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A PARABOLIC EQUATION FOR THE FRACTIONAL LAPLACIAN IN THE WHOLE SPACE: BLOW-UP OF NONNEGATIVE SOLUTIONS

ПАРАБОЛІЧНЕ РІВНЯННЯ ДЛЯ ДРОБОВОГО ЛАПЛАСІАНА В УСЬОМУ ПРОСТОРІ: ВИБУХ НЕВІД'ЄМНИХ РОЗВ'ЯЗКІВ

The main aim of the present paper is to investigate under what conditions the nonnegative solutions blow-up for the parabolic problem $\dfrac{\partial u}{\partial t} = -(-\triangle)^{\frac{\alpha}{2}}u + \dfrac{c}{|x|^\alpha}u$ in $\mathbb{R}^d \times (0,T)$, where $0 < \alpha < \min(2,d)$, $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Laplacian on $\mathbb{R}^d$ and the initial condition u_0 is in $L^2(\mathbb{R}^d)$.

Вивчено умови, за яких „вибухають” невід'ємні розв'язки параболічної задачі $\dfrac{\partial u}{\partial t} = -(-\triangle)^{\frac{\alpha}{2}}u + \dfrac{c}{|x|^\alpha}u$ в $\mathbb{R}^d \times$ $\times (0,T)$, де $0 < \alpha < \min(2,d)$, $(-\triangle)^{\frac{\alpha}{2}}$ — дробовий лапласіан на $\mathbb{R}^d$, а початкова умова u_0 належить $L^2(\mathbb{R}^d)$.

1. Introduction. This study aims at verifying that a similar critical behavior of the Cauchy problem holds when the classical Laplacian is replaced by the fractional Laplacian $-(-\triangle)^{\frac{\alpha}{2}}$ with $0 < \alpha <$ $< \min(2,d)$. In this context we discuss the question of blow-up results of nonnegative solutions for negatively perturbed Dirichlet fractional Laplacian on $\mathbb{R}^d$.

For every $0 < \alpha < \min(2,d)$, we put $L_0 := (-\triangle)^{\frac{\alpha}{2}}$. Let us consider the parabolic perturbed problem

$$\begin{gathered} -\frac{\partial u}{\partial t} = L_0 u - Vu \quad \text{in} \quad \mathbb{R}^d \times (0,T), \quad T > 0, \\ u(x,0) = u_0(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^d, \end{gathered} \tag{1.1}$$

where $u_0 \in L^2(\mathbb{R}^d)$, $u_0 \geq 0$ and V is nonnegative potential in $L^1_{\text{loc}}(\mathbb{R}^d)$. We focus on the special case $V_c = \dfrac{c}{|x|^\alpha}$, $c > c^* = \dfrac{2^\alpha \Gamma^2\left(\dfrac{d+\alpha}{4}\right)}{\Gamma^2\left(\dfrac{d-\alpha}{4}\right)}$.

This work addresses several important problems of the potential theory of fractional Laplacian. One of the results is the blow-up of nonnegative solution for a parabolic problem perturbed by potential. Our main findings were motivated by the result of J. A. Goldstein and Q. S. Zhang [16] for the Laplacian perturbed by a singular potential.

By using the idea in [4, 10, 16] where the problem was addressed and solved for the Dirichlet Laplacian (i.e., $\alpha = 2$), A. Ben Amor and T. Kenzizi [1] established conditions ensuring existence as well was blow-up of nonnegative solutions for a nonlocal case. The inspiring point for us was originally developed by P. Baras, J. A. Goldstein [4, 5] and J. A. Golstein, Q. S. Zhang [16] where the problem was addressed and solved for the Lapacian operator (i.e., $\alpha = 2$). The authors in [16] generalized the result of existence and nonexistence of nonnegative solutions in [4] to equations with variable coefficients in the principal part or to degenerate equations, one of the most important degenerate equations is the heat equation on the Heisenberg group. However, there is a substantial difference between the Laplacian and the fractional Laplacian. While the first one is local and therefore suitable for describing diffusions, the second one is nonlocal and commonly used for describing superdiffusions (Lévy flights). These differences are reflected in the Green formula, integration by part, Leibniz formula, $\ldots$. The fractional operator appears in numerous fields of mathematical physics, mathematical biology and mathematical finance and has attracted a lot of attention recently. We shall show that the method used in [4, 16] still apply in our setting.

2. Preliminaries and preparing results. To state our main results, it is convenient to introduce the following notations and definitions. In what follows, $\mathbb{R}^d$ denotes the Euclidean space of dimension $d \geq 1$, dy is the Lebesgue measure on $\mathbb{R}^d$. We shall write $\displaystyle\int \ldots$ as a shorthand form $\displaystyle\int_{\mathbb{R}^d} \ldots$.

Throughout this paper, letters k, C, c, $C', \ldots$ denote generic positive constants which may vary in value from line to line. $|A(x,r)|$ will denote the volume of the ball A centred at x and of radius r, $(a \wedge b) := \min(a,b)$ and $(a \vee b) := \max(a,b)$.

Consider the quadratic form $\mathcal{E}^\alpha$ defined in L^2 by

$$\mathcal{E}^\alpha(f,g) = \frac{1}{2}\mathcal{A}(d,\alpha)\int\int \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}dxdy,$$

$$D(\mathcal{E}^\alpha) = W^{\frac{\alpha}{2},2}(\mathbb{R}^d) := \{f \in L^2:\ \mathcal{E}[f] : \mathcal{E}(f,f) < \infty\},$$

where

$$\mathcal{A}(d,\alpha) = \frac{\alpha\Gamma\left(\dfrac{d+\alpha}{2}\right)}{2^{1-\alpha}\pi^{\frac{d}{2}}\Gamma\left(1-\dfrac{\alpha}{2}\right)} \quad \text{for} \quad 0<\alpha<\min(2,d).$$

It is crucial to remind the reader that $\mathcal{E}^\alpha$ is a transient Dirichlet form (see [15]) and is related (via Kato representation theorem) to the self-adjoint operator commonly named the fractional Laplacian on $\mathbb{R}^d$, and which we shall denote by $(-\triangle)^{\frac{\alpha}{2}}$. We note that the domain of $(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Sobolev space $W^{\frac{\alpha}{2},2}(\mathbb{R}^d)$. For smooth compactly supported function $\phi \in C_c^\infty(\mathbb{R}^d)$, as in [18] the fractional Laplacian is defined as the $L^2(\mathbb{R}^d)$-closure of the operator

$$(-\triangle)^{\frac{\alpha}{2}}\phi(x) = \lim_{\epsilon\longrightarrow 0}\int\limits_{|y|>\epsilon}[\phi(x+y)-\phi(x)]\nu(y)dy, \qquad x \in \mathbb{R}^d,$$

where ν is the Lévy measure given by the following density function:

$$\nu(y) = \frac{2^\alpha\Gamma\left(\dfrac{d+\alpha}{2}\right)}{\pi^{\frac{d}{2}}\left|\Gamma\left(\dfrac{-\alpha}{2}\right)\right|}|y|^{-d-\alpha}.$$

This definition is very useful to probability applications. Its Fourier transform (see [23]) is given by

$$\widehat{(-\triangle)^{\frac{\alpha}{2}}\phi}(\xi) = -|\xi|^\alpha\widehat{\phi}(\xi).$$

Moreover, if ϕ is regular enough and $\alpha \in (0,2)$, $(-\triangle)^{\frac{\alpha}{2}}\phi(x)$ can be computed by the formula

$$(-\triangle)^{\frac{\alpha}{2}}\phi(x) = c_{d,\alpha}\int \frac{\phi(x)-\phi(y)}{|x-y|^{d+\alpha}}dy, \tag{2.1}$$

where $c_{d,\alpha}$ is a constant depending only on d and α. The inverse of $(-\triangle)^{\frac{\alpha}{2}}$ is $(-\triangle)^{-\frac{\alpha}{2}}$. For $0<\alpha<\min(2,d)$, there is an integral formula which says that $(-\triangle)^{-\frac{\alpha}{2}}u$ is the convolution of the function u with the Riesz potential (see [23]):

$$(-\triangle)^{-\frac{\alpha}{2}}\phi(x) = c_{d,\alpha}\int \frac{\phi(x-y)}{|y|^{d-\alpha}}dy,$$

which holds a long as ϕ is integrable for the right-hand side to make sense. For instance, if $r>0$ and $\phi_r(x)=\phi(rx)$, then we obtain

$$(-\triangle)^{\frac{\alpha}{2}}\phi_r(x) = r^{\alpha}(-\triangle)^{\frac{\alpha}{2}}\phi(rx),\ x\in\mathbb{R}^d.$$

We let p_t the fractional heat kernel which is the fundamental solution to the equation

$$\frac{\partial p_t(x)}{\partial t} + (-\triangle)^{\frac{\alpha}{2}}p_t = 0,$$

$$p_0(x) = \delta_0(x),$$

with Fourier transform

$$\hat{p}_t(\xi) = \int p_t(x)e^{ix\xi}dx = e^{-t|\xi|^{\alpha}}, \qquad t>0, \quad x\in\mathbb{R}^d. \tag{2.2}$$

This yields

$$p_t(x) = (2\pi)^{-d}\int e^{-t|\xi|^{\alpha}}e^{-ix\xi}d\xi, \quad x\in\mathbb{R}^d.$$

Consequently, we get the scaling property

$$p_t(x) = t^{-\frac{d}{\alpha}}p_1(t^{-\frac{1}{\alpha}}x), \qquad t>0, \quad x\in\mathbb{R}^d, \tag{2.3}$$

and it is well-known (see [25]) that $p_1(x)\approx 1\wedge|x|^{-d-\alpha}$. Hence, the following inequality holds for some constant C:

$$C^{-1}\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\right) \le p_t(x) \le C\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\right), \qquad t>0, \quad x\in\mathbb{R}^d.$$

In particular, the maximum of p_t is $p_t(0) = 2^{1-\alpha}\pi^{-\frac{d}{2}}\alpha^{-1}\dfrac{\Gamma\left(\dfrac{d}{\alpha}\right)}{\Gamma\left(\dfrac{d}{2}\right)}t^{-\frac{d}{\alpha}}$. Furthermore, notice that the semigroup $P_t\phi(x)=\int p_t(x,y)\phi(y)dy$ has the fractional Laplacian as generator (see [2, 7, 24]).

Using (2.2), one proves that p satisfies the following equation:

$$\int_s^{\infty}\int p(u-s,x,z)\Big[\partial_u\phi(u,z) + \triangle_z^{\frac{\alpha}{2}}\phi(u,z)\Big]dzdu = -\phi(s,x),$$

where $p(t,x,y)=p_t(y-x)$, $s\in\mathbb{R}$, $x\in\mathbb{R}^d$ and $\phi\in C_c^{\infty}(\mathbb{R}\times\mathbb{R}^d)$.

Let $D \subseteq \mathbb{R}^d$ be an open set. We denote by p_D the heat kernel of the Dirichlet fractional Laplacian on D such that p_D is jointly continuous when $t \neq 0$, and we know from [9] (Theorem 1.1) that

$$0 \leq p_D(t,x,y) = p_D(t,y,x) \leq p(t,x,y), \quad t > 0, \ x, \ y \in \mathbb{R}^d.$$

In particular,

$$\int p_D(t,x,y) \leq 1.$$

In addition, we define the Green function for $(-\triangle)^{\frac{\alpha}{2}}$ on D by

$$G_D(x,y) = \int\limits_0^\infty p_D(t,x,y)dt,$$

and based on the scaling property of p_D, the following scaling of G_D is given by

$$G_{rD}(rx,ry) = r^{\alpha-d}G_D(x,y), \qquad r > 0, \ x, \quad y \in \mathbb{R}^d.$$

Now, as in [1, 17], let us recall the notion of solution for the heat equation (1.1).

Definition 2.1. *Let $0 < T \leq \infty$. A Borel measurable function $u\colon [0,T) \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is a solution of problem* (1.1) *if*:

1) $u \in L^1_{\text{loc}}\big((0,T) \times \mathbb{R}^d, dt \otimes V\,dx\big)$,

2) $u \in \mathcal{L}^2_{\text{loc}}\big([0,T], L^2_{\text{loc}}(\mathbb{R}^d)\big)$,

3) *for every $0 \leq t < T$, every $\Omega \subset \mathbb{R}^d$ and every $\phi \in C_c^\infty\big([0,T] \times \Omega\big)$, the following identity holds true*:

$$\int\limits_\Omega \big((u\phi)(t,x) - u_0(x)\phi(0,x)\big)dx + \int\limits_0^t \int\limits_\Omega u(s,x)\big(-\phi_s(s,x) + L_0\phi(s,x)\big) =$$

$$= \int\limits_0^t \int\limits_\Omega u(s,x)\phi(s,x)V(x)dxds.$$

Moreover, for bounded domain Ω and for fixed $\alpha \in (0, \min(2,d)]$, the spectrum of the $(-\triangle)^{\frac{\alpha}{2}}|_\Omega$ is discrete and consists of a sequence $\{\lambda_k(\alpha)\}_{k=1}^\infty$ of eigenvalues (with finite multiplicity) written in increasing order according to their multiplicity (see, for example, [6])

$$0 < \lambda_1(\alpha) < \lambda_2(\alpha) \leq \ldots \leq \lambda_k(\alpha) \ldots \nearrow +\infty.$$

In this context, Weyl's asymptotic formula for the fractional laplacian with Dirichlet boundary condition (see [8]) is taken as

$$\lambda_k(\alpha) \sim \frac{(2\pi)^\alpha k^{\frac{\alpha}{d}}}{(\omega_d|\Omega|)^{\frac{\alpha}{d}}} \quad \text{as} \quad k \longrightarrow +\infty,$$

where $\omega_d = \dfrac{\pi^{\frac{d}{2}}}{\Gamma\Big(1 + \dfrac{d}{2}\Big)}$ is the volume of the unit ball in $\mathbb{R}^d$.

3. Blow-up of nonnegative solutions. Inspired from the idea of J. A. Goldstein and Q. S. Zhang [16], we shall study the lower bound of the heat kernel p of the operator $L_0 - V.$ The idea of estimating $p_t(x,y)$ is similar to that in [16]. For any positive integer $k > [\alpha]+1,$ where $[\alpha]$ is the integer function, we introduce the regularized function

$$V_k(x) = \begin{cases} \dfrac{c}{|x|^\alpha}, & \text{if} \quad |x| \geq \dfrac{1}{k}, \\ ck^\alpha, & \text{if} \quad |x| \leq \dfrac{1}{k}. \end{cases}$$

Our main task is to proof that the blow up of nonnegative solutions of (1.1) is deeply related to

$$\lambda_k := \inf_{\phi \in C_c^\infty(\mathbb{R}^d)\setminus\{0\}} \frac{\mathcal{E}^\alpha[\phi] - \displaystyle\int \phi^2 V_k dx}{\displaystyle\int \phi^2 dx}. \tag{3.1}$$

Lemma 3.1. *The operator $H_k \equiv L_0 - V_k$ is not nonnegative.*

Proof. We consider two cases:

Case $\alpha = 2.$ The authors in [16] prove that the operator $H \equiv -\triangle - V$ is nonnegative and if $\alpha > 2,$ H is not nonnegative.

Case $0 < \alpha < 2.$ By using the following Hardy-type inequality:

$$\int |x|^{-\alpha} f^2(x) dx \leq C_{d,\alpha} \int\int \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} dxdy \quad \forall f \in C_c^\infty(\mathbb{R}^d),$$

we deduce that the operator $H \equiv L_0 - V$ is not nonnegative, i.e., there exists $f \in C_0^\infty(\mathbb{R}^d)$ such that

$$\int\int \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} dxdy < \int V(x) f^2(x) dx.$$

Since V_k is nondecreasing and $V_k \longrightarrow V$ a.e. as $k \longrightarrow +\infty,$ we have

$$\int\int \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}} dxdy < \int V_k(x) f^2(x) dx, \tag{3.2}$$

when k is sufficiently large.

On the other hand, by using (2.1), we get, for any $f,\ g \in C_0^\infty(\mathbb{R}^d),$

$$\langle L_0 f, g\rangle = c_{d,\alpha} \int\int \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}} dxdy. \tag{3.3}$$

Now, let us consider $u \in C_0^\infty(\mathbb{R}^d),$ $g(x) = g_\beta(x)$ and $f(x) = |u(x)|^2 g_\beta^{-1}(x),$ where $g_\beta(x) = $ $= \dfrac{\Gamma\left(\dfrac{d}{2}-\dfrac{\beta}{2}\right)}{\Gamma\left(\dfrac{\beta}{2}\right) 2^\beta \pi^{\frac{d}{2}}} |x|^{\beta-d}.$ Since the Fourier transform of g_β is $\widehat{g_\beta}(\xi) = |\xi|^{-\beta},$ so by Plancherel equality the left-hand side of (3.3) is equals to

$$\int |\xi|^{\frac{\alpha}{2}} \widehat{f}(\xi)\widehat{g}(\xi) d\xi = \int \widehat{f}(\xi)|\xi|^{\frac{\alpha}{2}-\beta} d\xi = \int |u(x)|^2 \frac{g_{\beta-\frac{\alpha}{2}}(x)}{g_{\frac{\beta}{2}}(x)} dx.$$

By simple computation, the right-hand side of (3.3) becomes

$$c_{d,\alpha}\int\int\left(|u(x)-u(y)|^2-\left|\frac{u(x)}{g_\beta(x)}-\frac{u(y)}{g_\beta(y)}\right|^2 g_\beta(x)g_\beta(y)\right)\frac{1}{|x-y|^{d+\alpha}}dxdy.$$

Let consider $v(x)=\dfrac{u(x)}{g_\beta(x)}$, so

$$\langle L_0u,u\rangle-\langle L_0f,g\rangle=c_{d,\alpha}\int\int\frac{|v(x)-v(y)|^2}{|x-y|^{d+\alpha}}g_\beta(x)g_\beta(y)dxdy=H_\alpha[u].$$

Hence,

$$\langle L_0u,u\rangle-c\int\frac{|u(x)|^2}{|x|^\alpha}dx=H_\alpha[u]-(c-c^*)\int\frac{|u(x)|^2}{|x|^\alpha}dx.$$

Taking $u=g_\beta$, we have

$$\langle L_0g_\beta,g_\beta\rangle-c\int\frac{|g_\beta(x)|^2}{|x|^\alpha}dx=\int|\xi|^{\frac{\alpha}{2}-\beta}d\xi-c\int\frac{|g_\beta(x)|^2}{|x|^\alpha}dx=$$

$$=(c-c^*)\int\frac{|g_\beta(x)|^2}{|x|^\alpha}dx<0,$$

which implies that L_0-V_1 is not nonnegative and, consequently, L_0-V_k is not nonnegative for every $k\geq 1$.

Lemma 3.1 is proved.

As a consequence of Lemma 3.1, we deduce that the operator L_0-V_k has a nonnegative eigenvalue.

On the other hand, by Weyl's theorem [26] on essential spectrum and the fact that

$$V_k(x)\to 0\quad\text{as}\quad |x|\to\infty,$$

we have

$$\sigma_{\mathrm{ess}}(H)=\sigma_{\mathrm{ess}}(L_0)=\sigma(L_0)=[0,+\infty[.$$

Hence, the operator $A=(-\triangle)^{\frac{\alpha}{2}}$ is nonnegative.

Lemma 3.2. *The operator $\chi_{B_R}(\Lambda-L_0)^{-1}$ is compact, where χ_{B_R} is the characteristic function of the ball $B_R:=B(0,R)\subset\mathbb{R}^d$.*

Proof. Since the domain $D(L_0)$ (on the L^2 space based on a bounded subset of $\mathbb{R}^d$, equipped with its graph norm) is compactly embedded in $L^2(B_R)$, we deduce the result by using the Kato–Rellich's theorem.

***Remark* 3.1.** 1. Let $p_{k,t}$ be the heat kernel of the operator $H_k\equiv L_0-V_k$ for a positive integer k. Note that, for all $k\in\mathbb{N}$, we have

$$H_{k+1}(p_{k+1,t}-p_{k,t})+\frac{\partial}{\partial t}(p_{k+1,t}-p_{k,t})=p_{k,t}(V_{k+1}-V_k)\quad\text{in}\quad\mathbb{R}^d\times(0,+\infty),$$

$$p_{k+1,0}-p_{k,0}=0\quad\text{for a.e.}\quad x\in\mathbb{R}^d.$$

Then, by using the comparison principle, we deduce that

$$p_{k+1,t} \geq p_{k,t}, \quad \text{when} \quad k \quad \text{increases}.$$

Based on

$$u_k(x,t) = e^{-tH_k} u_0(x) = \int p_{k,t}(x,y) u_0(y) dy,$$

it is enough to establish that

$$\lim_{k \longrightarrow +\infty} p_{k,t}(x,y) = \infty$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$.

2. Since V_k is bounded, the estimation of $p_{k,t}(x,y)$ is deduced from the Kato theorem and the identity

$$u_k(x,t) = \int p_{k,t}(x,y) u_0(y) dy.$$

3. Let λ_k be the ground state energy of H_k and let Φ_k be the corresponding eigenfunction, then we have

$$\Phi_k \in L^2(\mathbb{R}^d), \ \|\Phi_k\|_{L^2} = 1, \ H_k \Phi_k = \lambda_k \Phi_k, \quad \text{and} \quad \Phi_k(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

4. The operator $p_{k,t}$ is continuous and satisfies

$$p_{k,t} \geq p_{k,t}^{B_R}.$$

Lemma 3.3. *Since $V_k(x) = k^\alpha V_1(kx)$, a scaling argument implies*

$$\Phi_k(x) = k^{\frac{d}{2}} \Phi_1(kx), \qquad \lambda_k = k^\alpha \lambda_1.$$

Proof. Let Φ_k be a normalized eigenfunction corresponding to the first eigenvalue λ_k, i.e.,

$$H_k \Phi_k = \lambda_k \Phi_k, \ \ \|\Phi_k\|_{L^2} = 1.$$

Note that $V_k = k^\alpha V_1(kx)$, we obtain

$$L_0 \Phi_k - V_k \Phi_k = \lambda_k \Phi_k. \tag{3.4}$$

On the other hand, we have

$$L_0 \Phi_1(kx) - V_1(x) \Phi_1(kx) = \lambda_1 \Phi_1(kx).$$

Hence,

$$k^{-\alpha} L_0[\Phi_1(kx)] - V_1(x) \Phi_1(kx) = \lambda_1 \Phi_1(kx).$$

Let now $\widetilde{\Phi}(x) = \Phi_1(kx)$, then we obtain

$$L_0 \widetilde{\Phi}(y) - V_k(y) \widetilde{\Phi}(y) = k^\alpha \lambda_1 \widetilde{\Phi}(y).$$

Note that $\widetilde{\Phi}$ is a solution of (3.4), which implies that $\mu\widetilde{\Phi}$ is a solution of (3.4) for all $\mu \in \mathbb{R}$ and $\|\mu\widetilde{\Phi}\|_{L^2} = 1$. Then

$$\mu^2 \int_{\mathbb{R}^d} |\Phi_1(kx)|^2 dx = 1.$$

Let $y = kx$, yields

$$\frac{\mu^2}{k^d} \int\limits_{R^d} |\Phi_1(y)|^2 dy = 1.$$

Hence,

$$\mu = k^{\frac{d}{2}}.$$

Then

$$\Phi_k(x) = k^{\frac{d}{2}} \widetilde{\Phi}(x) = k^{\frac{d}{2}} \Phi_1(kx).$$

Lemma 3.4. *The operator $p_{k,t}$ satisfies the following inequality*:

$$p_{k,t}(x) \geq e^{-t\lambda_k} \Phi_k^2(x) = k^d e^{-tk^\alpha \lambda_1} \Phi_1^2(kx). \tag{3.5}$$

Proof. For $R > 0$, let $p_{k,t}^{B_R}$ be the Dirichlet heat kernel of H_k on $B(0,R)$. Since the resolvent is compact, so by using the spectral decomposition [13, 21], we have

$$p_{k,t}^{B_R}(x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j^{k,R} t} \Phi_j^{k,R}(x) \Phi_j^{k,R}(y) =$$

$$= e^{-\lambda^{k,R} t} \Phi^{k,R}(x) \Phi_1^{k,R}(y) + \sum_{j=2}^{\infty} e^{-\lambda_j^{k,R} t} \Phi_j^{k,R}(x) \Phi_j^{k,R}(y),$$

where $\lambda^{k,R}$ and $\Phi^{k,R}$ are the ground state energy and the ground state, respectively.

Notice that, $\lambda_j^{k,R}$ and $\Phi_j^{k,R}$ are the other eigenvalues and normalized eigenfunctions, consequently,

$$p_{k,t}^{B_R}(x) \geq e^{-t\lambda^{k,R}} (\Phi^{k,R}(x))^2.$$

Since the proof is very similar to the one in [14, p. 94] (Theorem 1) we omit it, and, by using the comparison principle, we have

$$\lim_{R \longrightarrow +\infty} p_{k,t}^{B_R}(x) = p_{k,t}(x).$$

On the other hand, using the same idea as in the proof of [12, p. 128] (Theorem 6.2.3) and the fact that

$$\lambda^{k,R} = \inf \sigma(H_k^R) \qquad \text{and} \qquad \lambda_k = \inf \sigma(H_k),$$

we have

$$\lim_{R \longrightarrow +\infty} \lambda^{k,R} = \lambda_k.$$

We also need to show that $\lim_{R \longrightarrow +\infty} \Phi^{k,R}(x) = \Phi_k(x)$ for all $x \in \mathbb{R}^d$. To this end, observe that $\|\Phi^{k,R}\|_{L^2(B^R)} = 1$ and $V_k \in L^\infty$.

Let us recall that, $\Phi^{k,R}$ satisfy the equation

$$H_k \Phi^{k,R} = \lambda^{k,R} \Phi^{k,R} \quad \text{in} \quad B(0,R),$$

$$\Phi^{k,R} = 0 \quad \text{on } B^c(0,R).$$

Notice that, the standard subelliptic theory [22] shows that

$$\Phi^{k,R}(x) = \lambda^{k,R} \int G^{k,R}(x,y)\Phi^{k,R}(y)dy =$$

$$= \lambda^{k,R} \int\limits_{B^R} G^{k,R}(x,y)\Phi^{k,R}(y)dy \quad \forall x \in B^R,$$

where $G^{k,R}$ is the Green function of H_k on B^R. So by using Hölder inequality there exists $C_k > 0$ depending only on k such that

$$\|\Phi^{k,R}\|_{L^\infty(B^R)} \le C_k. \tag{3.6}$$

It is easily seen that, if R is sufficiently large, the sequence $\lambda^{k,R}$ is nonincreasing for some $\delta > 0$ and $\lim_{R \longrightarrow +\infty} \lambda^{k,R} = \lambda_k$.

On the other hand, by (3.1) and (3.2), we have $\lambda_k < 0$.

Now, since $\lambda^{k,R} \longrightarrow \lambda_k$ as $R \longrightarrow +\infty$, so there exists $N \in \mathbb{N}$ such that, for all $R \ge N$, $\lambda^{k,R} < 0$. Hence, by using the fact that $(\lambda^{k,R})_R$ is nonincreasing, we get $\lambda_k \le \lambda^{k,R} \le \lambda^{k,N} < 0$. Let $\delta > 0$ such that $\lambda^{k,N} = -2\delta$, then we have

$$\lambda_k \le \lambda^{k,R} \le -2\delta < 0.$$

Thus, by the decay property of V_k, there exists $R_0 > 0$ such that

$$V_k(x) + \lambda^{k,R} < -\delta, \quad \text{when} \quad |x| \ge R_0. \tag{3.7}$$

Consequently, this shows that, the operator $-L_0 + V_k + \lambda^{k,R}$ satisfies the maximum principle in $B^c(0, R_0)$.

Let u_0 be the weak solution of

$$-L_0 u_0 + V_k u_0 + \lambda^{k,R} u_0 = 0 \quad \text{in} \quad B^c(0,R),$$

$$u_0(x) = C_k \quad \text{in} \quad B(0,R),$$

$$u_0(x) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow +\infty,$$

which implies, with (3.7), that

$$L_0 u_0 + \delta u_0 \le 0 \ \text{ in the distribution sense.}$$

Let consider

$$\Psi^{k,R} = \Phi^{k,R} - u_0.$$

Hence,

$$H_k \Psi^{k,R} \ge -L_0 u_0 - \delta u_0 \ge 0.$$

Moreover, by using [19] (Corollary 3), there exists $c_\delta > 0$ such that

$$u_0(x) \le c_\delta (1 + |x|)^{-d-\alpha},$$

whence

$$\Phi^{k,R}(x) \leq u_0(x) \leq c_\delta(1+|x|)^{-d-\alpha},$$

when $|x| \geq R_0$ and R is sufficiently large. Thus, there exists a constant $c' > 0$ such that

$$\|\Phi^{k,R}\|_{L^\infty} \leq c'. \tag{3.8}$$

Furthermore, by (3.6) and (3.8), we can extract a subsequence still denoted by $\Phi^{k,R}$ converging weakly to a nonzero function Φ_k. On the other hand, taking ρ_k as a mollifier function, we obtain

$$\|(\Phi^{k,R} - \Phi_k)\rho_k\|_{L^1} \longrightarrow 0 \quad \text{as} \quad R \longrightarrow +\infty.$$

Therefore,

$$\Phi^{k,R} \longrightarrow \Phi_k \quad \text{a.e. as} \quad R \longrightarrow +\infty.$$

Thus, we get

$$\Phi^{k,R} \longrightarrow \Phi_k \quad \text{pointwise as} \quad R \longrightarrow +\infty.$$

Lemma 3.4 is proved.

Lemma 3.5. *There exists* $\beta > 0$, $\eta > 0$ *such that*

$$\Phi_1(x) \geq \beta e^{-\eta|x|^{\frac{\alpha}{2}}} \quad \forall\, x \in \mathbb{R}^d. \tag{3.9}$$

Proof. Since V_1 is a bounded function, by using [11] (Remark 4.4), we show that the Harnack inequality holds for the first eigenfunction Φ_1 of the operator $H_1 \equiv L_0 - V_1$, so there exists a universal constant $\mathcal{H} > 0$, which can be made explicitly in [20] such that

$$\sup_{y \in B(x, \frac{2 \vee 2^{1/\alpha}}{3})} \Phi_1(y) \leq \mathcal{H} \inf_{y \in B(x, \frac{2 \vee 2^{1/\alpha}}{3})} \Phi_1(y) \quad \forall x \in \mathbb{R}^d.$$

Let z and m be the smallest integer such that $m \geq |z|^{\frac{\alpha}{2}} + 1$, by chain arguments we only need to give a lower bound of $\Phi_1(z)$.

It's known that, the segment $\{\tau z / 0 \leq \tau \leq 1\}$ can be covered by at most m interconnected balls of radius $\dfrac{2 \wedge 2^{\frac{1}{\alpha}}}{3}$. Then we obtain

$$\Phi_1(0) \leq C^{|z|^{\frac{\alpha}{2}}+1}\Phi_1(z),$$

which implies that

$$\Phi_1(z) \geq \Phi_1(0)C^{-(|z|^{\frac{\alpha}{2}}+1)} \geq \Phi_1(0)e^{-(|z|^{\frac{\alpha}{2}}+1)\ln C} \geq$$

$$\geq \Phi_1(0)e^{-\ln C}e^{-|z|^{\frac{\alpha}{2}}\ln C} \geq \beta e^{-\eta|z|^{\frac{\alpha}{2}}}.$$

Lemma 3.5 is proved.

***Remark* 3.2.** Referring back now to the previous study of the inequality (3.5), by using inequality (3.9) and the fact that $\lambda_1 < 0$, we deduce that

$$p_{k,t}(x) \geq k^d e^{-tk^\alpha\lambda_1}\Phi_1^2(kx) \geq \beta^2 k^d e^{-tk^\alpha\lambda_1} e^{-2\eta k^{\frac{\alpha}{2}}|x|^{\frac{\alpha}{2}}} \geq$$

$$\geq \beta^2 k^d \exp\Big(-tk^\alpha\lambda_1 - 2\eta k^{\frac{\alpha}{2}}|x|^{\frac{\alpha}{2}})\Big) \longrightarrow +\infty \quad \text{as} \quad k \longrightarrow +\infty \tag{3.10}$$

for all $t > 0$ and $x \in \mathbb{R}^d$.

The aim of this section is devoted to show that

$$\lim_{k\longrightarrow+\infty} p_{k,t}(x,y)=\infty \quad \text{for all} \quad t>0 \quad \text{and} \quad x,y\in\mathbb{R}^d.$$

For the proof, we need several lemmas.

Lemma 3.6. *Let $x,y\in\mathbb{R}^d$ such that $|x|,|y|\le a_1t^{\frac{1}{\alpha}}$ for some $a_1>0$. Then there exist positive constants C_1,η such that*

$$p_t(x,y)\ge C_1\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\right)^{-\eta}\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|y|^{d+\alpha}}\right)^{-\eta}p_{\frac{t}{8}}(z),$$

where η and C_1 are two positive constants. Here z is a point such that $|x-z|=(a_1t)^{\frac{1}{\alpha}}$.

Proof. Without loss of generality, we take $a_1=1$.

Let γ be a minimal geodesic connecting 0 and x. We extend γ to infinity, pick z on γ such that $|x-z|=t^{\frac{1}{\alpha}}$ and $|z|=|x|+|x-z|$. For simplicity, we parameterize γ by arc-length such that $\gamma(0)=x$ and $\gamma\big(t^{1/\alpha}\big)=z$. Here we remark that the ratio between the distance d and the length of minimal geodesic is bounded away from zero at infinity. So the above choice is always possible up to a constant multiple, hence we can take that constant to be 1 for simplicity. For nonnegative integer i, we write $y_i=\gamma(2^i\delta|x|)$ with $\delta\in\left[\dfrac{1}{\alpha},1\right]$ to be determined later.

Clearly,

$$y_i,\ y_{i+1}\in\overline{B}(y_{i+1},2^i\delta|x|)\subset B\left(y_{i+1},2^i\delta|x|\frac{11}{10}\right).$$

First, observe that

$$|y_i-y_{i+1}|=\big|\gamma(2^i\delta|x|)-\gamma(2^{i+1}\delta|x|)\big|\le 2^i\delta|x|$$

and, for $y'\in B\left(y_{i+1},2^i\delta|x|\dfrac{11}{10}\right)$, we have

$$|y'|\ge|y_{i+1}|-|y'-y_{i+1}|,$$

which implies

$$|y'-y_{i+1}|\le 2^i\delta|x|\frac{11}{10}.$$

Since $y_{i+1}=\gamma(2^{i+1}\delta|x|)$, we have

$$|y_{i+1}|\ge 2^{i+1}\delta|x|.$$

Hence,

$$|y'|\ge|y_{i+1}|-|y'-y_{i+1}|\ge\left(2^{i+1}-2^i\frac{11}{10}\right)\delta|x|=2^i\delta|x|\frac{9}{10}.$$

Therefore, there exists $C>0$ such that

$$\beta_i=\sup_{y'\in B(y_{i+1},2^i\delta|x|\frac{11}{10})}|V(y'|\le$$

$$\leq \sup_{y'\in B(y_{i+1},2^i\delta|x|\frac{11}{10})} \frac{C}{|y'|^\alpha} \leq \frac{aC}{2^{\alpha i}\delta^\alpha |x|^\alpha}.$$

Let u be a positive solution of the problem (1.1) for $V = V_c = \dfrac{c}{|x|^\alpha}$. By the Harnack's inequality [11], for $x', y' \in \overline{B}(y_{i+1}, 2^i\delta|x|)$ and $s > s'$, we have

$$u(x',s) \geq u(y',s')\left(1 \wedge \frac{s-s'}{|x'-y'|^{d+\alpha}}\right)\left(1 \wedge \frac{1}{\left((s-s')\left(\beta_i + \dfrac{1}{s'}\right)\right)^{d+\alpha}}\right).$$

Taking now $y' = y_{i+1},\ x' = y_i,\ s' = 2t - 2^{\alpha(1+i)}\delta^\alpha|x|^\alpha$ and $s = 2t - 2^{\alpha i}\delta^\alpha|x|^\alpha$, we obtain

$$u(y_i, 2t - 2^{\alpha i}\delta^\alpha|x|^\alpha) \geq u(y_{i+1}, 2t - 2^{\alpha(1+i)}\delta^\alpha|x|^\alpha)\left(1 \wedge \frac{1}{2^{id}\delta^d|x|^d}\right) \times$$

$$\times \left(1 \wedge \frac{1}{\left(2^{\alpha i}\delta^\alpha|x|^\alpha\left(\beta_i + \dfrac{1}{2t - 2^{\alpha(1+i)}\delta^\alpha|x|^\alpha}\right)\right)^{d+\alpha}}\right).$$

Thereby, we derive

$$u(y_i, 2t - 2^{\alpha i}\delta^\alpha|x|^\alpha) \geq c_2 u(y_{i+1}, 2t - 2^{\alpha(1+i)}\delta^\alpha|x|^\alpha). \tag{3.11}$$

Since $\dfrac{t^{\frac{1}{\alpha}}}{|x|} \geq 1$, there exists $\delta \in \left[\dfrac{1}{\alpha}, 1\right]$ such that

$$k = \log_2\left(\frac{t^{\frac{1}{\alpha}}}{\delta|x|^{d+\alpha}}\right) \quad \text{is an integer.}$$

Observe that, for such an integer k we get

$$y_k = \gamma(2^k\delta|x|) = \gamma\big(t^{\frac{1}{\alpha}}\big) = z, \qquad 2t - 2^{\alpha k}\delta^\alpha|x|^\alpha = t.$$

Now, iterating (3.11) k times, we obtain

$$u(z,t) \leq C_2^k u(y_0, 2t - \delta^\alpha|x|^\alpha) \leq$$

$$\leq C_2^{c\log_2\left(\frac{t^{\frac{1}{\alpha}}}{\delta|x|^{d+\alpha}}\right)} u(x,2t) \leq$$

$$\leq \left(C_3 e^{C_4 a}\right)^{c\log_2\left(\frac{t^{\frac{1}{\alpha}}}{\delta|x|^{d+\alpha}}\right)} u(x,2t).$$

Therefore, there exists $\eta = C_5 + aC_6 > 0$ such that

$$C\left(\frac{t^{\frac{1}{\alpha}}}{|x|^{d+\alpha}}\right)^{-\eta}u\left(z,\frac{t}{2}\right)\leq u(x,t).$$

Notice that in the above, $\delta\in\left[\frac{1}{\alpha},1\right]$ is absorbed into C.

Repeating the above process for z_1 satisfying $|y-z_1|=t^{\frac{1}{\alpha}}$ and $|z_1|=|y|+|y-z_1|$, we have

$$C\left(\frac{t^{\frac{1}{\alpha}}}{|y|^{d+\alpha}}\right)^{-\eta}u\left(z_1,\frac{t}{4}\right)\leq u\left(y,\frac{t}{2}\right).$$

Let us fix $y\in\mathbb{R}^d$ and set $u(x,t)=p_t(x,y)$. Applying the above inequality to the first entry of p_t, we obtain

$$p_t(x,y)\geq C\left(\frac{t^{\frac{1}{\alpha}}}{|x|^{d+\alpha}}\right)^{-\eta}p_{\frac{t}{2}}(y)\geq$$

$$\geq C\left(\frac{t^{\frac{1}{\alpha}}}{|x|^{d+\alpha}}\right)^{-\eta}\left(\frac{t^{\frac{1}{\alpha}}}{|y|^{d+\alpha}}\right)^{-\eta}p_{\frac{t}{4}}(z_1,z).$$

Next we need to find a lower bound for $p_{\frac{t}{4}}(z_1,z)$. Based on our construction, we have $t^{\frac{1}{\alpha}}\leq|z|\leq$ $\leq 2t^{\frac{1}{\alpha}}$ and $t^{\frac{1}{\alpha}}\leq|z_1|\leq 2t^{\frac{1}{\alpha}}$.

Clearly we can form a chain of fixed number of parabolic cubes satisfying:

1. Each cube is of size $\frac{t^{\frac{1}{\alpha}}}{4}$ in the spacial direction and $\frac{t^{\frac{2}{\alpha}}}{16}$ in the time direction.

2. The first cube covers $\left(z_1,\frac{t^{\frac{2}{\alpha}}}{4}\right)$ and the last one covers $\left(z,\frac{t^{\frac{2}{\alpha}}}{8}\right)$.

3. Adjacent cubes have a gap of $ct^{\frac{2}{\alpha}}$ in the time direction and the centers of the adjacent cubes have a distance no greater than $ct^{\frac{1}{\alpha}}$ in the spacial direction.

4. For each (z,t) in the cubes, $|z|\geq ct^{\frac{1}{\alpha}}$ for some $c>0$.

Notice that, along this chain, by statement 4, we obtain

$$V(z)\leq\frac{c_1}{|z|^{\alpha}}\leq\frac{c_1}{c^{\alpha}t}.$$

On the other hand, by the parabolic Harnack principle [11] (Proposition 4.3), for any (z_i,τ_i) in the ith cube, we have

$$p_{\tau_{i+1}}(z_{i+1},z)\leq c_1p_{\tau_i}(z_i,z)\quad\forall z_{i+1}\in B\left(z,(n-i)ct^{\frac{1}{\alpha}}\right),$$

where n is the number of cubes.

Now, multiplying the above together, we obtain

$$p_{\frac{t}{8}}(z)\leq c_1p_{\frac{t}{4}}(z_1)\quad\forall z_i\in B\left(z,nct^{\frac{1}{\alpha}}\right),\quad 1\leq i\leq n-1.$$

Then we derive that

$$p_t(x,y)\geq C_1\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|x|^{d+\alpha}}\right)^{-\eta}\left(t^{-\frac{d}{\alpha}}\wedge\frac{t}{|y|^{d+\alpha}}\right)^{-\eta}p_{\frac{t}{8}}(z).$$

Lemma 3.7. *Assume that* $|x-y| \geq a_1 t^{\frac{1}{\alpha}},\ |x| \geq a_2 t^{\frac{1}{\alpha}}$ *and* $|y| \geq a_2 t^{\frac{1}{\alpha}}$ *for some* $a_1,\ a_2 > 0$. *Then there exist two positive constants* c_1 *and* c_2 *such that*

$$p_t(x,y) \geq c_1 \left(1 \wedge \left(\frac{t}{|x-y|^\alpha} \right)^{-d-\alpha} \right)^{c_2} p_{\frac{t}{2}}(y). \tag{3.12}$$

Proof. Without loss of generality we take $a_1 = a_2 = 1$. Since $|x-y| \geq t^{\frac{1}{\alpha}}$, we can form a chain of parabolic cubes such that:

1. Each cube is of size $\dfrac{\delta t}{|x-y|}$ in the spacial direction and $\left(\dfrac{\delta t}{|x-y|}\right)^\alpha$ in the time direction.
2. The first cube covers (x,t) and the last one covers $\left(y, \dfrac{t}{2}\right)$.
3. Adjacent cubes have a gap of $c\left(\dfrac{\delta t}{|x-y|}\right)^\alpha$ in the time direction and the centers of the adjacent cubes have a distance no greater than $c\dfrac{\delta t}{|x-y|}$ in the spacial direction.
4. For each (z,t) in the cubes, $|z| \geq c t^{\frac{1}{\alpha}}$ for some $c > 0$ depending on b and δ.
5. The number of cubes along this chain is chosen as $k = -c \ln \left(1 \wedge \left(\dfrac{t}{|x-y|^\alpha} \right)^{-d-\alpha} \right)$. In the above δ is a fixed number.

Notice that, along this chain, by using statements 4 and 3, we have

$$V(z) \leq \frac{c_1}{|z|^\alpha} \leq \frac{c_1}{c^\alpha t} \leq \frac{c'}{t},$$

which implies that

$$p_{\tau_{i+1}}(z_{i+1}, z) \leq C p_{\tau_i}(z_i, z).$$

Now, multiplying the above together, by statement 5 we obtain

$$p_{\frac{t}{2}}(y) \leq C^k p_t(x,y),$$

whence it follows (3.12).

Theorem 3.1. *Assume that* $c > c^*$. *Then the heat equation* (1.1) *has no nonnegative solution except* $u \equiv 0$.

Proof. We consider two cases:

Case 1: $|x-y| \leq t^{\frac{1}{\alpha}}$.

(a) For $|x| \leq 2t^{\frac{1}{\alpha}}$ and $|y| \leq 2t^{\frac{1}{\alpha}}$. In view of Lemma 3.6, there exist two positive constants C_1, η such that

$$p_t(x,y) \geq C_1 \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right)^{-\eta} \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|y|^{d+\alpha}} \right)^{-\eta} p_{\frac{t}{8}}(z),$$

where z satisfies $|x-z| = (a_1 t)^{\frac{1}{\alpha}}$. Thus, by using the scaling property (2.3) and (3.10), we obtain

$$p_t(x,y) \geq C_1 t^{2\frac{d\eta}{\alpha}} \left(1 \wedge \left(\frac{t}{|x|} \right)^{-d-\alpha} \right)^{-\eta} \left(1 \wedge \left(\frac{t}{|y|} \right)^{-d-\alpha} \right)^{-\eta} \beta^2 k^d e^{-tk^\alpha \frac{\lambda_1}{8} - 2\eta k^{\frac{\alpha}{2}} |z|^{\frac{\alpha}{2}}}.$$

Consequently, we have

$$p_{k,t}(x,y) \longrightarrow +\infty \quad \text{as} \quad k \longrightarrow +\infty.$$

(b) For $y = 0$ and $|x| \neq 0$. By using the reproducing formula (or semigroup property) and a comparison argument, we get

$$p_{k,t}(x) = \int p_{k,\frac{t}{2}}(x,z)p_{k,\frac{t}{2}}(z)dz \geq$$

$$\geq \int\limits_{1\leq|z|\leq 2} p_{k,\frac{t}{2}}(x,z)p_{0,\frac{t}{2}}(z)dz \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

where p_0 is the fundamental solution of the problem (1.1) when $V \equiv 0$.

(c) For $|x| \geq 2t^{\frac{1}{\alpha}}$. First, notice that $|x-y| \geq \| y| - |x\|$ and $|y| \geq |x| - t^{\frac{1}{\alpha}} \geq t^{\frac{1}{\alpha}}$. Hence, by using the parabolic Harnack principle in [11] (Proposition 4.3), we obtain

$$p_{k,\frac{t}{2}}(y,\omega) \leq c_1 p_{k,t}(x,y) \quad \text{for} \quad \omega \in B\left(y, \frac{2}{3}t^{\frac{1}{\alpha}}\right),$$

where $c_1 > 0$ is independent of x, y and t. Consequently, there exists a positive constant $c > 0$ such that

$$p_{k,\frac{t}{2}}(y) \leq c p_{k,t}(x,y).$$

Finally, by (3.10), we obtain

$$p_{k,t}(x) \geq c\beta^2 k^d e^{-tk^\alpha \frac{\lambda_1}{2} - 2\eta k^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}}} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow +\infty.$$

Case 2: $|x-y| \geq t^{\frac{1}{\alpha}}$. First, notice that $|y| \leq |x|$ or $|y| \geq |x|$, therefore we have either $|x| \geq \frac{1}{2}t^{\frac{1}{\alpha}}$ or $|y| \geq \frac{1}{2}t^{\frac{1}{\alpha}}$. If both inequalities hold, Lemma 3.7 implies

$$p_{k,t}(x,y) \geq c_1 \left(1 \wedge \left(\frac{t}{|x-y|^\alpha}\right)^{-d-\alpha}\right)^{c_2} p_{k,\frac{t}{2}}(y).$$

Hence, as in the first part, we see that $p_{k,t}(x,y) \longrightarrow \infty$ as $k \longrightarrow \infty$.

(a) For $|x| \geq \frac{1}{2}t^{\frac{1}{\alpha}}$ and $|y| \geq \frac{1}{2}t^{\frac{1}{\alpha}}$. Pick now a point z such that $|y-z| = \frac{t^{\frac{1}{\alpha}}}{4}$ and $|z| = |y| + |y-z|$. By using the Harnack chain argument again as in the proof of Lemma 3.6, we obtain

$$p_{k,t}(x,y) \geq c\left(\frac{t^{\frac{1}{\alpha}}}{|y|^{d+\alpha}}\right)^{-\eta} p_{k,\frac{t}{2}}(x,z). \tag{3.13}$$

Since $|x-z| \geq \frac{t^{\frac{1}{\alpha}}}{4}$, $|x| \geq \frac{t^{\frac{1}{\alpha}}}{2}$ and $|z| \geq \frac{t^{\frac{1}{\alpha}}}{4}$, Lemma 3.7 implies that

$$p_{k,\frac{t}{2}}(x,z) \geq c_3 \left(1 \wedge \left(\frac{t}{|x-z|^\alpha}\right)^{-d-\alpha}\right)^{c_4} p_{k,\frac{t}{4}}(z).$$

On the other hand, notice that $|x-z| \leq |x-y| + \frac{1}{4}t^{\frac{1}{\alpha}}$, hence there exists $c > 0$ such that $|x-z|^\alpha \leq c|x-y|^\alpha$. Therefore,

$$p_{k,\frac{t}{2}}(x,z) \geq c_5 \left(1 \wedge \left(\frac{t}{|x-y|^\alpha}\right)^{-d-\alpha}\right)^{c_4} p_{k,\frac{t}{4}}(z). \tag{3.14}$$

Combining now (3.13) and (3.14), we derive

$$p_{k,t}(x,y) \geq c \left(\frac{t^{\frac{1}{\alpha}}}{|y|^{d+\alpha}}\right)^{-\eta} \left(1 \wedge \left(\frac{t}{|x-y|^\alpha}\right)^{-d-\alpha}\right)^{c_4} p_{k,\frac{t}{4}}(z) \geq$$

$$\geq c \left(1 \wedge \left(\frac{t^{\frac{1}{\alpha}}}{|y|^{d+\alpha}}\right)^{-\eta}\right) \left(1 \wedge \left(\frac{t}{|x-y|^\alpha}\right)^{-d-\alpha}\right)^{c_4} p_{k,\frac{t}{4}}(z),$$

which implies, by using (3.10), that

$$p_{k,t}(x,y) \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty.$$

(b) For $y = 0$. As in the first part, we have

$$p_{k,t}(x) = \int p_{k,\frac{t}{2}}(x,z) p_{k,\frac{t}{2}}(z) dz \geq \int\limits_{1 \leq |z| \leq 2} p_{k,\frac{t}{2}}(x,z) p_{0,\frac{t}{2}}(z) dz \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

which is the desired result.

Theorem 3.1 is proved.

References

1. *Ben Amor A. B., Kenzizi T.* The heat equation for the Dirichlet fractional Laplacian with negative potentials: eistence and blow-up of nonnegative solutions // Acta Math. Sinica (Engl. Ser.). – 2017.
2. *Bogdan K., Byczkowski T.* Potential theory of Schrödinger operator on bounded Lipschitz domains // Stud. Math. – 1999. – **133**, № 1. – P. 53 – 92.
3. *Bogdan K., Byczkowski T., Kulczycki T., Ryznar M., Song R., Vondraček Z.* Potential analysis of stable processes and its extensions // Lect. Notes Math. – 2009. – **1980**.
4. *Baras P., Goldstein J. A.* The heat equation with a singular potential // Trans. Amer. Math. Soc. – 1984. – **284**, № 1. – P. 121 – 139.
5. *Baras P., Goldstein J. A.* Remarks on the inverse square in quantum mechanics // North-Holland Math. Stud. – 1984. – **92**. – P. 31 – 35.
6. *Blumenthal R., Getoor R.* The asymptotic distribution of the eigenvalues for a class of Markov operators // Pacif. J. Math. – 1959. – **9**. – P. 399 – 408.
7. *Berg C., Forst G.* Potential theory on locally compact abelian groups. – New York: Springer-Verlag, 1975.
8. *Bannuelos R., Kulczycki T.* Trace estimates for stable processes // Probab. Theory and Relat. Fields. – 2008. – **142**. – P. 313 – 338.
9. *Chen Z. Q., Kim P., Song R.* Heat kernel estimates for the Dirichlet fractional Laplacian // J. Eur. Math. Soc. – 2010. – **12**. – P. 1307 – 1329.
10. *Cabré X., Martel Y.* Existence versus explosion instantanée pour des equations de la chaleur linéaires avec potentiel singulier // C. R. Math. Acad. Sci. Paris. – 1999. – **329**, № 11. – P. 973 – 978.
11. *Chen Z. Q., Kumagai T.* Heat kernel estimates for stable-like processes on d-sets // Stochast. Process and Appl. – 2003. – **108**. – P. 27 – 62.
12. *Davies E. B.* Spectral theory and differential operators. – Cambridge: Cambridge Univ. Press, 1995.
13. *Davies E. B.* Heat kernels and spectral theory. – Cambridge: Cambridge Univ. Press, 1989.
14. *Doob J.* Classical potential theory and its probabilistic counterpart. – Springer, 1983.

15. *Fukushima M., Oshima Y., Takeda M.* Dirichlet forms and symmetric Markov processes // De Gruyter Stud. Math. – 2011. – **19**. – 489 p.
16. *Goldstein J. A., Zhang Q. S.* On a degenerate heat equation with a singular potential // J. Funct. Anal. – 2001. – **186**. – P. 342 – 359.
17. *Kenzizi T.* Existence of nonnegative solutions for a fractional parabolic equation in the whole space // Ukr. Math. J. – 2019. – **71**, № 8. – P. 1064 – 1072.
18. *Kwaśnicki M.* Ten equivalent definitions of the fractional Laplace operator // arXiv: 1507.07356v2[math.AP], 14 Sep. (2015).
19. *Kaleta K., Kulczycki T.* Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians // Potential Anal. – 2010. – **33**, № 4. – P. 313 – 339.
20. *Bonforte M., Grillo G., Vazquez J. L.* Quantitative local bounds for subcritical semilinear elliptic equations // Milan J. Math. – 2012. – **80**. – P. 65 – 118.
21. *Park H., Song R.* Trace estimates for relativistic stable processes // Potential Anal. – 2014. – **41**. – P. 1273 – 1291.
22. *Roach G. F.* Green's functions: introductory theory with appications. – Van Nostrand, 1970.
23. *Silvestre L.* Regularity of the obstacle problem for a fractional power of the Laplace operator // Commun. Pure and Appl. Math. – 2007. – **60**, № 1. – P. 67 – 112.
24. *Yosida K.* Functional analysis // Classics Math. – Berlin: Springer-Verlag, 1995.
25. *Zolotarev V. M.* One-dimensional stable distributions. – Moscow: Nauka, 1983.
26. *Weyl H.* Über beschränkte quadratische Formen, deren differenz vollstetig ist // Rend. Circ. Math. Palermo. – 1909. – **27**. – P. 373 – 392.

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