$, $2\alpha_1 = n_1/(n_1+2)$, $2\beta_1 = m_1/(m_1+2)$,

$$0<\alpha_1<\beta_1<\frac{1}{2}, \qquad \frac{1}{2}<\alpha_0<1. \tag{11}$$

We assume that C_1 and C_2 are crossed points of the characteristics AC and BC with characteristics by leaving from a point $E(\kappa_1,0) \in AE$, respectively. Function $\sigma(x) \in C^2\,[0,\,\kappa_1]$ is a map which displays set of points of the segment $[0,\ \kappa_1]$ to set of points of the segment $[\kappa_1,\ 1]$, and $\sigma'(x)<0$, $\sigma(0)=1$, $\sigma(\kappa_1)=\kappa_1$. Example of such function is $\sigma(x) = 1-k_0\,x$, $k_0 = (1-\kappa_1)/\kappa_1$.

Problem $\boldsymbol{TF}$. To find a function $u(x,y)$ with following properties:

1) $u(x,y) \in C(\bar{\Omega})$;

2) $u(x,y) \in C^{2,1}_{x,y}\,(\Omega_1 \cup A_0B_0 \cup AB) \cap C^2\,(\Omega_2 \backslash (EC_1 \cup EC_2)$ satisfies Eq. (10) in domains Ω_1 and $\Omega_2 \backslash\,(EC_1 \cup EC_2)$;

3) $u(x,y)$ satisfies the boundary conditions

$$u|_{AA_0} = \varphi_1\,(y)\,, \quad u|_{BB_0} = \varphi_2\,(y)\,, \quad 0 \leq y \leq h_2, \tag{12}$$

$$u|_{AC_1} = \psi_1(x), \quad 0 \le x \le \left(\frac{q_1}{2}\kappa\right)^{1/q_1}, \tag{13}$$

$$\mu\, u(x,0) - u(\sigma(x),0) = \psi_2(x), \quad 0 \le x \le k_1; \tag{14}$$

4) $u_y \in C\left(\Omega_2 \cup AE \cup EB\right)$, $y^{-m_0}u_y \in C\left(\Omega_1 \cup AE \cup EB\right)$ and on the intervals AE and EB satisfy gluing condition

$$\lim_{y\to -0} \frac{\partial u\,(x,y)}{\partial y} = \lim_{y\to +0} y^{-m_0} \frac{\partial u\,(x,y)}{\partial y}, \quad x \in (0,k_1) \cup (k_1,h_1), \tag{15}$$

where $\varphi_j(y),\ \psi_j(x),\ j=1,2$, are given functions and $\psi_1(0)=\varphi_1(0)$,

$$\varphi_1(y), \varphi_2(y) \in C\left[0,h_2\right] \cap C^1\left(0,h_2\right), \tag{16}$$

$$\psi_1(x) \in C^3 \left[0; \left(\frac{q_1}{2}\kappa\right)^{1/q_1}\right], \tag{17}$$

$$\psi_2(x) \in C\left[0,k_1\right] \cap C^2\left(0,k_1\right). \tag{18}$$

Now, we recall the following designations which will play an important role in investigation of this problem. Let $\Omega_{21}\,[\Omega_{22}]$ be domain, restricted with characteristics $AC_1,\ EC_1\ [BC_2,\ EC_2]$ of Eq. (10) at $x>0,\ y<0$ and segment $AE\ [EB]$, $\Omega_{23} = \Omega_2 \backslash \left(\Omega_{21} \cup \Omega_{22}\right)$.

3.1. Reduction of main functional relations. It is known that functional relation between $\tau(x)$ and $\nu(x)$ transferred from the parabolic part Ω_1 (hyperbolic part Ω_2) to the line $y=0$ plays in important role in the proof of uniqueness and existence of solution.

It is known that the solution of the Cauchy problem for the Eq. (10) in domain Ω_2 satisfying conditions

$$u(x,-0) = \tau^-(x), \quad 0 \le x \le 1; \quad u_y(x,-0) = \nu^-(x), \quad 0 < x < 1, \tag{19}$$

can be presented in the form [11]

$$u(x,y) = \frac{\Gamma(2\alpha_1)}{\Gamma^2(\alpha_1)} \left(\frac{x^{q_1}}{q_1}\right)^{-\alpha_1} \int\limits_0^1 \left[\frac{(-y)^{p_1}}{p_1}(2z-1) + \frac{x^{q_1}}{q_1}\right]^{\alpha_1} [z(1-z)]^{\beta_1 - 1} \times$$

$$\times \tau^- \left\{\left[\frac{q_1}{p_1}(-y)^{p_1}(2z-1) + x^{q_1}\right]^{\frac{1}{q_1}}\right\} F(\alpha_1, 1-\alpha_1, \beta_1, \rho)dz -$$

$$- \frac{\Gamma(1-2\alpha_1)}{\Gamma^2(1-\alpha_1)} p_1((-y)^{p_1})^{1-2\beta_1} \int\limits_0^1 \left[\frac{(-y)^{p_1}}{p_1}(2z-1) + \frac{x^{q_1}}{q_1}\right]^{\alpha_1} [z(1-z)]^{-\beta_1} \times$$

$$\times \nu^- \left\{\left[\frac{q_1}{p_1}(-y)^{p_1}\,(2z-1) + x^{q_1}\right]^{\frac{1}{q_1}}\right\} F(\alpha_1, 1-\alpha_1, \beta_1, \rho)dz, \tag{20}$$

where

$$\rho = \frac{q_1(-y)^{\frac{1}{p_1}} z(1-z)}{{p_1}^2 x^{q_1} \left[\dfrac{1}{p_1}(-y)^{p_1}\,(2z-1) + \dfrac{1}{q_1}x^{q_1}\right]}.$$

Due to conditions (9) and (13) from (20), by using formula (8), we get

$$\tilde{\psi}_1(x) = \gamma_1 (x^{2q_1})^{\frac{2-\alpha_1-3\beta_1}{2}} F_{0x}\begin{bmatrix} \frac{\beta_1-\alpha_1}{2}, & \frac{\alpha_1+\beta_1-1}{2} \\ \beta_1, & x^{2q_1} \end{bmatrix} (x^{2q_1})^{\frac{\alpha_1+\beta_1-2}{2}} \tau^-(x) -$$

$$-\gamma_2 (x^{2q_1})^{\frac{\beta_1-\alpha_1}{2}} F_{0x}\begin{bmatrix} \frac{1-\beta_1-\alpha_1}{2}, & \frac{\alpha_1-\beta_1}{2} \\ 1-\beta_1, & x^{2q_1} \end{bmatrix} (x^{2q_1})^{\frac{\alpha_1-\beta_1-1}{2}} \nu^-(x), \tag{21}$$

where $0 < x < k_1$, $\tilde{\psi}_1(x) = \psi_1\left[\left(\frac{x^{q_1}}{2}\right)^{\frac{1}{q_1}}\right]$, $\gamma_1 = \frac{\Gamma(2\alpha_1)}{\Gamma^2(\alpha_1)} 2^{\alpha_1-\beta_1}$, $\gamma_2 = \frac{\Gamma(1-2\alpha_1)}{\Gamma(1-\alpha_1)} \times$ $\times \left(\frac{p_1}{q_1}\right)^{1-2\alpha_1} 2^{\alpha_1+3\beta_1-2}$.

Applying operator

$$\frac{d}{d(x^{2q_1})} (x^{2q_1})^{\frac{1-\alpha_1-\beta_1}{2}} F_{0x}\begin{bmatrix} \frac{\alpha_1+\beta_1-1}{2}, & \frac{\alpha_1+\beta_1}{2} \\ \beta_1, & x^{2q_1} \end{bmatrix} (x^{2q_1})^{\frac{2\alpha_1-1}{2}}$$

to both parts of the equality (21), finally we obtain functional relation between $\tau^-(x)$ and $\nu^-(x)$ transferred from hyperbolic domain Ω_{21}^- on the line $y = 0$:

$$\tilde{\nu}^-(x) = \frac{\gamma_1}{\gamma_2} q^{2\beta_1-1} x^{\frac{1-2\alpha_1}{2}} \frac{d}{dx} x^{\frac{1-2\beta_1}{2}} F_{0x}\begin{bmatrix} \alpha_1+\beta_1, & \frac{2\beta_1-1}{2} \\ 2\beta_1, & x \end{bmatrix} x^{\frac{2\alpha_1-1}{2}} \tilde{\tau}^-(x) -$$

$$-\tilde{\psi}_1(x), \qquad 0 < x < k_1, \tag{22}$$

where $k_1 = q_1 k$,

$$\tilde{\tau}^-(x) = u\left[\left((q_1k)^2 x\right)^{1/2q_1}, 0\right], \qquad \tilde{\nu}^-(x) = u_y\left[\left((q_1k)^2 x\right)^{1/2q_1}, -0\right].$$

In the sequel, replacing x to $(h_1 - \sigma(x))/k_0$ in (22), we deduce

$$\tilde{\nu}^-\left(\frac{h_1-\sigma(x)}{k_0}\right) = \chi_1 (h_1-\sigma(x))^{\frac{1-2\alpha_1}{2}} \frac{d}{d\left(\frac{h_1-\sigma(x)}{k_0}\right)} (h_1-\sigma(x))^{\frac{1-2\beta_1}{2}} \times$$

$$\times \int\limits_{\sigma(x)}^{h_1} \frac{(z-\sigma(x))^{2\beta_1-1}}{(h_1-z)^{\frac{1-2\alpha_1}{2}}} \tilde{\tau}^-(z) F\left(\alpha_1+\beta_1, \ \frac{2\beta_1-1}{2}, \ 2\beta_1, \ \frac{z-\sigma(x)}{h_1-\sigma(x)}\right) dz -$$

$$-\tilde{\psi}_1(x), \quad 0 < \frac{h_1-\sigma(x)}{k_0} < k_1, \quad k_1 < \sigma(x) < h_1, \tag{23}$$

where

$$\chi_1 = -\frac{\gamma_1 q^{2\beta_1-1}}{\gamma_2 \Gamma(2\beta_1) {k_0}^{2\beta_1+1}}.$$

Based on the conditions of the Problem TF at $y \to +0$ from Eq. (10) in Ω_1, on the $y = 0$, we have

$$\tau^{+''}(x) - x^{n_0}\nu^+(x) = 0, \qquad x \in (0, k_1), \tag{24}$$

$$\tau(0) = \varphi_1(0), \qquad \tau(k_1) = \frac{\varphi_2(k_1)}{\mu - 1}, \tag{25}$$

where

$$\nu^+(x) = \lim_{y \to +0} y^{-m_0} u_y(x, 0).$$

Solving problem (24), (25), we find functional relation between $\tau^+(x)$ and $\nu^+(x)$ from the domain Ω_1 brought on AE :

$$\tau^+(x) = \int_0^{k_1} G_1(x,t)\, t^{n_0}\nu^+(t)dt + f_1(x), \quad 0 \le x \le k_1, \tag{26}$$

where

$$G_1(x,t) = \begin{cases} \dfrac{t}{k_1}(x - k_1), & 0 \le t \le x, \\ \dfrac{x}{k_1}(t - k_1), & x \le t \le k_1, \end{cases} \tag{27}$$

$$f_1(x) = \left[\frac{\psi_2(k_1)}{\mu - 1} - \varphi_1(0)\right] \frac{x}{k_1} + \varphi_1(0). \tag{28}$$

Similarly, on a piece EB, due to conditions of the Problem TF at $y \to +0$ from Eq. (10) we get

$$\tau''^{+}(x) - x^{n_0}\nu^+(x) = 0, \qquad x \in (k_1, h_1), \tag{29}$$

$$\tau(k_1) = \frac{\psi_2(k_1)}{\mu - 1}, \qquad \tau(h_1) = \varphi_2(0). \tag{30}$$

From (29), (30), we obtain functional relation between $\tau^+(x)$ and $\nu^+(x)$, brought on EB from the domain Ω_1^+:

$$\tau^+(x) = \int_{k_1}^{h_1} G_2(x,t)\, t^{n_0}\nu^+(t)dt + f_2(x), \quad k_1 \le x \le h_1, \tag{31}$$

where

$$G_2(x,t) = \begin{cases} \dfrac{t - k_1}{h_1 - k_1}(x - h_1), & k_1 \le t \le x, \\ \dfrac{x - k_1}{h_1 - k_1}(t - h_1), & x \le t \le h_1, \end{cases} \tag{32}$$

$$f_2(x) = \frac{x - k_1}{h_1 - k_1}\left[\varphi_2(0) - \frac{\psi_2(k_1)}{\mu - 1}\right] + \frac{\psi_2(k_1)}{\mu - 1}. \tag{33}$$

3.2. The uniqueness of solution.

Theorem 1. *If conditions* (11) *hold and*

$$\mu \in (0,\, 1), \tag{34}$$

then the solution of Problem TF is unique.

Proof. It is obvious that if solution of the homogenous problem identically equals to zero, then the solution of non homogenous problem is unique. Hence, it is enough to prove that the solution $u(x,y)$ of Problem TF identically equals to zero at $\varphi_1(y)\equiv\varphi_2(y)\equiv 0$ and $\psi_1(x)\equiv\psi_2(x)\equiv 0$. There holds the following preliminary assertion.

Lemma 2. *Solution of Problem TF at $\psi_1(x)\equiv\psi_2(x)\equiv 0$ reaches own positive maximum and negative minimum in the closed domain only on $\overline{AA_0}\cup\overline{BB_0}$.*

Proof. Based on the extreme principle for parabolic equation (see [9, 26]) the solution $u(x,y)$ of Eq. (10) in closed domain reaches own positive maximum and negative minimum on $\overline{AA_0}\cup\overline{BB_0}\cup\overline{J}$. We will proof that the solution $u(x,y)$ of the Eq. (10) does not reach positive maximum (negative minimum) on the intervals $AE,\ EB$ and on the point $E(k_1,0)$. Let us consider following cases:

I. In the sequel we assume that $u(x,y)$ reaches positive maximum (negative minimum) on some points $P(x_0,0)$ of the interval AE. Obviously, the equality (22) at $\psi_1(x)\equiv 0$ will be represented in the form [19]

$$\frac{\Gamma(2\beta_1)\gamma_2}{\gamma_1}\nu^-(x)=\frac{\Gamma(2\beta_1)\Gamma\left(\dfrac{1+2\alpha_1}{2}\right)}{\Gamma(\alpha_1+\beta_1)\Gamma\left(\dfrac{1+2\beta_1}{2}\right)}\left(x^{2q_1}\right)^{\frac{2\beta_1-1}{2}}\tau^-(x)+$$

$$+\frac{1-2\beta_1}{\left(x^{2q_1}\right)^{\alpha_1+\beta_1-1}}\int\limits_0^x\left(t^{2q_1}\right)^{\frac{2\alpha_1-1}{2}}\left[\tau^-(x)-\tau^-(t)\right]\left(x^{2q_1}-t^{2q_1}\right)^{2\beta_1-2}\times$$

$$\times F\left(\beta_1+\alpha_1-1,\ \frac{2\beta_1-1}{2},\ 2\beta_1-1,\ \frac{x^{2q_1}-t^{2q_1}}{x^{2q_1}}\right)\frac{2q_1}{t^{1-2q_1}}dt,\qquad x\in(0,k_1). \tag{35}$$

Estimating hypergeometrical function, taking into account the formula (7), (11) (see [3], § 2.8, formulae (46)) and

$$2\beta_1-1-\frac{2\beta_1-1}{2}-(\alpha_1+\beta_1-1)=\frac{1-2\alpha_1}{2}>0,\qquad 0\le\frac{x^{2\,q_1}-t^{2\,q_1}}{x^{2\,q_1}}\le 1,$$

we have

$$F\left(\beta_1+\alpha_1-1,\ \frac{2\beta_1-1}{2},\ 2\beta_1-1,\ \frac{x^{2q_1}-t^{2q_1}}{x^{2q_1}}\right)\ge$$

$$\ge\frac{\Gamma(2\beta_1-1)\Gamma\left(\dfrac{1-2\alpha_1}{2}\right)}{\Gamma(\beta_1-\alpha_1)\Gamma\left(\dfrac{2\beta_1-1}{2}\right)}>0. \tag{36}$$

By virtue of (36), (11) owing to

$$\tau^-(x_0)>0\quad(\tau^-(x_0)<0),\qquad\qquad \tau^-(x_0)-\tau^-(t)>0\quad\left(\tau^-(x_0)-\tau^-(t)<0\right),$$

from (35) we deduce that on the point $P(x_0,0)$ of positive maximum (negative minimum)

$$\nu^-(x_0)>0\ \ (\nu^-(x_0)<0). \tag{37}$$

From here and from (15) taking into account (19), we receive

$$\nu^{+}(x_0) > 0 \ (\nu^{+}(x_0) < 0\,).$$

In fact, on the point of positive maximum (negative minimum) $\tau''^{+}(x_0) < 0 \ \left(\tau''^{+}(x_0) > 0\right)$ from (24) we get $\nu^{+}(x_0) < 0 \ (\nu^{+}(x_0) > 0)$, and this inequality contradicts inequality (37). Thus, the solution $u(x,y)$ of Eq. (10) does not reach positive maximum (negative minimum) on the interval AE.

II. Now, we assume that $u(x,y)$ on some point $Q(x_0,0)$ of interval EB reaches positive maximum (negative minimum). From (14) at $\psi_2(x) \equiv 0$ we have

$$\mu\tau(x) = \tau(\sigma(x)), \qquad x \in [0,\ \kappa_1]\,. \tag{38}$$

Let x_1 $(x_1 \in (0,k_1))$ be a solution of equation (38), then owing to $\sigma(x_1) = x_0$ $(x_0 \in (k_1,h_1))$ from (38) we obtain

$$\mu\tau(x_1) = \tau(\sigma(x_1)) = \tau(x_0). \tag{39}$$

By virtue of (34), from (39), we inter that $\tau(x_0) \le \tau(x_1)$, Hence, we will conclude that the point x_1 is extreme point of the function $\tau(x)$ in interval AE. This conclusion contradicts to the previous discussions (see case I). Thus, $u(x,y)$ does not reach positive maximum (negative minimum) on the interval EB.

III. Let us assume that $u(x,y)$ reaches positive maximum (negative minimum) on the point $E(\kappa_1,0)$. Due to $\sigma(\kappa_1) = \kappa_1$ and (34), from (38) we get $\mu\tau(\kappa_1) = \tau(\sigma(\kappa_1)) = \tau(\kappa_1) \Rightarrow (1 - \mu)\tau(\kappa_1) = 0 \Rightarrow \tau(\kappa_1) = 0$, i.e., $u(\kappa_1,0) = 0$. Hence, $u(x,y)$ does not reach extreme on the point $E(\kappa_1,0)$.

Lemma 2 is proved.

Let us begin proving Theorem 1. On the base of Lemma 2 owing to (12) at $\varphi_1(y) \equiv \varphi_2(y) \equiv 0$ we have $u(x,y) \equiv 0$ in $\bar{\Omega}_1$. Then by virtue of continuity of solution of the problem TF in the domain $\overline{\Omega}$ due to gluing condition (15) and (22) we deduce that

$$u\,(x,0) = 0, \qquad (x,0) \in \bar{J}, \qquad u_y\,(x,0) = 0, \qquad (x,0) \in J. \tag{40}$$

By virtue of (40), from the solution of the Cauchy problem for Eq. (10) in domain Ω_2 we get $u(x,y) \equiv 0,\ (x,y) \in \overline{\Omega}_2$. Hence, $u(x,y) \equiv 0,\ (x,y) \in \overline{\Omega}$. Thus, solution of Problem TF is unique.

Theorem 1 is proved.

3.3. Existence of the solution.

Theorem 2. *If conditions* (11), (16), (17), (18), *and* (34) *hold, then the solution of Problem* TF *exists.*

Proof. Having excluded $\tau^{+}(x)$ from the functional relations (22) and (26) owing to (15) and first condition of Problem TF, we receive integral equation

$$\tilde{\nu}(x) = \int\limits_0^{k_1} S_1(x,t)\tilde{\nu}(t)dt + \tilde{F}_1(x), \qquad 0 < x < k_1, \tag{41}$$

where

$$S_1(x,t)=\bar{\gamma}x^{\frac{1-2\alpha_1}{2}}t^{\frac{n_0+1}{n_1+2}-1}\frac{d}{dx}x^{\frac{1-2\beta_1}{2}}\int\limits_0^x(x-z)^{2\beta_1-1}z^{\frac{2\alpha_1-1}{2}}\tilde{G}_1(x,z)\times$$

$$\times F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2},\ 2\beta_1,\ \frac{x-z}{x}\right)dz, \tag{42}$$

$$\tilde{F}_1(x)=\frac{\gamma_1 q^{2\beta_1-1}}{\gamma_2\Gamma(2\beta_1)x^{\frac{2\alpha_1-1}{2}}}\frac{d}{dx}x^{\frac{1-2\beta_1}{2}}\int\limits_0^x\frac{(x-z)^{2\beta_1-1}}{t^{\frac{1-2\alpha_1}{2}}}\times$$

$$\times F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2},\ 2\beta_1,\ \frac{x-z}{x}\right)\left[\left\{\frac{\psi_2(k_1)}{\mu-1}-\varphi_1(0)\right\}\frac{z}{k_1}+\varphi_1(0)\right]dz-$$

$$-\frac{1}{\gamma_2}\left((q_1k)^2x\right)^{\frac{1-\alpha_1+\beta_1}{2}}\tilde{\psi}_1(x), \tag{43}$$

$$\bar{\gamma}=\frac{\gamma_1(q_1k)^{2\alpha_0}}{2\gamma_2 q_1^{2-2\beta_1}\Gamma(2\beta_1)},\qquad \tilde{G}_1(x,z)=G_1\left[\left((q_1k)^2x\right)^{\frac{1}{2q_1}},\left((q_1k)^2z\right)^{\frac{1}{2q_1}}\right].$$

By virtue of (11), (16), (17), (18) taking into account replacement $z=xu$, from (42), (43) it follows that

$$S_1(x,t)=\bar{\gamma}x^{\frac{1-2\alpha_1}{2}}t^{\frac{n_0+1}{n_1+2}-1}\frac{d}{dx}\left(x^{\alpha_1+\beta_1}\times\right.$$

$$\left.\times\int\limits_0^1\frac{(1-u)^{2\beta_1-1}}{u^{\frac{1-2\alpha_1}{2}}}F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2},\ 2\beta_1,\ 1-u\right)\tilde{G}_1(x,xu)\,du)\right), \tag{44}$$

$$\tilde{F}_1(x)=\frac{\gamma_1 q^{2\beta_1-1}}{\gamma_2\Gamma(2\beta_1)}x^{\frac{1-2\alpha_1}{2}}\frac{d}{dx}x^{\alpha_1+\beta_1}\times$$

$$\times\int\limits_0^1(1-u)^{2\beta_1-1}u^{\frac{2\alpha_1-1}{2}}F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2},\ 2\beta_1,\ 1-u\right)\tilde{f}_1(u)du-$$

$$-\frac{1}{\gamma_2}\left((q_1k)^2x\right)^{\frac{1-\alpha_1+\beta_1}{2}}\tilde{\psi}(x). \tag{45}$$

Further, from (44) we have

$$S_1\ (x,t)=\bar{\gamma}(\alpha_1+\beta_1)x^{\frac{2\beta_1-1}{2}}t^{\frac{n_0+1}{n_1+2}-1}\times$$

$$\times\int\limits_0^1\frac{(1-u)^{2\beta_1-1}}{u^{\frac{1-2\alpha_1}{2}}}F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2},\ 2\beta_1,\ 1-u\right)\tilde{G}_1(x,xu)du+$$

$$+\bar{\gamma}x^{\frac{2\beta_1+1}{2}}t^{\frac{n_0+1}{n_1+2}-1}\times$$

$$\times \int_0^1 \frac{(1-u)^{2\beta_1-1}}{u^{\frac{1-2\alpha_1}{2}}} F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2},\ 2\beta_1,\ 1-u\right) \frac{d}{dx}\tilde{G}_1(x,xu)du =$$

$$= K_1(x,t)+K_2(x,t).$$

From here, owing to (11), (27) and basing on conditions of the hypergeometric functions [3], we estimate kernel and right-hand side of Eq. (41):

$$|K_1(x,t)| \le \left|\bar{\gamma}(\alpha_1+\beta_1)x^{\frac{2\beta_1-1}{2}} t^{\frac{n_0+1}{n_1+2}-1}\right| \times$$

$$\times \left|\int_0^1 (1-u)^{2\beta_1-1} u^{\frac{2\alpha_1-1}{2}} F\left(\alpha_1+\beta_1,\ \frac{2\beta_1-1}{2}, 2\beta_1, 1-u\right)\tilde{G}_1(x,xu)du\right| \le$$

$$\le \text{const}\left|\frac{t^{\frac{n_0+1}{n_1+2}-1}}{x^{\frac{1-2\beta_1}{2}}}\right| \left|\int_0^1 \frac{(1-u)^{2\beta_1-1}}{u^{\frac{1-2\alpha_1}{2}}} F\left(\alpha_1+\beta_1, \frac{2\beta_1-1}{2}, 2\beta_1, 1-u\right)du\right| \le$$

$$\le \text{const}\left|\frac{t^{\frac{n_0+1}{n_1+2}-1}}{x^{\frac{1-2\beta_1}{2}}}\right| \left|\int_0^1 (1-u)^{2\beta_1-1} u^{\frac{2\alpha_1-1}{2}} du\right| \le$$

$$\le \text{const}\left|x^{\frac{2\beta_1-1}{2}} t^{\frac{n_0+1}{n_1+2}-1}\right| B\left(\alpha_1+\frac{1}{2}, 2\beta_1\right) \le \text{const}\cdot x^{\beta_1-\frac{1}{2}} t^{\alpha_0-1},$$

$$|K_2(x,t)| \le \left|\bar{\gamma} x^{\frac{2\beta_1+1}{2}} t^{\frac{n_0+1}{n_1+2}-1}\right| \times$$

$$\times \left|\int_0^1 \frac{u^{\frac{2\alpha_1-1}{2}}}{(1-u)^{1-2\beta_1}} F\left(\alpha_1+\beta_1, \frac{2\beta_1-1}{2}, 2\beta_1, 1-u\right) \frac{d}{dx}\tilde{G}_1(x,xu)du\right| \le$$

$$\le \text{const}\left|x^{\frac{2\beta_1+1}{2}} t^{\frac{n_0+1}{n_1+2}-1} \int_0^1 u^{\frac{2\alpha_1-1}{2}} (1-u)^{2\beta_1-1} \frac{d}{dx}\tilde{G}_1(x,xu)du\right| \le$$

$$\le \text{const}\left|\frac{x^{\frac{2\beta_1+1}{2}}}{t^{1-\frac{n_0+1}{n_1+2}}}\right| \left|\int_0^1 \frac{u^{\frac{2\alpha_1-1}{2}}}{(1-u)^{1-2\beta_1}} \frac{d}{dx} G_1\left(\left[(q_1k)^2 x\right]^{\frac{1}{2q_1}}, \left[(q_1k)^2 t\right]^{\frac{1}{2q_1}}\right) du\right| \le$$

$$\le \text{const}\left|\frac{(q_1k)^{\frac{2}{n_1+2}}}{2q_1} x^{\frac{2\beta_1+1}{2}} t^{\frac{n_0+1}{n_1+2}-1} x^{-\left(\frac{2\alpha_1+1}{2}\right)}\right| \left|\int_0^1 \frac{u^{\frac{2\alpha_1-1}{2}}}{(1-u)^{1-2\beta_1}} du\right| \le$$

$$\le \text{const}\left|x^{\beta_1-\alpha_1} t^{\frac{n_0+1}{n_1+2}-1}\right| B\left(\alpha_1+\frac{1}{2},\ 2\beta_1\right) \le$$

$$\le \text{const}\cdot x^{\beta_1-\alpha_1} t^{\frac{n_0+1}{n_1+2}-1},$$

$$|S_1(x,t)| = |K_1(x,t) + K_2(x,t)| \leq \mathrm{const} \cdot x^{\beta_1 - \frac{1}{2}} t^{\alpha_0 - 1}. \tag{46}$$

Similarly, estimating (45), we obtain

$$\left|\tilde{F}_1(x)\right| \leq \mathrm{const} \cdot x^{\frac{2\beta_1 - 1}{2}}, \tag{47}$$

i.e., $\tilde{F}_1(x) \in C^2(0,1)$ and that this function can have a feature of an order less $\dfrac{1-2\beta_1}{2}$ at $x \to 0$, and it is limited at $x \to k_1$. Thus, by virtue of (46), (47) we conclude that Eq. (41) is Fredholm integral equation of second kind with a weak feature. Solvability of this equation follows from the uniqueness of solution of Problem TF. Excluding $\tau(x)$ from (31) and (23), we have integral equation with respect to $\nu(x)$:

$$\tilde{\nu}(x) = \int\limits_{k_1}^{h_1} S_2(\sigma(x), t)\, \tilde{\nu}(t) dt + \tilde{F}_2(\sigma(x)), \qquad k_1 < x < h_1, \tag{48}$$

where

$$S_2(\sigma(x), t) = \frac{\chi_1 (q_1 k)^2 \alpha_0}{2qt^{1-\alpha_0}} (h_1 - \sigma(x))^{\frac{1-2\alpha_1}{2}} \frac{d}{d\left(\dfrac{h_1 - \sigma(x)}{k_0}\right)} (h_1 - \sigma(x))^{\frac{1-2\beta_1}{2}} \times$$

$$\times \int\limits_{\sigma(x)}^{h_1} \frac{(h_1 - z)^{\frac{1-2\alpha_1}{2}}}{(z - \sigma(x))^{1-2\beta_1}} \tilde{G}_2(x,z) F\left(\alpha_1 + \beta_1, \frac{2\beta_1 - 1}{2}, 2\beta_1, \frac{z - \sigma(x)}{h_1 - \sigma(x)}\right) dz, \tag{49}$$

$$\tilde{F}_2(\sigma(x)) = \chi_1 (h_1 - \sigma(x))^{\frac{1-2\alpha_1}{2}} \frac{d}{d\left(\dfrac{h_1 - \sigma(x)}{k_0}\right)} (h_1 - \sigma(x))^{\frac{1-2\beta_1}{2}} \times$$

$$\times \int\limits_{\sigma(x)}^{h_1} \frac{(h_1 - z)^{\frac{1-2\alpha_1}{2}}}{(z - \sigma(x))^{1-2\beta_1}} F\left(\alpha_1 + \beta_1,\ \frac{2\beta_1 - 1}{2},\ 2\beta_1,\ \frac{z - \sigma(x)}{h_1 - \sigma(x)}\right) \tilde{f}_2(z) dz -$$

$$-\tilde{\psi}_1(x). \tag{50}$$

Replacing $z = \sigma(x) + (h_1 - \sigma(x))s$ in (49), we obtain

$$S_2(\sigma(x), t) = \frac{\chi_1 (q_1 k)^2 \alpha_0}{2qt^{\alpha_0 - 1}} (h_1 - \sigma(x))^{\frac{1-2\alpha_1}{2}} \frac{d}{d\left(\dfrac{h_1 - \sigma(x)}{k_0}\right)} (h_1 - \sigma(x))^{\alpha_1 + \beta_1} \times$$

$$\times \int\limits_0^1 s^{2\beta_1 - 1} (1-s)^{\frac{2\alpha_1 - 1}{2}} \tilde{G}_2(x, (\sigma(x) + (h_1 - \sigma(x))s)) \times$$

$$\times F\left(\alpha_1 + \beta_1,\ \frac{2\beta_1 - 1}{2},\ 2\beta_1,\ s\right) ds. \tag{51}$$

Taking into account a class of given functions from (51) and (50), we get

$$|S_2(\sigma(x), t)| \leq \text{const} \cdot t^{\alpha_0 - 1}(h_1 - \sigma(x))^{\beta_1 - \frac{1}{2}}, \tag{52}$$

$$\left|\tilde{F}_2(\sigma(x))\right| \leq \text{const} \cdot (h_1 - \sigma(x))^{\beta_1 - \frac{1}{2}}. \tag{53}$$

By virtue of (11), (52), Eq. (48) will be Fredholm integral equation of second kind with a weak feature [18]. Solvability of the Fredholm integral equation of the second kind follows from the uniqueness of solution of Problem TF and a solution be given by formula

$$\tilde{\nu}(x) = \tilde{F}_2(x) + \int\limits_{k_1}^{h_1} R(\sigma(x), t)\, \tilde{F}_2(\sigma(t))dt, \quad k_1 < x < h_1,$$

where $R(\sigma(x), t)$ is resolvent of kernel $S_2(\sigma(x), t)$.

After definition of $\nu^-(x),\ \tau^-(x),\ \tau^+(x)$ from (15), (22), (26) ((23), (31)), solution of Problem TF will be constructed in domain Ω_1 as a solution of the first boundary-value problem [11] for the equation (10) with conditions (12) and $u(x, 0) = \tau^+(x)$, $(x, 0) \in \bar{J}$, and in domain Ω_2 as a solution of the Cauchy problem for Eq. (10) with initial dates (19).

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Received 01.10.16,
after revision — 11.07.17