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SYSTEMS OF VARIATIONAL INEQUALITIES AND MULTIPLE-SET SPLIT EQUALITY FIXED-POINT PROBLEMS FOR COUNTABLE FAMILIES OF MULTIVALUED TYPE-ONE MAPPINGS OF THE DEMICONTRACTIVE TYPE

СИСТЕМИ ВАРІАЦІЙНИХ НЕРІВНОСТЕЙ ТА ЗАДАЧІ ПРО НЕРУХОМУ ТОЧКУ З БАГАТОМНОЖИННОЮ РОЗЩЕПЛЕНОЮ РІВНІСТЮ ДЛЯ ЗЛІЧЕННИХ СІМЕЙ БАГАТОЗНАЧНИХ НАПІВСТИСКАЮЧИХ ВІДОБРАЖЕНЬ ПЕРШОГО ТИПУ

Our main aim is to introduce an iterative algorithm for the approximation of a common solution to a split-equality problem for finite families of variational inequalities and the split equality fixed-point problem. By using our iterative algorithm, we state and prove a strong convergence theorem for the approximation of an element in the intersection of the set of solutions of the split-equality problem for finite families of variational inequalities and the set of solutions of the split equality fixed-point problem for countable families of multivalued type-one mappings of the demicontractive type. Finally, we apply our result to study related problems. Our result supplements and extends some recent results in the literature.

Запропоновано ітеративний алгоритм для наближення спільного розв'язку задачі про розщеплену рівність для скінченних сімей варіаційних нерівностей та задачі про нерухому точку з розщепленою рівністю. За допомогою розробленого алгоритму сформульовано та доведено теорему про сильну збіжність для наближення елемента, що належить перетину множини розв'язків задачі про розщеплену рівність для скінченних сімей варіаційних нерівностей та множини розв'язків задачі про нерухому точку з розщепленою рівністю для злічених сімей багатозначних напівстискаючих відображень першого типу. Результат, що отримано, застосовано до вивчення споріднених задач. Наш результат доповнює та узагальнює деякі нові результати в літературі.

1. Introduction. Let (X, d) be a metric space and $CB(X)$ be the family of all closed and bounded subsets of X. Let $\mathcal{H}$ denote the Hausdorff metric induced by the metric d, then, for all $A,\ B \in$ $\in CB(X)$,

$$\mathcal{H}(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \tag{1.1}$$

where $d(a, B) := \inf_{b \in B} d(a, b)$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H and 2^C be the family of all nonempty subsets of C. Let $T : C \to 2^C$ be a multivalued mapping, then $P_T x := \{u \in Tx : \|x - u\| = d(x, Tx)\}$. A point $x \in C$ is called a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, then x is called a strict fixed point of T. We denote the set of fixed point of T by $F(T)$.

A multivalued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\mathcal{H}(Tx, Ty) \leq L\|x - y\|, \quad x, y \in C. \tag{1.2}$$

In (1.2), if $L \in (0, 1)$, then T is called a strict contraction while T is called nonexpansive if $L = 1$. T is said to be

(i) *of type-one* if

$$\|u - v\| \leq \mathcal{H}(Tx, Ty) \quad \forall x, y \in C, \quad u \in P_T x, \quad v \in P_T y,$$

(ii) *quasinonexpansive* if $F(T) \neq \varnothing$ and

$$\mathcal{H}(Tx, Ty) \leq \|x - y\| \quad \forall x \in C, \quad y \in F(T),$$

(iii) *k-strictly pseudocontractive* in the sense of [23] if there exist $k \in (0,1)$ such that, for all $x,\ y \in C$ and $u \in Tx$, there exists $v \in Ty$ satisfying $\|u - v\| \leq \mathcal{H}(Tx, Ty)$ and

$$\mathcal{H}^2(Tx, Ty) \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2,$$

(iv) *demicontractive-type* in the sense of [24] if $F(T) \neq \varnothing$ and

$$\mathcal{H}^2(Tx, Ty) \leq \|x - y\|^2 + kd^2(x, Tx), \qquad x \in C, \quad y \in F(T) \quad \text{and} \quad k \in (0,1).$$

***Remark* 1.1.** Every multivalued quasinonexpansive mapping is a multivalued demicontractive-type mapping.

We give an example to show that the converse of Remark 1.1 is not true.

***Example* 1.1.** Let $H = \mathbb{R}$ (endowed with the usual metric) and $T\colon \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$Tx = \begin{cases} \left[-(\alpha+1)x,\ -\dfrac{2\alpha+1}{2}x\right], & x \in [0, \infty), \\ \\ \left[-\dfrac{2\alpha+1}{2}x,\ -(\alpha+1)x\right] & \forall \alpha > 0, \quad x \in (-\infty, 0). \end{cases}$$

Then $F(T) = \{0\}$. For each $x \in (-\infty, 0) \cup (0, \infty)$,

$$\mathcal{H}^2(Tx, T0) = |-(\alpha+1)x - 0|^2 = (\alpha+1)^2|x - 0|^2 = |x - 0|^2 + (\alpha^2 + 2\alpha)|x - 0|^2. \tag{1.3}$$

Also

$$d^2(x, Tx) = \left|x + \frac{2\alpha+1}{2}x\right|^2 = \left(\frac{2\alpha+3}{2}\right)^2 \left|x - 0\right|^2,$$

which implies

$$\left|x - 0\right|^2 = \frac{4}{(2\alpha+3)^2} d^2(x, Tx). \tag{1.4}$$

Substituting (1.4) into (1.3), we obtain

$$\mathcal{H}^2(Tx, T0) = |x - 0|^2 + \frac{4(\alpha^2 + 2\alpha)}{(2\alpha+3)^2} d^2(x, Tx),$$

which implies that T is a demicontractive-type multivalued mapping with $k = \dfrac{4(\alpha^2 + 2\alpha)}{(2\alpha+3)^2} \in$ $\in (0,1)$ for all $\alpha > 0$. However, we see in (1.3) that T is not a quasinonexpansive multivalued mapping. Hence, the class of quasinonexpansive mappings is properly contained in the class of demicontractive-type multivalued mappings.

A single valued mapping $T\colon C \to C$ is said to be L-Lipschitzian if there exist a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\| \ \ \forall\, x,\ y \in C$. If $L = 1$, then T is called a nonexpansive mapping. T is called an α-inverse strongly monotone (or α-cocoercive) operator if there exists $\alpha > 0$ such that $\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2 \ \ \forall\, x, y \in C$. If $\alpha = 1$, then T is called a firmly nonexpansive mapping (see [22] for more information on firmly nonexpansive mappings).

***Remark* 1.2.** It is obvious that any α-inverse strongly monotone operator T is $\dfrac{1}{\alpha}$-Lipschitz continuous.

A single valued mapping $T\colon C \to C$ is said to be *strongly nonexpansive* if T is nonexpansive and for all bounded sequences $\{x_n\}$, $\{y_n\}$ in C, $\lim_{n\to\infty}(\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0$ implies $\lim_{n\to\infty} \|(x_n - y_n) - (Tx_n - Ty_n)\| = 0$. This class of mapping was first introduced by Bruck and Riech in [7]. T is said to be *cutter* if $\langle x - Tx, z - Tx\rangle \leq 0$ for all $x \in C$ and $z \in F(T)$.

A mapping $T\colon C \to C$ is said to be averaged nonexpansive if, for all $x,\ y \in C$, $T = (1-\alpha)I + \alpha S$ holds for a nonexpansive operator $S\colon C \to C$ and $\alpha \in (0,1)$. The term "averaged mapping" was coined by Biallon et al. [4].

***Remark* 1.3.** In a Hilbert space, T is firmly nonexpansive if and only if it is averaged with $\alpha = \dfrac{1}{2}$.

The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| = \inf\{\|x - y\|:\ y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x\rangle \leq 0$, for all $z \in C$, and P_C is a firmly nonexpansive mapping. We also know that if f is β-inverse strongly monotone mapping with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is averaged nonexpansive (see [14], Lemma 2.9). Hence, $P_C(I - \lambda f)$ is firmly nonexpansive. For more information on metric projections, see ([22], Section 3).

Let $M\colon H \to 2^H$ be a set-valued operator defined on a real Hilbert space H. M is called a maximal monotone operator if M is monotone, i.e.,

$$\langle u - v, x - y\rangle \geq \quad \forall x, y \in H,\ u \in M(x) \text{ and } y \in M(y),$$

and the graph $G(M)$ of M defined by

$$G(M) := \{(x, y) \in H \times H:\ u \in M(x)\},$$

is not properly contained in the graph of any other monotone operator. It is easy to see that a monotone operator is maximal if and only if for each $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0\ \forall\,(v, y) \in G(M) \implies u \in M(x)$.

Let C be a nonempty, closed and convex subset of H. The normal cone of C at the point $z \in C$ is defined

$$N_C z := \{d \in H : \langle d, y - z\rangle \leq 0 \ \text{ for all } \ y \in C\}.$$

The Split Feasibility Problem (SFP) introduced in 1994 by Censor and Elfving [10] is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.5}$$

where C and Q are nonempty closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ real matrix. The SFP has wide applications in many fields such as phase retrieval, medical image reconstruction, signal processing and radiation therapy treatment planning (see, for example, [5, 9–11, 37, 42] and the references therein).

Censor et al. [11] introduced the Multiple-Sets Split Feasibility Problem (MSSFP) which is to find

$$x^* \in C = \bigcap_{i=1}^{N} C_i \text{ such that } Ax^* \in Q = \bigcap_{j=1}^{M} Q_j,$$

where N and M are positive integers, $\{C_1, \ldots, C_N\}$ and $\{Q_1, \ldots, Q_N\}$ are nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A\colon H_1 \to H_2$ is a bounded linear map (see also [33] for more information on MSSFP).

In 2009, Censor and Segal [12] presented an important form of the SFP called the Split Common Fixed Point Problem (SCFPP), which is to find a point

$$x^* \in F(T) \quad \text{such that} \quad Ax^* \in F(S), \tag{1.6}$$

where T and S are some nonlinear operators on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, A is a real $m \times n$ matrix. They presented the following algorithm for solving the SCFPP:

$$x_{n+1} = T(x_n + \gamma A^T(S - I)Ax_n) \quad \forall n \geq 1, \quad x_1 \in \mathbb{R}^n, \tag{1.7}$$

where $\gamma \in \left(0, \dfrac{2}{\|A\|^2}\right)$. They also established a convergence result for this algorithm.

Recently, Moudafi and Al-Shemas [32] introduced the following Split Equality Fixed Point Problem (SEFPP) which generalizes the SFP (1.5): find

$$x \in C\colon = F(T), \quad y \in Q := F(S) \quad \text{such that} \quad Ax = By, \tag{1.8}$$

where $C \subset H_1,\ Q \subset H_2$ are two nonempty, closed and convex sets, $A\colon H_1 \to H_3,\ B\colon H_2 \to H_3$ are two bounded linear operators, $F(T)$ and $F(S)$ denotes the sets of fixed points of operators T and S defined on H_1 and H_2, respectively. Note that if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in (1.8), then problem (1.8) reduces to problem (1.5). Furthermore, Moudafi established the weak convergence result for problem (1.8). Some other authors have studied SEFPP for single-valued mappings in Hilbert spaces (see, for example, [6, 17, 21, 31, 34, 35, 38, 41, 46]).

The approximation of fixed point of multivalued mappings with respect to Hausdorff metric has been an area of great research interest due to its numerous applications in various fields such as game theory and mathematical economics. Thus, it is ideal to extend the known results on SEFPP for single-valued mappings to multivalued mappings.

Wu et al. [43] introduced the Multiple-Set Split Equality Fixed Point Problem (MSSEFPP) for finite families of multivalued quasinonexpansive mappings, which is to find

$$x \in C = \bigcap_{j=1}^{N} F(R_1^j) \quad \text{and} \quad y \in Q = \bigcap_{j=1}^{N} F(R_2^j) \quad \text{such that} \quad Ax = By, \tag{1.9}$$

where N is a positive integer, $A\colon H_1 \to H_3$ and $B\colon H_2 \to H_3$ are two bounded linear operators, $R_i^j\colon H \to CB(H_i),\ i = 1, 2,\ j = 1, 2, \ldots, N,$ is a family of multivalued quasinonexpansive mappings. They established strong convergence result to a solution of problem (1.9).

Chang et al. [15] studied the MSSFP for a countable family of multivalued quasinonexpansive mappings S_i and a total asymptotically strict pseudocontractive mapping T, which is to find

$$x^* \in C = \bigcap_{i=1}^{\infty} F(S_i) \quad \text{such that} \quad Ax^* \in Q = F(T), \tag{1.10}$$

where $A\colon H_1 \to H_2$ is a bounded linear map.

Shehu [40] introduced the following MSSEFPP for infinite families of multivalued quasinonexpansive mappings: find

$$x \in \bigcap_{i=1}^{\infty} F(S_i) \quad \text{and} \quad y \in \bigcap_{i=1}^{\infty} F(T_i) \quad \text{such that} \quad Ax = By, \tag{1.11}$$

where $A\colon H_1 \to H_3$ and $B\colon H_2 \to H_3$ are bounded linear operators, $S_i\colon H_1 \to CB(H_1)$ and $T_i\colon H_2 \to CB(H_2),\ i = 1, 2, \ldots,$ are two infinite families of multivalued quasinonexpansive mappings. With these assumptions, he proposed the following algorithm for finding a solution of problem (1.11):

$$\begin{aligned} u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} &= t_n u + (a_{0,n} - t_n)u_n + \sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n}, \quad w_{i,n} \in S_i u_n, \\ v_n &= y_n - \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} &= t_n v + (a_{0,n} - t_n)v_n + \sum_{i=1}^{\infty} \alpha_{i,n} z_{i,n}, \quad z_{i,n} \in T_i u_n, \end{aligned} \tag{1.12}$$

where

$$\{\gamma_n\} \in \left(\varepsilon,\ \frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \varepsilon\right), \quad n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n\colon Ax_n - By_n \neq \neq 0\}$. Shehu [40] established the strong convergence result for problem (1.11) using algorithm (1.12).

Based on the works of Chidume et al. [17], Wu et al. [43], Chang et al. [15] and Chidume et al. [18] introduced the following algorithm for solving the MSSEFPP for countable families of multivalued demicontractive mappings:

$$\begin{aligned} x_{n+1} &= a_0 \left(x_n - \gamma A^*(Ax_n - By_n)\right) + \sum_{i=1}^{\infty} \alpha_i z_n^i, \\ y_{n+1} &= a_0 \left(y_n - \gamma B^*(Ax_n - By_n)\right) + \sum_{i=1}^{\infty} \alpha_j w_n^j \quad \text{for all} \quad n \geq 1, \end{aligned} \tag{1.13}$$

where $z_n^i \in S_i\left(x_n - \gamma A^*(Ax_n - By_n)\right),\ w_n^j \in T_i\left(y_n - \gamma B^*(Ax_n - By_n)\right),\ A\colon H_1 \to H_3$ and $B\colon H_2 \to H_3$ are bounded linear maps, $S_i\colon H_1 \to CB(H_1),\ i = 1, 2, \ldots,$ and $T_j\colon H_2 \to CB(H_2),$ $j = 1, 2, \ldots,$ are two families of multivalued demicontractive mappings. Chidume et al. [18] proved weak and strong convergence result for problem (1.11) using the iterative scheme (1.13).

The theory of Variational Inequality Problems (VIP) is well known, developed and has been applied to solve numerous problems in many fields such as; sciences, social sciences, engineering and management. There are several monographs on variational inequalities, we mention here a few

[1, 3, 20, 26, 27]. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $f: H \to H$ be an operator. The VIP defined for C and f is to find $x^* \in C$ such that

$$x^* \in (VI(C,f)), \quad \text{i.e.,} \quad \langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C. \tag{1.14}$$

Let $T: H \to H$ be an operator such that $F(T) \neq \varnothing$ and $f: H \to H$ be an operator. The Hierarchical Variational Inequality Problem (HVIP) is to find $x^* \in F(T)$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in F(T). \tag{1.15}$$

Let f be an α-inverse strongly monotone operator on C and $N_C z$ be the normal cone of C at the point $z \in C$, we define the following set-valued operator $M: C \to 2^C$ by

$$Mz = fz + N_C z.$$

Then M is maximal monotone. Furthermore, $0 \in M(x^*) \Longleftrightarrow x^* \in \mathrm{VI}(C,f)$ (see [39], Theorem 3).

Several other methods for solving (1.14) and (1.15) have been investigated in the literature (see, for example, [2, 28 – 30, 25, 45] and the references therein).

In 2014, Ansari et al. [1] introduced a split-type problem by combining a Split Fixed Point Problem (SFPP) and a HVIP; thus, presenting the Split Hierarchical Variational Inequality Problem (SHVIP), which is to determine $x^* \in F(T)$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in F(T), \tag{1.16}$$

and such that $Ax^* \in F(S)$ satisfies

$$\langle h(Ax^*), y - Ax^* \rangle \geq 0, \quad y \in F(S), \tag{1.17}$$

where H_1, H_2 are two real Hilbert spaces, $T: H_1 \to H_1$ is a strongly nonexpansive operator such that $F(T) \neq \varnothing$, $S: H_2 \to H_2$ is a strongly nonexpansive cutter operator such that $F(S) \neq \varnothing$, $A: H_1 \to H_2$ is a bounded linear operator with $R(A) \cap F(S) \neq \varnothing$, f(resp., h) is a monotone and continuous operator on H_1(resp., H_2). With these assumptions, they proposed the following iterative scheme for finding a solution of problem (1.16), (1.17):

$$\begin{gathered} x_1 \in H_1, \\ y_n := x_n - \gamma A^*(I - S(I - \beta_k h))Ax_n, \\ x_{n+1} := T(I - \alpha_n f)y_n, \end{gathered} \tag{1.18}$$

where $\gamma \in \left(0, \dfrac{2}{\|A\|^2}\right)$, $\{\alpha_n\}$, $T\{\beta_k\} \subset (0, +\infty)$. They proved that the sequence generated by (1.18) converges weakly to a solution of (1.16), (1.17).

Censor et al. [14] introduced the general Common Solutions to Variational Inequalities Problem (CSVIP), which consist of finding common solutions to unrelated variational inequalities for a finite number of sets. That is, find $x^* \in \bigcap_{i=1}^N K_i$ such that, for each $i = 1, 2, \ldots, N$,

$$\langle A_i(x^*), x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in K_i, \quad i = 1, 2, \ldots, N, \tag{1.19}$$

where $A_i\colon H \to H$ is any operator for each $i = 1,2,\ldots,N$ and K_i is a nonempty, closed and convex subset of H. They obtained the solution of problem (1.19) by considering first, a case where $i = 1,2$ and later obtain the result of the problem for $i = 1,2,\ldots,N$.

Let H_1, H_2, H_3 be real Hilbert spaces and for each $l = 1,2,\ldots,N,\ r = 1,2,\ldots,m$, let C_l and Q_r be nonempty, closed and convex subsets of $H_1,\ H_2$, respectively. Let $T_i\colon H_1 \to CB(H_1)$, $i = 1,2,\ldots,$ and $S_j\colon H_2 \to CB(H_2),\ j = 1,2,\ldots,$ be two countable families of multivalued mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \varnothing$ and $\bigcap_{j=1}^{\infty} F(S_j) \neq \varnothing$. Let $f_l\colon C_l \to C_l,\ h_r\colon Q_r \to Q_r$ be α_l, (resp., μ_r)-inverse strongly monotone operators, and $A\colon H_1 \to H_3,\ B\colon H_2 \to H_3$ be bounded linear operators. Motivated by the works of Zhoa [46], Shehu [40], Chidume et al. [18], Ansari et al. [1] and Censor et al. [14], we study the following problem: find

$$(\bar{x},\bar{y}) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$$

such that

$$\langle f_l(\bar{x}), x - \bar{x}\rangle \geq 0 \quad \forall\, x \in C_l, \quad l = 1,2,\ldots,N, \tag{1.20}$$

$$\langle h_r(\bar{y}),\ y - \bar{y}\rangle \geq 0 \quad \forall\, y \in Q_r, \quad r = 1,2,\ldots,m, \quad \text{and such that} \quad A\bar{x} = B\bar{y}. \tag{1.21}$$

Problem (1.20), (1.21) is equivalent to finding $(\bar{x},\bar{y}) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ such that

$$(\bar{x},\bar{y}) \in \bigcap_{l=1}^{N} VI(C_l, f_l) \times \bigcap_{r=1}^{m} VI(Q_r, h_r) \quad \text{and} \quad A\bar{x} = B\bar{y}. \tag{1.22}$$

Furthermore, we propose an iterative scheme and using the iterative scheme, we state and prove a strong convergence result for the approximation of a solution of (1.22). Finally, we applied our result to study related problems. Our theorem extends and complements the result of Shehu [40], Ansari et al. [1], Censor et al. [14] and a host of other results.

2. Preliminaries. We state some known and useful results which will be needed in the proof of our main theorem. We denote the strong and weak convergence by "$\longrightarrow$" and "$\rightharpoonup$", respectively, and the solution set of (1.22) by Γ defined by

$$\Gamma := \left\{ (\bar{x},\bar{y}) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j) : (\bar{x},\bar{y}) \in \bigcap_{l=1}^{N} VI(C_l, f_l) \times \bigcap_{r=1}^{m} VI(Q_r, h_r) \text{ and } A\bar{x} = B\bar{y} \right\}.$$

Let H be a real Hilbert space and $T\colon H \to 2^H$ be a multivalued mapping. Then T is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset H$ such that $x_n \rightharpoonup x^*$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, we have that $x^* \in Tx^*$ (i.e., $x^* \in F(T)$).

Lemma 2.1. *Let H be a Hilbert space, then*

$$2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \quad \forall\, x,y \in H.$$

Lemma 2.2 [19]. *Let H be a real Hilbert space and $\{x_i\}_{i\geq 1}$ be a bounded sequence in H. For $\alpha_i \in (0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$, the following identity holds*:

$$\left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\|^2 = \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \sum_{1\leq i<j<\infty} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3 [18]. *Let K be a nonempty subset of a real Hilbert space H and let $T\colon K \to CB(K)$ be a multivalued k-demicontractive mapping. Assume that for every $p \in F(T)$, $Tp = \{p\}$. Then*

$$\mathcal{H}(Tx, Tp) \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\|x-p\| \quad \forall x \in K,\ p \in F(T).$$

Lemma 2.4 [16]. *Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Let f be a mapping of C into H and $x^* \in C$, then for $\rho > 0$, $x^* = P_C(I-\rho f)x^*$ if and only if $x^* \in VI(C, f)$.*

Lemma 2.5 [16]. *Let H be a Hilbert space, then, for all $x,\ y \in H$ and $\alpha \in (0,1)$, we have*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.6 [44]. *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main result.

Theorem 3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces and, for each $l = 1, 2, \ldots, N$, $r = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T_i\colon H_1 \to CB(H_1)$, $i = 1, 2, \ldots,$ and $S_j\colon H_2 \to CB(H_2)$, $j = 1, 2, \ldots,$ be two families of multivalued type-one demicontractive-type mappings, with constants k_i and k_j, respectively, such that T_i and S_j are demiclosed at 0. Let $f_l\colon C_l \to C_l$, $h_r\colon Q_r \to Q_r$ be μ_l (resp., ν_r)-inverse strongly monotone operators and $A\colon H_1 \to H_3$, $B\colon H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \ne \varnothing$ and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that, for some $\varepsilon > 0$,*

$$\gamma_n \in \left(\varepsilon,\ \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \varepsilon\right), \quad n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n\colon Aw_n - Bz_n \ne 0\}$.

Let $u,\ x_1 \in H_1$ and $v,\ y_1 \in H_2$ be arbitrary and the sequence $(\{x_n\},\ \{y_n\})$ be generated by

$$\begin{gathered} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ z_n = (1-\alpha_n)y_n + \alpha_n v, \\ u_n = P_{C_N}(I-\lambda f_N)\circ P_{C_{N-1}}(I-\lambda f_{N-1})\circ\ldots\circ P_{C_1}(I-\lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{Q_m}(I-\lambda h_m)\circ P_{Q_{m-1}}(I-\lambda h_{m-1})\circ\ldots\circ P_{Q_1}(I-\lambda h_1)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = \beta_0 u_n + \sum_{i=1}^{\infty}\beta_i g_n^i, \\ y_{n+1} = \beta_0 v_n + \sum_{j=1}^{\infty}\beta_j h_n^j \quad \textit{for all} \quad n \ge 1, \end{gathered} \tag{3.1}$$

where $0 < \lambda < 2\mu,\ 2\nu,\ \mu := \min\{\mu_l,\ l = 1, 2, \ldots, N\},\ \nu := \min\{\nu_r,\ r = 1, 2, \ldots, m\}$ *and* A^*, B^* *are the adjoint of* A *and* B, *respectively,* $g_n^i \in P_{T_i} u_n,\ z_n^j \in P_{S_j} v_n$ *and* $P_{T_i} u_n := \{g_n^i \in T_i u_n : \|g_n^i - u_n\| = d(u_n, T_i u_n)\}$, *with conditions*:

(i) $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $k \in (0,1)$, *where* $k := \max\{k_1, k_2\},\ k_1 = \sup_{i\geq 1}\{k_i\},\ k_2 = \sup_{j\geq 1}\{k_j\} \in (0,1)$,

(iii) $\beta_0 \in (k,1),\ \beta_i, \beta_j \in (0,1),\ i,j = 1,2,\ldots,$ *such that* $\sum_{i=0}^{\infty} \beta_i = 1$ *and* $\sum_{j=0}^{\infty} \beta_j = 1$,

(iv) *for each* $x^* \in \bigcap_{i=1}^{\infty} F(T_i);\ T_i x^* = \{x^*\}$ *and for each* $y^* \in \bigcap_{j=1}^{\infty} F(S_j);\ S_j y^* = \{y^*\}$.

Then $(\{x_n\}, \{y_n\})$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Γ.

Proof. First, we show that, for each $i = 1, 2, \ldots,$ $\{g_n^i\}$ is bounded. By using Lemma 2.3, we have

$$\|g_n^i - x^*\| \leq \mathcal{H}(T_i u_n, T_i x^*) \leq \frac{1+\sqrt{k_1}}{1-\sqrt{k_1}} \|u_n - x^*\| := P_n.$$

Hence, $\{g_n^i\}_{i\geq 1}$ is bounded. Similarly, $\{h_n^j\}_{j\geq 1}$ is bounded.

Let $(x^*, y^*) \in \Gamma$, $\Phi^N = P_{C_N}(I - \lambda f_N) \circ P_{C_{N-1}}(I - \lambda f_{N-1}) \circ \ldots \circ P_{C_1}(I - \lambda f_1)$, where $\Phi^0 = I$ and $\Psi^m = P_{Q_m}(I - \lambda h_m) \circ P_{Q_{m-1}}(I - \lambda h_{m-1}) \circ \ldots \circ P_{Q_1}(I - \lambda h_1)$, where $\Psi^0 = I$, then, from (3.1), we obtain

$$\|u_n - x^*\|^2 = \|\Phi^N (w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*\|^2 =$$
$$= \|P_{C_N}(I - \lambda f_N)\Phi^{N-1} (w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*\|^2 \leq$$
$$\leq \|\Phi^{N-1} (w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*\|^2 \leq \ldots$$
$$\ldots \leq \|w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*\|^2 =$$
$$= \|w_n - x^*\|^2 - 2\gamma_n \langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle + \gamma_n^2 \|A^*(Aw_n - By_n)\|^2. \qquad (3.2)$$

From Lemma 2.1 and noting that A^* is the adjoint of A, we get

$$-2\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle = -2\langle Aw_n - Ax^*, Aw_n - Bz_n\rangle =$$
$$= -\|Aw_n - Ax^*\|^2 - \|Aw_n - Bz_n\|^2 + \|Bz_n - Ax^*\|^2. \qquad (3.3)$$

Substituting (3.3) into (3.2), we have

$$\|u_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \gamma_n \|Aw_n - Ax^*\|^2 - \gamma_n \|Aw_n - Bz_n\|^2 +$$
$$+ \gamma_n \|Bz_n - Ax^*\|^2 + \gamma_n^2 \|A^*(Aw_n - Bz_n)\|^2. \qquad (3.4)$$

Similarly, from (3.1), we obtain

$$\|v_n - y^*\|^2 \leq \|z_n - y^*\|^2 - \gamma_n \|Bz_n - By^*\|^2 - \gamma_n \|Aw_n - Bz_n\|^2 +$$
$$+ \gamma_n \|Aw_n - By^*\|^2 + \gamma_n^2 \|B^*(Aw_n - Bz_n)\|^2. \qquad (3.5)$$

From (3.1), Lemma 2.5 and adding inequality (3.4) and (3.5) together with the fact that $Ax^* = By^*$, we get

$$\|u_n - x^*\|^2 + \|v_n - y^*\|^2 \leq \|w_n - x^*\|^2 + \|z_n - y^*\|^2 - \gamma_n \big[2\|Aw_n - Bz_n\|^2 -$$

$$-\gamma_n\left(\|A^*(Aw_n-Bz_n)\|^2+\|B^*(Aw_n-Bz_n)\|^2\right)\Big]\le$$
$$\le\|w_n-x^*\|^2+\|z_n-y^*\|^2=$$
$$=\|(1-\alpha_n)x_n+\alpha_n u-x^*\|^2+\|(1-\alpha_n)y_n+\alpha_n v-y^*\|^2=$$
$$=\|(1-\alpha_n)(x_n-x^*)+\alpha_n(u-x^*)\|^2+\|(1-\alpha_n)(y_n-y^*)+\alpha_n(v-y^*)\|^2\le$$
$$\le(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\|u-x^*\|^2+(1-\alpha_n)\|y_n-y^*\|^2+\alpha_n\|v-y^*\|^2=$$
$$=(1-\alpha_n)\left[\|x_n-x^*\|^2+\|y_n-y^*\|^2\right]+\alpha_n\left[\|u-x^*\|^2+\|v-y^*\|^2\right]. \qquad (3.6)$$

From (3.1), Lemma 2.2 and the fact that T is of type-one demicontractive-type mapping, we have

$$\|x_{n+1}-x^*\|^2=\|\beta_0 u_n+\sum_{i=1}^{\infty}\beta_i g_n^i-x^*\|^2=$$
$$=\|\beta_0(u_n-x^*)+\sum_{i=1}^{\infty}\beta_i(g_n^i-x^*)\|^2=$$
$$=\beta_0\|u_n-x^*\|^2+\sum_{i=1}^{\infty}\beta_i\|g_n^i-x^*\|^2-\sum_{i=1}^{\infty}\beta_0\beta_i\|u_n-g_n^i\|^2-\sum_{1\le i<j<\infty}\beta_i\beta_j\|g_n^i-g_n^j\|^2\le$$
$$\le\beta_0\|u_n-x^*\|^2+\sum_{i=1}^{\infty}\beta_i\mathcal{H}^2(T_iu_n,T_ix^*)-\sum_{i=1}^{\infty}\beta_0\beta_i\|u_n-g_n^i\|^2\le$$
$$\le\beta_0\|u_n-x^*\|^2+\sum_{i=1}^{\infty}\beta_i\left[\|u_n-x^*\|^2+k_1d^2(u_n,T_iu_n)\right]-\sum_{i=1}^{\infty}\beta_0\beta_i\|u_n-g_n^i\|^2=$$
$$=\beta_0\|u_n-x^*\|^2+\sum_{i=1}^{\infty}\beta_i\left[\|u_n-x^*\|^2+k_1\|u_n-g_n^i\|^2\right]-\sum_{i=1}^{\infty}\beta_0\beta_i\|u_n-g_n^i\|^2=$$
$$=\|u_n-x^*\|^2+(k_1-\beta_0)\sum_{i=1}^{\infty}\beta_i\|u_n-g_n^i\|^2\le$$
$$\le\|u_n-x^*\|^2. \qquad (3.7)$$

Similarly, we obtain

$$\|y_{n+1}-y^*\|^2\le\|v_n-y^*\|^2. \qquad (3.8)$$

Adding (3.7) and (3.8), and using (3.6), we get

$$\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\le\|u_n-x^*\|^2+\|v_n-y^*\|^2\le$$
$$\le(1-\alpha_n)\left[\|x_n-x^*\|^2+\|y_n-y^*\|^2\right]+\alpha_n\left[\|u-x^*\|^2+\|v-y^*\|^2\right]\le$$
$$\le\max\left\{\|x_n-x^*\|^2+\|y_n-y^*\|^2,\|u-x^*\|^2+\|v-y^*\|^2\right\}\le\ldots$$
$$\ldots\le\max\left\{\|x_0-x^*\|^2+\|y_0-y^*\|^2,\|u-x^*\|^2+\|v-y^*\|^2\right\}.$$

Therefore, $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is bounded. Consequently, $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$, and $\{By_n\}$ are bounded.

We consider two cases.

Case 1. Assume that $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is monotone decreasing, then $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is convergent, thus

$$\lim_{n\to\infty} \left[\left(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\right) - \left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right)\right] = 0.$$

From (3.6) and (3.8), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \le \|w_n - x^*\|^2 + \|z_n - y^*\|^2 - \gamma_n\big[2\|Aw_n - Bz_n\|^2 - \\ -\gamma_n \left(\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2\right)\big] \le \\ \le (1-\alpha_n)\left[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right] + \alpha_n\left[\|u - x^*\|^2 + \|v - y^*\|^2\right] - \\ -\gamma_n\big[2\|Aw_n - Bz_n\|^2 - \gamma_n\big(\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2\big)\big],\end{aligned} \tag{3.9}$$

which implies

$$\begin{aligned}\gamma_n^2\left(\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2\right) \le \\ \le (1-\alpha_n)\left[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right] + \alpha_n\left[\|u - x^*\|^2 + \|v - y^*\|^2\right] - \\ -\left[\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\right] \to 0, \quad \text{as} \quad n \to \infty.\end{aligned} \tag{3.10}$$

By the condition

$$\gamma_n \in \left(\varepsilon,\ \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \varepsilon\right), \quad n \in \Omega,$$

we obtain $\left(\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2\right) \to 0$ as $n \to \infty$. Since $Aw_n - Bz_n = 0$, if $n \notin \Omega$, we get

$$\lim_{n\to\infty} \|A^*(Aw_n - Bz_n)\|^2 = \lim_{n\to\infty} \|B^*(Aw_n - Bz_n)\|^2 = 0. \tag{3.11}$$

From (3.1), we have

$$\lim_{n\to\infty} \|w_n - x_n\|^2 = \lim_{n\to\infty} \alpha_n^2 \|u - x_n\|^2 = 0 \tag{3.12}$$

and

$$\lim_{n\to\infty} \|z_n - y_n\|^2 = \lim_{n\to\infty} \alpha_n^2 \|v - y_n\|^2 = 0. \tag{3.13}$$

Let

$$a_n = w_n - \gamma_n A^*(Aw_n - Bz_n)$$

and

$$b_n = z_n + \gamma_n B^*(Aw_n - Bz_n).$$

Then

$$\lim_{n\to\infty}\|a_n - w_n\|^2 = \lim_{n\to\infty}\gamma_n^2\|A^*(Aw_n - Bz_n)\|^2 = 0 \tag{3.14}$$

and

$$\lim_{n\to\infty}\|b_n - z_n\|^2 = \lim_{n\to\infty}\gamma_n^2\|B^*(Aw_n - Bz_n)\|^2 = 0. \tag{3.15}$$

From (3.12) and (3.14), we obtain

$$\lim_{n\to\infty}\|a_n - x_n\|^2 = 0. \tag{3.16}$$

Also, from (3.13) and (3.15), we get

$$\lim_{n\to\infty}\|b_n - y_n\|^2 = 0. \tag{3.17}$$

From (3.1) and Lemma 2.1, we have

$$\begin{gathered}\|u_n - x^*\|^2 = \|P_{C_N}(I-\lambda f_N)\Phi^{N-1}a_n - x^*\|^2 \leq \\ \leq \langle u_n - x^*, \Phi^{N-1}a_n - x^*\rangle = \\ = \frac{1}{2}\left[\|u_n - x^*\|^2 + \|\Phi^{N-1}a_n - x^*\|^2 - \|u_n - \Phi^{N-1}a_n\|^2\right],\end{gathered} \tag{3.18}$$

which implies

$$\|u_n - \Phi^{N-1}a_n\|^2 \leq \|\Phi^{N-1}a_n - x^*\|^2 - \|u_n - x^*\|^2. \tag{3.19}$$

Similarly, we obtain

$$\|v_n - \Psi^{m-1}b_n\|^2 \leq \|\Psi^{m-1}b_n - y^*\|^2 - \|v_n - y^*\|^2. \tag{3.20}$$

Adding (3.19) and (3.20), we get

$$\begin{gathered}\|u_n - \Phi^{N-1}a_n\|^2 + \|v_n - \Psi^{m-1}b_n\|^2 \leq \\ \leq \|\Phi^{N-1}a_n - x^*\|^2 + \|\Psi^{m-1}b_n - y^*\|^2 - \left(\|u_n - x^*\|^2 + \|v_n - y^*\|^2\right) \leq \\ \leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - \left(\|u_n - x^*\|^2 + \|v_n - y^*\|^2\right) \leq \\ \leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - \left(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\right) = \\ = \|a_n - x^*\|^2 - \|x_n - x^*\|^2 + \|b_n - y^*\|^2 - \|y_n - y^*\|^2 + \|x_n - x^*\|^2 + \\ + \|y_n - y^*\|^2 - \left(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\right) \to 0 \quad \text{as} \quad n\to\infty,\end{gathered} \tag{3.21}$$

which implies

$$\lim_{n\to\infty}\|\Phi^N a_n - \Phi^{N-1}a_n\| = \lim_{n\to\infty}\|\Psi^m b_n - \Phi^{m-1}b_n\| = 0. \tag{3.22}$$

By the same argument as (3.18) – (3.21), we have

$$\|\Phi^{N-1}a_n - \Phi^{N-2}a_n\|^2 + \|\Psi^{m-1}b_n - \Psi^{m-2}b_n\|^2 \leq$$
$$\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - \left(\|\Phi^{N-1}a_n - x^*\|^2 + \|\Psi^{m-1}b_n - y^*\|^2\right) \leq$$
$$\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - \left(\|u_n - x^*\|^2 + \|v_n - y^*\|^2\right) \leq$$
$$\leq \|a_n - x^*\|^2 + \|b_n - y^*\|^2 - \left(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\right) \to 0, \tag{3.23}$$

which implies

$$\lim_{n\to\infty} \|\Phi^{N-1}a_n - \Phi^{N-2}a_n\| = \lim_{n\to\infty} \|\Psi^{m-1}b_n - \Psi^{m-2}b_n\| = 0. \tag{3.24}$$

Continuing in the same manner, we obtain

$$\lim_{n\to\infty} \|\Phi^{N-2}a_n - \Phi^{N-3}a_n\| = \ldots = \lim_{n\to\infty} \|\Phi^2 a_n - \Phi^1 a_n\| = 0 \tag{3.25}$$

and

$$\lim_{n\to\infty} \|\Psi^{m-2}b_n - \Psi^{m-3}b_n\| = \ldots = \lim_{n\to\infty} \|\Psi^2 b_n - \Psi^1 b_n\| = 0. \tag{3.26}$$

From (3.22), (3.24), (3.25) and (3.26), we conclude that

$$\lim_{n\to\infty} \|\Phi^l a_n - \Phi^{l-1}a_n\| = 0, \quad l = 1, 2, \ldots, N, \tag{3.27}$$

and

$$\lim_{n\to\infty} \|\Psi^r b_n - \Psi^{r-1}b_n\| = 0, \quad r = 1, 2, \ldots, m. \tag{3.28}$$

Since f_l and h_r are Lipschitz continuous (by Remark 1.2), from (3.27) and (3.28), we have

$$\lim_{n\to\infty} \|f_l\Phi^l a_n - f_l\Phi^{l-1}a_n\| = 0 \tag{3.29}$$

and

$$\lim_{n\to\infty} \|h_r\Psi^r b_n - h_r\Psi^{r-1}b_n\| = 0. \tag{3.30}$$

Also

$$\|u_n - a_n\| \leq \|u_n - \Phi^{N-1}a_n\| + \|\Phi^{N-1}a_n - \Phi^{N-2}a_n\| +$$
$$+ \|\Phi^{N-2}a_n - \Phi^{N-3}a_n\| + \ldots + \|\Phi^1 a_n - a_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

which implies

$$\lim_{n\to\infty} \|u_n - a_n\| = 0. \tag{3.31}$$

Similarly, we have

$$\lim_{n\to\infty} \|v_n - b_n\| = 0. \tag{3.32}$$

From (3.14) and (3.31), we obtain

$$\lim_{n\to\infty} \|u_n - w_n| = 0. \tag{3.33}$$

Also, from (3.15) and (3.32), we get

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.34}$$

From (3.12) and (3.33), we have

$$\lim_{n\to\infty} \|x_n - u_n\| \leq \lim_{n\to\infty} \big[\|x_n - w_n\| + \|w_n - u_n\|\big] = 0. \tag{3.35}$$

Similarly, from (3.13) and (3.34), we obtain

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{3.36}$$

From (3.7), we get

$$\sum_{i=1}^{\infty} \beta_i(\beta_0 - k_1)\|u_n - g_n^i\|^2 \leq \|u_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.37}$$

Similarly, we have

$$\sum_{j=1}^{\infty} \beta_j(\beta_0 - k_2)\|v_n - h_n^j\|^2 \leq \|v_n - y^*\|^2 - \|y_{n+1} - y^*\|^2. \tag{3.38}$$

Adding (3.37) and (3.38), and from (3.6) we obtain

$$\sum_{i=1}^{\infty} \beta_i(\beta_0 - k_1)\|u_n - g_n^i\|^2 + \sum_{j=1}^{\infty} \beta_j(\beta_0 - k_2)\|v_n - h_n^j\|^2 \leq$$

$$\leq \|u_n - x^*\|^2 + \|v_n - y^*\|^2 - \big(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\big) \leq$$

$$\leq (1 - \alpha_n)\left[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right] +$$

$$+\alpha_n\left[\|u - x^*\|^2 + \|v - y^*\|^2\right] - \big(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\big)$$

and, for each $i,\ j = 1, 2, \ldots,$ we get

$$\beta_i(\beta_0 - k_1)\|u_n - g_n^i\|^2 + \beta_j(\beta_0 - k_2)\|v_n - h_n^j\|^2 \leq (1 - \alpha_n)\left[\|x_n - x^*\|^2 + \|y_n - y^*\|^2\right] +$$

$$+\alpha_n\left[\|u - x^*\|^2 + \|v - y^*\|^2\right] - \big(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2\big) \to 0, \quad \text{as} \quad n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \beta_i(\beta_0 - k_1)\|u_n - g_n^i\|^2 = \lim_{n\to\infty} \beta_j(\beta_0 - k_2)\|v_n - h_n^j\|^2 = 0.$$

By condition (iii), we have

$$\lim_{n\to\infty} \|u_n - g_n^i\|^2 = \lim_{n\to\infty} \|v_n - h_n^j\|^2 = 0.$$

Hence, we obtain

$$\lim_{n\to\infty} d(u_n, T_i u_n) = \lim_{n\to\infty} \|u_n - g_n^i\| = 0 \tag{3.39}$$

and

$$\lim_{n\to\infty} d(v_n, S_j v_n) = \lim_{n\to\infty} \|v_n - h_n^j\| = 0. \tag{3.40}$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ (without loss of generality, still denoted by $\{x_n\}$) such that $\{x_n\}$ converges weakly to $\bar{x} \in \bigcap_{l=1}^N C_l$. By (3.35) and (3.12), we have that $\{u_n\}$ and $\{w_n\}$ converges weakly to $\bar{x}$ and by the demicloseness of T_i at 0 and (3.39), we get that $\bar{x} \in F(T_i)$ for each $i = 1, 2, \ldots$. Similarly, since $\{y_n\}$ is bounded, there exists a subsequence of $\{y_n\}$ (without loss of generality, still denoted by $\{y_n\}$) such that $\{y_n\}$ converges weakly to $\bar{y} \in \bigcap_{r=1}^m Q_r$. By (3.36) and (3.13), we obtain that $\{v_n\}$ and $\{z_n\}$ converges weakly to $\bar{y}$ and by the demicloseness of S_j at 0 and (3.40), we have that $\bar{y} \in F(S_j)$, for each $j = 1, 2, \ldots$. Hence, $(\bar{x}, \bar{y}) \in \bigcap_{i=1}^\infty F(T_i) \times \bigcap_{j=1}^\infty F(S_j)$.

Next, we show that $A\bar{x} = B\bar{y}$. Since A and B are bounded linear operators, we get that $Aw_n \rightharpoonup A\bar{x}$ and $Bz_n \rightharpoonup B\bar{y}$. Using the condition on $\{\gamma_n\}$ and (3.11) in (3.9), we obtain

$$\lim_{n\to\infty} \|Aw_n - Bz_n\|^2 = 0.$$

By weakly semicontinuity of the norm, we have

$$\|A\bar{x} - B\bar{y}\| \le \liminf_{n\to\infty} \|Aw_n - Bz_n\| = 0.$$

That is,

$$A\bar{x} = B\bar{y}.$$

We now show that $(\bar{x}, \bar{y}) \in \bigcap_{l=1}^N VI(C_l, f_l) \times \bigcap_{r=1}^m VI(Q_r, h_r)$, that is $\bar{x}$ satisfies $\langle f_l(\bar{x}), x - \bar{x}\rangle \ge$ $\ge 0 \; \forall x \in \bigcap_{l=1}^N C_l$, and $\bar{y}$ satisfies $\langle h_r(\bar{y}), y - \bar{y}\rangle \ge 0 \; \forall y \in \bigcap_{r=1}^m Q_r$.

Let $N_{C_l} z$ be the normal cone of C_l at a point $z \in C_l$, $l = 1, 2, \ldots, N$, we define the following set-valued operator $M_l \colon C_l \to 2^{C_l}$, for each $l = 1, 2, \ldots, N$, by

$$M_l z = f_l z + N_{C_l} z.$$

Then M_l is maximal monotone for each $l = 1, 2, \ldots, N$. Let $(z, w) \in G(M_l)$, then $w - f_l z \in N_{C_l} z$. For $\Phi^l a_n \in C_l$, we have

$$\langle z - \Phi^l a_n, w - f_l z\rangle \ge 0, \quad l = 1, 2, \ldots, N. \tag{3.41}$$

From $\Phi^l a_n = P_{C_l}(I - \lambda f_l)\Phi^{l-1} a_n$, we obtain $\langle z - \Phi^l a_n, \Phi^l a_n - (\Phi^{l-1} a_n - \lambda f_l \Phi^{l-1} a_n)\rangle \ge 0$ for each $l = 1, 2, \ldots, N$, which implies

$$\left\langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} + f_l \Phi^{l-1} a_n \right\rangle \ge 0$$

for each $l = 1, 2, \ldots, N$. From (3.41), we get

$$\langle z - \Phi^l a_n, w\rangle \geq \langle z - \Phi^l a_n, f_l z\rangle \geq$$

$$\geq \langle z - \Phi^l a_n, f_l z\rangle - \left\langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} + f_l \Phi^{l-1} a_n \right\rangle =$$

$$= \left\langle z - \Phi^l a_n, f_l z - f_l \Phi^{l-1} a_n - \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} \right\rangle =$$

$$= \langle z - \Phi^l a_n, f_l z - f_l \Phi^l a_n\rangle + \langle z - \Phi^l a_n, f_l \Phi^l a_n - f_l \Phi^{l-1} a_n\rangle - \left\langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} \right\rangle \geq$$

$$\geq \langle z - \Phi^l a_n, f_l \Phi^l a_n - f_l \Phi^{l-1} a_n\rangle - \left\langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} \right\rangle. \tag{3.42}$$

Using (3.27) and (3.29) together with the fact that $\{u_n\} = \{\Phi^l a_n\}$ converges weakly to $\bar{x}$, from (3.42), we obtain that $\langle z - \bar{x}, w\rangle \geq 0$. Also M_l is maximal monotone for each $l = 1, 2, \ldots, N$, this gives us that $\bar{x} \in M_l^{-1}(0)$, which implies that $0 \in M_l(\bar{x})$ for each $l = 1, 2, \ldots, N$. Hence, $\bar{x} \in \bigcap_{l=1}^N VI(C_l, f_l)$, that is $\langle f_l(\bar{x}), z - \bar{x}\rangle \geq 0 \ \forall z \in \bigcap_{l=1}^N C_l$. In the same manner, we have that $\langle h_r(\bar{y}), y - \bar{y}\rangle \geq 0 \ \forall y \in \bigcap_{r=1}^m Q_r$. Hence, we obtain that $(\bar{x}, \bar{y}) \in \Gamma$.

Next, we show that $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$. From (3.6), we get

$$\|x_{n+1} - \bar{x}\|^2 + \|y_{n+1} - \bar{y}\|^2 \leq \|w_n - \bar{x}\|^2 + \|z_n - \bar{y}\|^2 =$$

$$= (1 - \alpha_n)^2 \|x_n - \bar{x}\|^2 + \alpha_n^2 \|u - \bar{x}\|^2 + 2(1 - \alpha_n)\alpha_n \langle x_n - \bar{x}, u - \bar{x}\rangle +$$

$$+ (1 - \alpha_n)^2 \|y_n - \bar{y}\|^2 + \alpha_n^2 \|v - \bar{y}\|^2 + 2(1 - \alpha_n)\alpha_n \langle y_n - \bar{y}, v - \bar{y}\rangle \leq$$

$$\leq (1 - \alpha_n)\left[\|x_n - \bar{x}\|^2 + \|y_n - \bar{y}\|^2\right] + \alpha_n\big[\alpha_n \|u - \bar{x}\|^2 +$$

$$+ 2(1 - \alpha_n)\langle x_n - \bar{x}, u - \bar{x}\rangle + \alpha_n \|v - \bar{y}\|^2 + 2(1 - \alpha_n)\langle y_n - \bar{y}, v - \bar{y}\rangle\big]. \tag{3.43}$$

Applying Lemma 2.6 to (3.43), we have that $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2. Assume that $\{\|x_n - x^*\|^2 + \|y_n - y^*\|^2\}$ is not monotone decreasing. Set $\Gamma_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ and let $\tau \colon \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} \colon k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \quad \text{for all} \quad n \geq n_0.$$

From (3.10), we have

$$\gamma_n^2 \left(\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2\right) \leq$$

$$\leq [\|x_{\tau(n)} - x^*\|^2 + \|y_{\tau(n)} - y^*\|^2] - [\|x_{\tau(n)+1} - x^*\|^2 +$$

$$+ \|y_{\tau(n)+1} - y^*\|^2] + \alpha_{\tau(n)}[\|u - x^*\|^2 + \|v - y^*\|^2] \leq$$

$$\leq \alpha_{\tau(n)}[\|u - x^*\|^2 + \|v - y^*\|^2].$$

Therefore,

$$\gamma_{\tau(n)}^2 \left(\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2\right) \to 0 \quad \text{as} \quad n \to \infty.$$

By the condition on $\{\gamma_{\tau(n)}\}$, we obtain

$$\left(\|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 + \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Note that $Aw_{\tau(n)} - Bz_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$. Hence,

$$\lim_{n\to\infty} \|A^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 = 0$$

and

$$\lim_{n\to\infty} \|B^*(Aw_{\tau(n)} - Bz_{\tau(n)})\|^2 = 0.$$

Following the same line of argument as in case 1, we can show that

$$\lim_{n\to\infty} \|\Phi^l a_{\tau(n)} - \Phi^{l-1} a_{\tau(n)}\| = \lim_{n\to\infty} \|\Psi^r b_{\tau(n)} - \Psi^{r-1} b_{\tau(n)}\| = 0,$$

$$l = 1, 2, \ldots, N, \quad r = 1, 2, \ldots, m,$$

$$\lim_{n\to\infty} d(u_{\tau(n)}, T_i u_{\tau(n)}) = \lim_{n\to\infty} d(v_{\tau(n)}, S_j v_{\tau(n)}) = 0 \quad \text{and} \quad (\{x_{\tau(n)}\}, \{y_{\tau(n)}\})$$

converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.

Now, for all $n \geq n_0$, from (3.43), we have

$$0 \leq [\|x_{\tau(n)+1} - x^*\|^2 + \|y_{\tau(n)+1} - y^*\|^2] - [\|x_{\tau(n)} - x^*\|^2 + \|y_{\tau(n)} - y^*\|^2] \leq$$

$$\leq (1 - \alpha_{\tau(n)})[\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2] - [\|x_{\tau(n)} - x^*\|^2 + \|y_{\tau(n)} - y^*\|^2] +$$

$$+\alpha_{\tau(n)}[\alpha_{\tau(n)}[\|u - \bar{x}\|^2 + \|v - \bar{y}\|^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle)],$$

which implies

$$\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2 \leq \alpha_{\tau(n)}[\|u - \bar{x}\|^2 + \|v - \bar{y}\|^2] +$$

$$+2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle) \to 0.$$

Hence,

$$\lim_{n\to\infty} (\|x_{\tau(n)} - \bar{x}\|^2 + \|y_{\tau(n)} - \bar{y}\|^2) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is clear that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.

Consequently, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty} \Gamma_n = 0$. That is $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

Theorem 3.1 is proved.

***Corollary* 3.1.** *Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T_i \colon H_1 \to CB(H_1),\ i = 1, 2, \ldots,$ and $S_j \colon H_2 \to CB(H_2),\ j = 1, 2, \ldots,$ be two families of multivalued type-one demicontractive-type mappings with constants k_i and k_j, respectively, such that for $i, j = 1, 2, \ldots, T_i$ and S_j are demiclosed at 0. Let $f \colon C \to C,\ h \colon Q \to Q$ be μ (resp., ν)-inverse strongly monotone operators and $A\colon H_1 \to H_3,\ B \colon H_2 \to H_3$ be bounded linear operators. Assume that the solution set*

$$\Gamma^* := \left\{ (\bar{x}, \bar{y}) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{i=1}^{\infty} F(S_j) \colon (\bar{x}, \bar{y}) \in VI(C, f) \times VI(Q, h) \ \ \text{and} \ \ A\bar{x} = B\bar{y} \right\} \neq \varnothing$$

and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon,\ \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \varepsilon \right), \quad n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n \colon Aw_n - Bz_n \neq \neq 0\}$.

Let $u,\ x_1 \in H_1$ and $v,\ y_1 \in H_2$ be arbitrary and the sequence $(\{x_n\},\ \{y_n\})$ be generated by

$$w_n = (1 - \alpha_n)x_n + \alpha_n u,$$

$$z_n = (1 - \alpha_n)y_n + \alpha_n v,$$

$$u_n = P_C(I - \lambda f)(w_n - \gamma_n A^*(Aw_n - Bz_n)),$$

$$v_n = P_Q(I - \lambda h)(z_n + \gamma_n B^*(Aw_n - Bz_n)),$$

$$x_{n+1} = \beta_0 u_n + \sum_{i=1}^{\infty} \beta_i g_n^i,$$

$$y_{n+1} = \beta_0 v_n + \sum_{j=1}^{\infty} \beta_j h_n^j \quad \text{for all} \quad n \geq 1,$$

where $0 < \lambda < 2\mu, 2\nu$ and A^*, B^* are the adjoint of A and B, respectively, $g_n^i \in P_{T_i} u_n,\ z_n^j \in \in P_{S_j} v_n,\ P_{T_i} u_n := \{g_n^i \in T_i u_n \colon \|g_n^i - u_n\| = d(u_n, T_i u_n)\}$, with conditions:

(i) *$\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,*

(ii) *$k \in (0, 1)$, where $k := \max\{k_1, k_2\},\ k_1 = \sup_{i\geq 1}\{k_i\},\ k_2 = \sup_{j\geq 1}\{k_j\} \in (0, 1)$,*

(iii) *$\beta_0 \in (k, 1),\ \beta_i, \beta_j \in (0, 1),\ \ i, j = 1, 2, \ldots,$ such that $\sum_{i=0}^{\infty} \beta_i = 1$ and $\sum_{j=0}^{\infty} \beta_j = 1$,*

(iv) *for each $x^* \in \bigcap_{i=1}^{\infty} F(T_i);\ T_i x^* = \{x^*\}$ and for each $y^* \in \bigcap_{j=1}^{\infty} F(S_j);\ S_j y^* = \{y^*\}$.*

Then $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$ in Γ^*.

***Corollary* 3.2.** *Let H_1, H_2 and H_3 be real Hilbert spaces and for each $l = 1, 2, \ldots, N,\ r = = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T_i \colon H_1 \to CB(H_1),\ i = 1, 2, \ldots,$ and $S_j \colon H_2 \to CB(H_2),\ j = 1, 2, \ldots,$ be two families of multivalued type-one quasinonexpansive mappings, such that T_i and S_j are demiclosed at 0. Let $f_l \colon C_l \to C_l,\ h_r \colon Q_r \to Q_r$ be μ_l (resp., ν_r)-inverse strongly monotone operators and $A \colon H_1 \to H_3$, $B \colon H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \varnothing$, and for each*

$x^* \in \bigcap_{i=1}^{\infty} F(T_i);\ T_i x^* = \{x^*\}$ *for each* $y^* \in \bigcap_{j=1}^{\infty} F(S_j),\ S_i y^* = \{y^*\}$. *Let the stepsize sequence* $\{\gamma_n\}$ *be chosen in such a way that*

$$\gamma_n \in \left(\varepsilon,\ \frac{2\|Aw_n - Bz_n\|^2}{\|A^*(Aw_n - Bz_n)\|^2 + \|B^*(Aw_n - Bz_n)\|^2} - \varepsilon\right), \quad n \in \Omega,$$

otherwise $\gamma_n = \gamma(\gamma$ *being any nonegative value), where the set of indexes*

$$\Omega = \{n\colon Aw_n - Bz_n \neq 0\}.$$

Let $u,\ x_1 \in H_1$ *and* $v,\ y_1 \in H_2$ *be arbitrary and the sequence* $(\{x_n\},\ \{y_n\})$ *be generated by*

$$w_n = (1-\alpha_n)x_n + \alpha_n u,$$

$$z_n = (1-\alpha_n)y_n + \alpha_n v,$$

$$u_n = P_{C_N}(I - \lambda f_N)\circ P_{C_{N-1}}(I - \lambda f_{N-1})\circ \ldots \circ P_{C_1}(I - \lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)),$$

$$v_n = P_{Q_m}(I - \lambda h_m)\circ P_{Q_{m-1}}(I - \lambda h_{m-1})\circ \ldots \circ P_{Q_1}(I - \lambda h_1)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \tag{3.44}$$

$$x_{n+1} = \beta_0 u_n + \sum_{i=1}^{\infty} \beta_i g_n^i,$$

$$y_{n+1} = \beta_0 v_n + \sum_{j=1}^{\infty} \beta_j h_n^j \quad \textit{for all} \quad n \geq 1,$$

where $0 < \lambda < 2\mu, 2\nu,\ \mu := \min\{\mu_l, l = 1, 2, \ldots, N\},\ \nu := \min\{\nu_r, r = 1, 2, \ldots, m\}$ *and* A^*, B^* *are the adjoint of* A *and* B, *respectively,* $g_n^i \in P_{T_i}u_n,\ z_n^j \in P_{S_j}v_n$ *and* $P_{T_i}u_n := \{g_n^i \in T_i u_n\colon \|g_n^i - u_n\| = d(u_n, T_i u_n)\}$. *Suppose* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $(\{x_n\}, \{y_n\})$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Γ.

4. Applications. ***4.1. Application to multiple-set split equality convex minimization problem.*** Let H_1, H_2 and H_3 be real Hilbert spaces and for each $l = 1, 2, \ldots, N,\ r = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $T_i\colon H_1 \to CB(H_1)$, $i = 1, 2, \ldots,$ and $S_j\colon H_2 \to CB(H_2),\ j = 1, 2, \ldots,$ be two countable families of multivalued type-one demicontractive-type mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \varnothing$ and $\bigcap_{j=1}^{\infty} F(S_j) \neq \varnothing$. Let $f_l\colon C_l \to C_l$, $h_r\colon Q_r \to Q_r$ be convex continuously differentiable operators and $A\colon H_1 \to H_3,\ B\colon H_2 \to H_3$ be bounded linear operators. Consider the following problem which we call the Multiple-Sets Split Equality Fixed Point Convex Minimization Problem (MSSEFPCMP): find $(\bar{x}, \bar{y}) \in \bigcap_{i=1}^{\infty} F(T_i) \times \times \bigcap_{j=1}^{\infty} F(S_j)$ such that for each $l = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, m$,

$$\bar{x} = \arg\min_{x\in C_l} f_l(x), \tag{4.1}$$

$$\bar{y} = \arg\min_{y\in Q_r} h_r(y) \qquad \text{and} \qquad A\bar{x} = B\bar{y}. \tag{4.2}$$

We can formulate the MSSSEFPCMP (4.1), (4.2) as follows; find $(\bar{x}, \bar{y}) \in \bigcap_{i=1}^{\infty} F(T_i) \times \bigcap_{j=1}^{\infty} F(S_j)$ such that, for each $l = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, m$,

$$\langle \nabla f_l(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \forall x \in C_l,$$

$$\langle \nabla h_r(\bar{y}), y - \bar{y} \rangle \geq 0 \quad \forall y \in Q_r \quad \text{and} \quad A\bar{x} = B\bar{y},$$

where ∇f_l and ∇h_r are the gradient of f_l and h_r, respectively. If we assume that for each $l = 1, 2, \ldots, N,\ r = 1, 2, \ldots, m,\ \nabla f_l$ and ∇h_r are inverse strongly monotone, then, we can apply algorithm (3.1) to obtain the solution of SESFPCMP (4.1), (4.2). Furthermore, by applying Theorem 3.1, we have that the sequence $(\{x_n\}, \{y_n\})$ converges to a solution of SESFPCMP (4.1), (4.2).

4.2. Application to systems of split equality variational inequalities over the solution set of monotone variational inclusion problem. Let H_1, H_2 and H_3 be real Hilbert spaces. Let $A\colon H_1 \to H_3,\ B\colon H_2 \to H_3$ be bounded linear operators and $\phi_l\colon H_1 \to H_1,\ \psi_r\colon H_2 \to H_2$ be α_l (resp., μ_r)-inverse strongly monotone mappings and $M_l\colon H_1 \to 2^{H_1},\ K_r\colon H_2 \to 2^{H_2}$ be maximal monotone mappings, for $l = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, m$. We consider the following System of Monotone Variational Inclusion Problem (SMVIP) which is to find $\bar{x} \in H_1$ such that for each $l = 1, 2, \ldots, N$,

$$0 \in f_l(\bar{x}) + M_l(\bar{x}).$$

Let $\mathrm{SOL}(\phi_l, M_l)$ be the solution set of SMVIP. The operator $J_\sigma^{M_l}(I - \lambda\phi_l)$ is single valued, averaged nonexpansive operator and $F(J_\sigma^{M_i}(I - \lambda\phi_l)) = \mathrm{SOL}(\phi_l, M_l),\ l = 1, 2, \ldots, N$, where $\sigma > 0,\ \lambda \in \in (0, 2\alpha_l)$ and $J_\sigma^{M_l}(I - \lambda f_l)$ is the resolvent of M_l with parameter σ (see, for example, [1, 36]).

Let us consider the following Systems of Split Equality Variational Inequality Problem (SSEVIP) which is to find $(\bar{x}, \bar{y}) \in \mathrm{SOL}(\phi_l, M_l) \times \mathrm{SOL}(\psi_r, K_r),\ (l = 1, 2, \ldots, N,\ \ r = 1, 2, \ldots, m)$ such that

$$\langle f_l(\bar{x}), x - \bar{x} \rangle \quad \forall\, x \in C_l, \tag{4.3}$$

$$\langle h_r(\bar{y}), y - \bar{y} \rangle \quad \forall\, y \in Q_r \quad \text{and} \quad A\bar{x} = B\bar{y}. \tag{4.4}$$

We know that every averaged nonexpansive mapping with nonempty fixed point set is quasinonexpansive and that single valued operators are special cases of multivalued mappings. By using these facts and adding the assumption that the resolvent operators are of type-one, we can apply algorithm (3.44) and Corollary 3.2 to obtain a solution of problem (4.3), (4.4).

Remark **4.1.** Our result extends and complements some recent results by making the following contributions:

1. We saw that the example of the multivalued mapping considered in Example 1.1 is not quasinonexpansive. Hence, the class of multivalued quasinonexpansive mappings considered in [40] is a proper subclass of the class of multivalued mappings considered in this paper.

2. In [18], the author imposed the hemi-compactness condition on the multivalued mappings to obtain strong convergence result. However, our result showed that this condition can be dispensed with.

3. In [1], the author proved weak convergence result for SHVIP, while in this paper, we obtained strong convergence result for systems of SEVIP. Furthermore, the class of operators considered in this paper is more general than the class of operators considered in [1].

4. In [14], the author obtained a general common solution to VIP, while in this paper, we obtained a common solution to both MSSEFPP and systems of SEVIP.

5. Our example (Example 1.1) generalizes the example given in [18].

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References

1. *Ansari Q. H., Nimana N., Petrot N.* Split hierarchical variational inequality problems and related problems // Fixed Point Theory and Appl. – 2014. – **208**. – P. 1186 – 1687.
2. *Ansari Q. H., Ceng L-C., Gupta H.* Triple hierarchical variational inequalities // Nonlinear Analysis: Appl. Theory, Optimization and Appl. – Berlin: Springer, 2014.
3. *Ansari Q. H., Lalitha C. S., Mehta M.* Generalized convexity, nonsmooth variational inequalities and optimization. – Boca Raton: CRS Press, 2014.
4. *Baillon J.-B., Bruck R. E., Reich S.* On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces // Houston J. Math.. – 1978. – **4**. – P. 1 – 9.
5. *Byrne C.* A unified treatment for some iterative algorithms in signal processing and image reconstruction // Inverse Problems. – 2004. – **20**. – P. 103 – 120.
6. *Byrne C., Censor Y., Gabali A., Reich S.* The split common null point problem // J. Nonlinear and Convex Anal. – 2012. – **13**. – P. 759 – 775.
7. *Bruck R. E., Rieich S.* Nonexpansive projections and resolvents of accretive operators in Banach spaces // Houston J. Math. – 1977. – **3**. – P. 459 – 470.
8. *Bruck R. E., Rieich S.* On the asymptotic behaviour of nonexpansive mapping and semigroups in Banach spaces // Houston J. Math. – 1978. – **4**. – P. 1 – 9.
9. *Censor Y., Bortfeld T., Martin B., Trofimov A.* A unified approach for inversion problems in intensity modulated radiation therapy // Phys. Med. Biol. – 2006. – **51**. – P. 2353 – 2365.
10. *Censor Y., Elfving T.* A multiprojection algorithm using Bregman projections in product space // Numer. Algorithms. – 1994. – **8**. – P. 221 – 239.
11. *Censor Y., Elfving T., Kopf N., Bortfield T.* The multiple-sets split feasibility problem and its applications for inverse problems // Inverse Problem. – 2005. – **21**. – P. 2071 – 2084.
12. *Censor Y., Segal A.* The split common fixed point problem for directed operators // J. Convex Anal. – 2009. – **2**. – P. 587 – 600.
13. *Censor Y., Gibali A., Reich S.* Algorithms for the split variational inequality problem // Numer. Algorithms. – 2012. – **59**. – P. 301 – 323.
14. *Censor Y., Gibali A., Reich S.* A von Neumann alternating method for finding common solutions to variational inequalities // arXiv:1202.0611V1 [math.OC], 3 Feb. (2012).
15. *Chang S. S., Lee J., Chan C. K., Wang L., Qin L. J.* The split feasibility problem for quasinonexpansive multivalued mappings and total asymptotically strict pseeudo-contractive mappings // Appl. Math. and Comput. – 2013. – **219**. – P. 10416 – 10424.
16. *Chidume C. E.* Geometric properties of Banach spaces and nonlinear iterations // Lect. Notes Math. – 2009. – **1965**.
17. *Chidume C. E., Ndambomve P., Bello A. U.* The split equality fixed point problem for demicontractive mappings // J. Nonlinear Anal. and Optim. – 2015. – **1**. – P. 61 – 69.
18. *Chidume C. E., Ndambomve P., Bello A. U., Okpala M. E.* The multiple-sets split equality fixed point problem for countable families of multivalued demicontractive mappings // Int. J. Math. Anal. – 2015. – **9**. – P. 453 – 469.
19. *Chidume C. E., Okpala M. E.* Fixed point iteration for a countable family of multivalued strictly pseudocontractive-type mappings. – SpringerPlus, 2015.
20. *Facchinei F., Pang J. S.* Finite-dimensional variational inequalities and complementarity problems. – New York: Springer, 2003.
21. *Fang Y., Wang L., Zi X.* Strong and weak convergence theorems for a new split feasibility problem // Int. Math. Forum. – 2013. – **33**. – P. 1621 – 1627.
22. *Goebel K., Reich S.* Uniform convexity, hyperbolic geometry and nonexpansive mappings. – New York: Marcel Dekker, 1984.

23. *Isiogugu F. O.* Demiclosedness principle and approximation theorems for certain classes of multivalued mappings in Hilbert spaces // Fixed Point Theory and Appl. – 2013. – **61**.
24. *Isiogugu F. O., Osilike M. O.* Convergence theorems for new classes of multivalued hemicontractive-type mappings // Fixed Point Theory and Appl. – 2014. – **93**.
25. *Jitpeera T., Kuman P.* Algorithm for solving the variational inequality problem over the triple hierarchical problem // Abstr. and Appl. Anal. – 2012. – Article ID 827156.
26. *Kassay G., Reich S., Sabach S.* Iterative methods for solving systems of variational inequalities in reflexive Banach spaces // SIAM J. Optim. – 2011. – **21**. – P. 1319 – 1344.
27. *Kinderlehrer D., Stampacchia G.* An introduction to variational inequalities and their applications. – New York: Acad. Press, 1980.
28. *Lemoto S., Hishinuma K., Liduka H.* Approximate solutions to variational inequality over a fixed point set of a strongly nonexpansive mapping // Fixed Point Theory and Appl. – 2014. – Article ID 51.
29. *Liduka H.*, A new iterative algorithm for the variational inequality problem over the fixed point set of a firmly nonexpansive mapping // Optimization. – 2012. – **59**. – P. 873 – 885.
30. *Liduka H.* Fixed point optimization algorithm and its application to power control in CDMA data network // Math. Program. – 2012. – **133**. – P. 227 – 242.
31. *Moudafi A.* The split common fixed point problem for demicontractive mappings // Inverse Problems. – 2010. – 6 p.
32. *Moudafi A., Al-Shemas E.* Simultaneous iterative methods for split equality problem // Trans. Math. Program. and Appl. – 2013. – **2**. – P. 591 – 597.
33. *Masad E., Reich S.* A note on the multiple-set split convex feasibility problem in Hilbert spaces // J. Nonlinear and Convex Anal. – 2007. – **8**. – P. 367 – 371.
34. *Mewomo O. T., Ogbuisi F. U.* Convergence analysis of an iterative method for solving multiple-set split feasibility problems in certain Banach spaces // Quaest. Math. – 2018. – **41**, № 1. – P. 129 – 148.
35. *Ogbuisi F. U., Mewomo O. T.* On split generalized mixed equilibrium problems and fixed-point problems with no prior knowledge of operator norm // J. Fixed Point Theory and Appl. – 2017. – **19**, № 3. – P. 2109 – 2128.
36. *Ogbuisi F. U., Mewomo O. T.* Iterative solution of split variational inclusion problem in real Banach space // Afr. Mat. – 2017. – **28**, № 1-2. – P. 295 – 309.
37. *Ogbuisi F. U., Mewomo O. T.* Convergence analysis of common solution of certian nonlinear problems // Fixed Point Theory. – 2018. – **19**, № 1. – P. 335 – 358.
38. *Okpala M. E., Nwaeze E., Ozoigbo G. E.* Split equality fixed point problems for Lipschitz hemi-contractive mappings // Adv. Fixed Point Theory. – 2015. – **5**, № 3. – P. 329 – 341.
39. *Rockafellar R. T.*, On the maximality of sums of nonlinear monotone operators // Trans. Amer. Math. Soc. – 1970. – **149**. – P. 75 – 288.
40. *ShehuY.* Iterative approximation for split equality fixed point problem for family of multivalued mappings // Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A. Math. – 2016. – **109**, № 2. – P. 627 – 643.
41. *Shehu Y., Mewomo O. T.* Further investigation into split common fixed point problem for demicontractive operators // Acta Math. Sinica (Engl. Ser.). – 2016. – **32**, № 11. – P. 1357 – 1376.
42. *Shehu Y., Mewomo O. T., Ogbuisi F. U.* Further investigation into approximation of a common solution of fixed point problems and split feasibility problems // Acta Math. Sci. Ser. B. (Engl. Ed.). – 2016. – **36**, № 3. – P. 913 – 930.
43. *Wu Y., Chen R., Shi L. Y.* Split equality problem and multiple-sets split equality problem for quasinonexpansive multivalued mappings // J. Inequal. and Appl. – 2014. – **428**.
44. *Xu H. K.* Iterative algorithms for nonlinear operators // J. London Math. Soc. – 2002. – **2**. – P. 240 – 256.
45. *Yamada I.* The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings // Inherent Parallel Algorithms in Feasibility and Optimization and Their Applications. – Amsterdam: Elsevier, 2011. – P. 473 – 504.
46. *Zhao J.* Solving split equality fixed point problem of quasinonexpansive mappings without prior knowledge of the operator norms // Optimization. – 2014.

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