UDC 512.5

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# COHOMOLOGY OF $q$-DEFORMED WITT – VIRASORO SUPERALGEBRAS OF THE HOM TYPE

# КОГОМОЛОГІЯ $q$-ДЕФОРМОВАНИХ СУПЕРАЛГЕБР ВІТТА – ВІРАСОРО ТИПУ ХОМА

We study Virasoro-type extensions of the $q$-deformed Witt Hom – Lie superalgebras. Moreover, we provide the cohomology of the $q$-deformed Witt – Virasoro superalgebras of the Hom type.

Вивчаються розширення типу Вірасоро для $q$-деформованих віттівських супералгебр Хома – Лі. Крім того, наведено когомологію $q$-деформованих супералгебр Вітта – Вірасоро типу Хома.

**1. Introduction.** The first instances of Hom-type algebras appeared in physics literature when studying quantum deformations. The first examples deal with oscillator algebra and Virasoro algebras. Various important examples of Lie superalgebras have been constructed starting from the Witt algebra $\mathcal{W}$. It is well-known that $\mathcal{W}$ (up to equivalence and rescaling) has a unique nontrivial one-dimensional central extension, the Virasoro algebra. This is not the case in the superalgebras case, very important examples are the Neveu – Schwarz and the Ramond superalgebras. For further generalizations, we refer to Schlichenmaier's book [13]. The Neveu – Schwarz and Ramond superalgebras are usually called super-Virasoro algebras since they can be viewed as superanalogs of the Virasoro algebra. Their corresponding second cohomology groups with values in the adjoint module are computed in [5]. One may found the adjoint valued second cohomology group computation of Witt and Virasoro algebras in [9, 10, 12]. The $q$-deformed Witt superalgebra $\mathcal{W}^q$ was defined in [1] as a main example of Hom – Lie superalgebras. The cohomology and deformations of $\mathcal{W}^q$ were studied in [2, 3]. The first and second cohomology groups of the $q$-deformed Heisenberg – Virasoro algebra of the Hom type are computed in [6].

In this paper, we aim to study extensions of Hom – Lie superalgebras and discuss mainly the case of $\mathcal{W}^q$ Hom-superalgebra. We provide a characterization of the Virasoro-type extensions of the $q$-Witt superalgebra and study their cohomology. In Section 2, we review the basics about Hom – Lie superalgebras and their cohomology. In Section 3, we give some observations about graded algebras and Hom – Lie algebras. In Section 4, we discuss their extensions; and in Section 5, we describe the $q$-Witt superalgebra extensions of the Virasoro type. Section 6 is dedicated to some computations derivations of Virasoro-type extensions of $q$-Witt superalgebras and in Section 7, we deal with adjoint cohomology.

**2. Preliminaries.** In this section, we recall definitions of Hom – Lie superalgebras, $q$-deformed Witt superalgebra and some basics about representations and cohomology. For more details we refer to [2].

**Definition 2.1.** *A Hom – Lie superalgebra is a triple $(\mathcal{G}, [\cdot,\cdot], \alpha)$ consisting of a superspace $\mathcal{G}$, an even bilinear map $[\cdot,\cdot] : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ and an even superspace homomorphism $\alpha : \mathcal{G} \to \mathcal{G}$ satisfying*

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

$$(-1)^{|x||z|}\big[\alpha(x),[y,z]\big]+(-1)^{|z||y|}\big[\alpha(z),[x,y]\big]+(-1)^{|y||x|}\big[\alpha(y),[z,x]\big]=0,$$

*for all homogeneous element $x,y,z$ in $\mathcal{G}$ and where $|x|$ denotes the degree of the homogeneous element $x$.*

***2.1. A q-deformed Witt superalgebra.*** A $q$-deformed Witt superalgebra $\mathcal{W}^q$ can be presented as the $\mathbb{Z}_2$-graded vector space with $\{L_n\}_{n\in\mathbb{Z}}$ as a basis of the even homogeneous part and $\{G_n\}_{n\in\mathbb{Z}}$ as a basis of the odd homogeneous part. It is equipped with the commutator

$$[L_n,L_m]=\big(\{m\}-\{n\}\big)L_{n+m}, \tag{2.1}$$

$$[L_n,G_m]=\big(\{m+1\}-\{n\}\big)G_{n+m}, \tag{2.2}$$

where $\{m\}$ denotes the $q$-number $m$, that is $\{m\}=\dfrac{1-q^m}{1-q}$. The other brackets are obtained by supersymmetry or are equal to $0$.

The even linear map $\alpha$ on $\mathcal{W}^q$ is defined on the generators by

$$\alpha(L_n)=(1+q^n)L_n,\qquad \alpha(G_n)=(1+q^{n+1})G_n. \tag{2.3}$$

For more details, we refer to [1].

***2.2. Cohomology of Hom – Lie superalgebras.*** Let $(\mathcal{G},[\cdot,\cdot],\alpha)$ be a Hom – Lie superalgebra and $V=V_0\oplus V_1$ be an arbitrary vector superspace. Let $\beta\in\mathcal{G}l(V)$ be an arbitrary even linear self-map on $V$ and

$$[\cdot,\cdot]_V:\mathcal{G}\times V\to V,$$

$$(g,v)\mapsto[g,v]_V$$

a bilinear map satisfying $[\mathcal{G}_i,V_j]_V\subset V_{i+j}$ where $i,j\in\mathbb{Z}_2$.

**Definition 2.2.** *The triple $(V,[\cdot,\cdot]_V,\beta)$ is called a representation of the Hom – Lie superalgebra $\mathcal{G}=\mathcal{G}_0\oplus\mathcal{G}_1$ or $\mathcal{G}$-module $V$ if the even bilinear map $[\cdot,\cdot]_V$ satisfies, for $x,y\in\mathcal{G}$ and $v\in V$,*

$$\big[[x,y],\beta(v)\big]_V=\big[\alpha(x),[y,v]\big]_V-(-1)^{|x||y|}\big[\alpha(y),[x,v]\big]_V. \tag{2.4}$$

***Remark* 2.1.** When $[\cdot,\cdot]_V$ is the zero-map, we say that the module $V$ is trivial.

***Remark* 2.2.** $(\mathcal{G},[\cdot,\cdot],\alpha)$ is a representation of the Hom – Lie superalgebra $(\mathcal{G},[\cdot,\cdot],\alpha)$, which we call the adjoint representation of $\mathcal{G}$.

**Definition 2.3** [2]**.** *The set $C^k(\mathcal{G},V)$ of $k$-cochains on space $\mathcal{G}$ with values in $V$ is the set of $k$-linear maps $f:\otimes^k\mathcal{G}\to V$ satisfying*

$$f(x_1,\ldots,x_i,x_{i+1},\ldots,x_k)=-(-1)^{|x_i||x_{i+1}|}f(x_1,\ldots,x_{i+1},x_i,\ldots,x_k)\quad \textit{for}\quad 1\le i\le k-1.$$

For $k=0$ we have $C^0(\mathcal{G},V)=V$. We denote the sets $C^k(\mathcal{G},\mathcal{G})$ by $C^k(\mathcal{G})$.

A $k$-cochain on $\mathcal{G}$ with values in $V$ is defined to be a $k$-Hom-cochain $f\in C^k(\mathcal{G},V)$ if it is compatible with $\alpha$ and $\beta$ in the sense that $\beta\circ f=f\circ\alpha$. Denote $C^k_{\alpha,\beta}(\mathcal{G},V)$ the set of $k$-Hom-cochain.

**Definition 2.4.** *We call* 1*-coboundary operator of Hom – Lie superalgebra* $\mathcal{G}$ *the map*

$$\delta^1_{\mathcal{G}} : C^1_{\alpha,\beta}(\mathcal{G}, V) \to C^2_{\alpha,\beta}(\mathcal{G}, V), \qquad f \mapsto \delta^1 f$$

*defined by*

$$\delta^1_{\mathcal{G}}(f)(x, y) = -f([x, y]) + (-1)^{|x||f|}[x, f(y)]_V - (-1)^{|y|(|f|+|x|)}[y, f(x)]_V. \tag{2.5}$$

**Definition 2.5.** *We call* 2*-coboundary operator of Hom – Lie superalgebra* $\mathcal{G}$ *the map*

$$\delta^2_{\mathcal{G}} : C^2_{\alpha,\beta}(\mathcal{G}, V) \to C^3_{\alpha,\beta}(\mathcal{G}, V), \qquad f \mapsto \delta^2 f$$

*defined by*

$$\delta^2_{\mathcal{G}}(f)(x, y, z) = -f([x, y], \alpha(z)) + (-1)^{|z||y|}f([x, z], \alpha(y)) + f(\alpha(x), [y, z])+$$

$$+(-1)^{|x||f|}[\alpha(x), f(y, z)]_V - (-1)^{|y|(|f|+|x|}[\alpha(y), f(x, z)]_V+$$

$$+(-1)^{|z|(|f|+|x|+|y|)}[\alpha(z), f(x, y)]_V. \tag{2.6}$$

A straightforward calculation shows that $\delta^2 \circ \delta^1 = 0$. We have with respect to the cohomology defined by the coboundary operators

$$\delta^k_{\mathcal{G}} : C^k_{\alpha,\beta}(\mathcal{G}, V) \to C^{k+1}_{\alpha,\beta}(\mathcal{G}, V), \quad k \in \{0, 1\}.$$

The $k$-cocycles space is defined as $Z^k(\mathcal{G}) = \ker\ \delta^k_{\mathcal{G}}$.

The $k$-coboundary space is defined as $B^k(\mathcal{G}) = \operatorname{Im} \delta^{k-1}_{\mathcal{G}}$.

The $k^{th}$ cohomology space is the quotient $H^k(\mathcal{G}) = Z^k(\mathcal{G})/B^k(\mathcal{G})$. It decomposes as well as even and odd $k^{th}$ cohomology spaces.

**3. Graded superalgebras.** In this section, we give some observations about graded algebras and provide a bracket characterizing Hom – Lie superalgebras.

**Definition 3.1.** *Let* $\Delta$ *be a commutative group. A graded vector superspace of type* $\Delta$ *is a vector superspace* $K = K_0 \oplus K_1$ *together with a family* $\{K^\alpha_0 \oplus K^\alpha_1\}_{\alpha\in\Delta}$ *of subspace of* $K$ *indexed by* $\Delta$*, such that* $K$ *is the direct sum of the family* $\{K^\alpha_0 \oplus K^\alpha_1\}_{\alpha\in\Delta}$ *of subspaces. We set* $K^\alpha = K^\alpha_0 \oplus K^\alpha_1$*. The elements of* $K^\alpha$ *are called homogeneous of degree* $\alpha$*.*

An even (resp. odd) linear map of graded superspace $K = \oplus_{\alpha\in\Delta}K^\alpha$ into a graded superspace $L = \oplus_{\alpha\in\Delta}L^\alpha$ is homogeneous of degree $\beta$ if, for every $\alpha \in \Delta$, we have $f(K^\alpha) \subset L^{\alpha+\beta}$.

***3.1. A*** $\mathbb{Z}_2 \times \mathbb{Z}$***-graded superspace.*** Let $K$ a superspace over the field $\mathbb{C}$. Consider the cochains $\varphi \in C^k(K)$, $\psi \in C^l(K)$ and $\gamma \in \operatorname{End}(K)_0$, $n = k + l - 1$. We define $\varphi \circ \psi \in C^n(K)$ by

$$\varphi \circ (\psi, \gamma)(v_1, \ldots, v_n) = \sum_{\sigma\in Sh(l,k-1)} \epsilon(\sigma)\varphi\Big(\psi\big(v_{\sigma(1)}, \ldots, v_{\sigma(l)}\big), \gamma(v_{\sigma(l+1)}), \ldots, \gamma(v_{\sigma(n)})\Big),$$

where $\epsilon(\sigma)$ is a sign determined by the rule $v_{\sigma(1)} \ldots v_{\sigma(n)} = \epsilon(\sigma)v_1 \ldots v_n$ and $Sh(l, k - l)$ are the permutations in $\sum_n$ which are increasing on the first $l$ and the last $k - 1$ elements.

The set $C^*(K) = \oplus^{+\infty}_{k=1}C^k(K)$ has a natural $\mathbb{Z}_2\times\mathbb{Z}$-grading, with the bidegree of a homogeneous element $\varphi \in C^n(K)$ given by $\operatorname{bideg}(\varphi) = (|\varphi|, n-1)$. There is a natural bracket operation in $C^*(K)$ given by

$$[\varphi, \psi]^* = \psi \circ (\varphi, \gamma) - (-1)^{|\varphi||\psi|+(k-1)(l-1)}\varphi \circ (\psi, \gamma^{k-1}). \tag{3.1}$$

By a simple calculation we obtain the following results.

**Proposition 3.1.** *Let $d \in C^2(K)$ be an even super-skew-symmetric bilinear operator. The triple $(K, d, \gamma)$ defines a Hom – Lie superalgebra if and only if $[d, d]^* = 0$.*

**Proposition 3.2.** *Let $(K, d, \gamma)$ be a Hom – Lie superalgebra and $f$ be a $k$-cochain. Then $\delta^k_{\mathcal{G}}(f) = [d, f]^*$.*

These results generalize the observations in the classical case which may be seen in [9, 11].

***3.2. Graded Hom – Lie superalgebras.*** A Hom – Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ is said to be $\mathbb{Z}$-graded if the superalgebra $\mathcal{G}$ is graded of type $\mathbb{Z}$, $\alpha(\mathcal{G}^n) \subset \mathcal{G}^n$ and $[\mathcal{G}^n, \mathcal{G}^m] \subset \mathcal{G}^{n+m}$ for all $n,\ m \in \mathbb{Z}$.

***Example* 3.1.** The $q$-deformed Witt superalgebra is $\mathbb{Z}$-graded by

$$\deg L_n = \deg G_n = n.$$

**4. Construction of Hom – Lie superalgebras by extensions.** An extension of a Hom – Lie superalgebra $(\mathcal{G}, [\cdot, \cdot], \alpha)$ by a representation $(V, [\cdot, \cdot]_V, \beta)$ is an exact sequence

$$0 \longrightarrow (V, \beta) \stackrel{i}{\longrightarrow} (K, \gamma) \stackrel{\pi}{\longrightarrow} (\mathcal{G}, \alpha) \longrightarrow 0$$

satisfying $\gamma\ o\ i = i\ o\ \beta$ and $\alpha\ o\ \pi = \pi\ o\ \gamma$. This extension is said to be central if $[K, i(V)]_K = 0$.

In particular, if $K = \mathcal{G} \oplus V$, $i(v) = v\ \forall v \in V$ and $\pi(x) = x\ \forall x \in \mathcal{G}$, then we have $\gamma(x, v) = (\alpha(x), \beta(v))$ and we denote

$$0 \longrightarrow (V, \beta) \longrightarrow (K, \gamma) \longrightarrow (\mathcal{G}, \alpha) \longrightarrow 0.$$

Now we shall determine the 2-cochains $d$ such that $(K, d, \gamma)$ is a Hom – Lie superalgebra.

For convenience, we introduce the following notation for certain cochains spaces on $K = \mathcal{G} \oplus V$, $\mathcal{C}(\mathcal{G}^n, \mathcal{G})$ and $\mathcal{C}(\mathcal{G}^k V^l, V)$, where $\mathcal{G}^k V^l$ is the subspace of $K^{k+l}$ determined by products of $k$ elements from $\mathcal{G}$ and $l$ elements from $V$.

For $f \in \mathcal{C}^2(K, K)$, we set $f = \widetilde{f} + \widehat{f} + \overline{f} + v + \widehat{v} + \overline{v}$, where $\widetilde{f} \in \mathcal{C}(\mathcal{G}^2, \mathcal{G})$, $\widehat{f} \in \mathcal{C}(\mathcal{G}V, \mathcal{G})$, $\overline{f} \in \mathcal{C}(V^2, \mathcal{G})$, $v \in \mathcal{C}(\mathcal{G}^2, V)$, $\widehat{v} \in \mathcal{C}(\mathcal{G}V, V)$ and $\overline{v} \in \mathcal{C}(V^2, V)$.

Let $d \in C^2(K)$ be an even super-skew-symmetric bilinear operator. If $(K, d, \gamma)$ is a Hom – Lie superalgebra and $V$ is an ideal in $K$ (i.e., $d(K, V) \subset V$), we obtain by using the above notation:

$\widehat{d} \equiv 0$,

$\overline{d} \equiv 0$,

$0 = [d, d]^*(x, y, z) = \Big([\widetilde{d}, \widetilde{d}]^2(x, y, z); 2[v, \widetilde{d}]^*(x, y, z) + 2[\widehat{v}, v]^*(x, y, z)\Big)$ (i.e., $[\widetilde{d}, \widetilde{d}]^*(x, y, z) \in \mathcal{G}$ and $\big(2[v, \widetilde{d}]^* + 2[\widehat{v}, v]^*\big)(x, y, z) \in V\ \forall x, y, z \in \mathcal{G}\big)$,

$0 = [d, d]^*(x, y, w) = \Big([\widehat{v}, \widehat{v}]^* + 2[\widetilde{d}, \widehat{v}]^* + 2[\overline{v}, v]^*\Big)(x, y, w)\ \forall (x, y, w) \in \mathcal{G}^2 \times V$,

$0 = [d, d]^*(x, v, w) = 2[\overline{v}, \widehat{v}]^*(x, v, w)\ \forall (x, v, w) \in \mathcal{G} \times V^2$,

$0 = [d, d]^*(u, v, w) = \dfrac{1}{2}[\overline{v}, \overline{v}]^*(u, v, w)\ \forall (u, v, w) \in V^3$.

Therefore we have the following theorem.

**Theorem 4.1.** *A triple $(K, d, \gamma)$ is a Hom – Lie superalgebra if and only if the following conditions are satisfied:*

$\big(\mathcal{G}, \widetilde{d}, \alpha\big)$ *is a Hom – Lie superalgebra,*

$[v, \widetilde{d}]^* + [\widehat{v}, v]^* \equiv 0$,

$\dfrac{1}{2}[\widehat{v}, \widehat{v}]^* + [\widetilde{d}, \widehat{v}]^* + [\overline{v}, v]^* \equiv 0$,

$[\overline{v},\widehat{v}]^*\equiv 0,$

$(V,\overline{v},\beta)$ *is a Hom – Lie superalgebra.*

***Corollary* 4.1.** *If* $\overline{v}\equiv 0$, *the triple* $(K,d,\gamma)$ *is a Hom – Lie superalgebra if and only if the following conditions are satisfied:*

$\big(\mathcal{G},\widetilde{d},\alpha\big)$ *is a Hom – Lie superalgebra,*

$\big(V,\widehat{v},\beta\big)$ *is a representation of* $\mathcal{G}$,

$v$ *is an even* 2*-cocycle on* $V$ $\big($*with the cohomology defined by* $\big(\mathcal{G},\widetilde{d},\alpha\big)$ *and* $\big(V,\widehat{v},\beta\big)\big)$.

***Proof.*** Let $(K,d,\gamma)$ be a Hom – Lie superalgebra and assume that $\overline{v}\equiv 0$. It follows from Theorem 4.1 that $\big(\mathcal{G},\widetilde{d},\alpha\big)$ is a Hom – Lie superalgebra and

$$\frac{1}{2}[\widehat{v},\widehat{v}]^*+\big[\widetilde{d},\widehat{v}\big]^*\equiv 0.$$

By using (3.1) (for $l=k=2,(v_1,v_2,v_3)=(x,y,u)$), we have

$$-(-1)^{|y||u|}\widehat{v}\big(\widehat{v}(x,u),\alpha(y)\big)+(-1)^{|x|(|u|+|y|)}\widehat{v}(\widehat{v}(y,u),\alpha(x))+\widehat{v}\big(\widetilde{d}(x,y),\beta(u)\big)=0.$$

Therefore,

$$\widehat{v}\big(\widetilde{d}(x,y),\beta(u)\big)=\widehat{v}\big(\alpha(x),\widehat{v}(y,u)\big)-(-1)^{|y||x|}\widehat{v}\big(\alpha(y),\widehat{v}(x,u)\big).$$

Then $\widehat{v}$ satisfies condition (2.4). Hence, $\widehat{v}$ is a representation of $\mathcal{G}$.

By Theorem 4.1, we have $\big(\big[v,\widetilde{d}\big]^*+[\widehat{v},v]^*\big)(x,y,z)=0\ \ \forall x,y,z\in\mathcal{G}$.

Then, by using (3.1), we obtain

$$v\big(\widetilde{d}(x,y),\alpha(z)\big)-(-1)^{|y||z|}v\big(\widetilde{d}(x,z),\alpha(y)\big)+(-1)^{|x|(|y|+|z|)}v\big(\widetilde{d}(y,z),\alpha(x)\big)+$$

$$+\widehat{v}\big(v(x,y),\alpha(z)\big)-(-1)^{|y||z|}\widehat{v}\big(v(x,z),\alpha(y)\big)+(-1)^{|x|(|y|+|z|)}\widehat{v}\big(v(y,z),\alpha(x)\big)=0.$$

Consequently, the even 2-cochain $\varphi$ is a 2-cocycle on $V$ $\big($with the cohomology defined by $\big(\mathcal{G},\widetilde{d},\alpha\big)\big)$.

**Theorem 4.2.** *Let* $(\mathcal{G},[\cdot,\cdot],\alpha)$ *be Hom – Lie superalgebra and* $(V,[\cdot,\cdot]_V,\beta)$ *be a representation. The even second cohomology space* $H^2_0(\mathcal{G},V)=Z^2_0(\mathcal{G},V)/B^2_0(\mathcal{G},V)$ *is in one-to-one correspondence with the set of the equivalence classes extensions of* $(\mathcal{G},[.,.],\alpha)$ *by* $(V,[\cdot,\cdot]_V,\beta)$.

***Proof.*** Let $(\mathcal{G}\oplus V,d,\gamma)$ and $(\mathcal{G}\oplus V,d',\gamma)$ be two extensions of $(\mathcal{G},[.,.],\alpha)$. So there are two even cocycles $\varphi$ and $\varphi'$ such as $d\big((x,u);(y,v)\big)=\Big([x,y],[x,v]_V+[u,y]_V+\varphi(x,y)\Big)$ and $d'\big((x,u);(y,v)\big)=\Big([x,y],[x,v]_V+[u,y]_V+\varphi'(x,y)\Big)$.

If $\varphi-\varphi'=\delta^1h(x,y)$, where $h\colon\mathcal{G}\to V$ is a linear map satisfying $h\circ\alpha=\beta\circ h$ (i.e., $\varphi-\varphi'\in B^2(\mathcal{G},V)$). Let us define $\Phi\colon(\mathcal{G}\oplus V,d,\gamma)\to(\mathcal{G}\oplus V,d',\gamma)$ by $\Phi(x,v)=(x,v-h(x))$. It is clear that $\Phi$ is bijective. Let us check that $\Phi$ is a Hom – Lie superalgebras homomorphism. We have

$$d\big(\Phi((x,v)),\Phi((y,w))\big)=d\big((x,v-h(x)),(y,w-h(y))\big)=$$

$$=\big([x,y],[x,w]_V+[v,y]_V-[x,h(y)]_V-[h(x),y]_V+\varphi(x,y)\big)=$$

$$=\big([x,y],[x,w]_V+[v,y]_V-\delta^1(h)(x,y)+\varphi(x,y)-h([x,y])\big)=$$

$$=\Phi\big(([x,y],f(x,y))\big)=$$

$$= \big([x,y],[x,w]_V + [v,y]_V + \varphi'(x,y) - h([x,y])\big) =$$

$$= \Phi\big([x,y],[x,w]_V + [v,y]_V + \varphi'(x,y)\big) =$$

$$= \Phi\big(d'((x,v),(y,w))\big).$$

We summarize the main facts in the following theorem.

**Theorem 4.3.** *Let $(V,[\cdot,\cdot]_V,\beta)$ be a representation of a Hom – Lie superalgebra $(\mathcal{G},[.,.],\alpha)$.*

(i) *Let*

$$0 \longrightarrow (V,\beta) \longrightarrow (\mathcal{G}\oplus V,\gamma) \longrightarrow (\mathcal{G},\alpha) \longrightarrow 0$$

*be an extension of $(\mathcal{G},[.,.],\alpha)$ by a representation $(V,[\cdot,\cdot]_V,\beta)$, where $\gamma$ is defined by $\gamma(x,u) = \big(\alpha(x),\beta(u)\big)$, for all $x\in\mathcal{G}$ and $u\in V$. Let $d\in C^2(\mathcal{G}\oplus V,\mathcal{G}\oplus V)_0$ be an even 2-cochain.*

*The triple $(\mathcal{G}\oplus V,d,\gamma)$ is a Hom – Lie superalgebra if and only if*

$$d\big((x,u);(y,v)\big) = \Big([x,y],[x,v]_V - (-1)^{|u||y|}[y,u]_V + \varphi(x,y)\Big), \tag{4.1}$$

*where $\varphi$ is a even 2-cocycle (i.e., $\varphi\in Z^2(\mathcal{G},V)_0$).*

(ii) *If $(\mathcal{G}\oplus V,[\cdot;\cdot]+[\cdot,\cdot]_V+\varphi,\gamma)$ and $\big(\mathcal{G}\oplus V,[\cdot;\cdot]+[\cdot,\cdot]_V+\varphi',\gamma\big)$ are two extensions of the Hom – Lie superalgebra $(\mathcal{G},[.,.],\alpha)$ by a representation $(V,[\cdot,\cdot]_V,\beta)$, then they are equivalent extensions precisely when $\varphi-\varphi'\in B^1(\mathcal{G},V)$.*

***Remark* 4.1.** If $\varphi=0$, the Hom – Lie superalgebra $(\mathcal{G}\oplus V,d,\gamma)$ is called the semidirect product of the Hom – Lie superalgebra $(\mathcal{G},d,\alpha)$ and $(V,[\cdot,\ \cdot]_V,\beta)$.

**5. $q$-Deformed Witt – Virasoro superalgebras of the Hom type.** The Virasoro algebra is the central extension of the Witt algebra with the choice the 2-cocycle defined by $\beta(m)=\dfrac{1}{12}(m^3-m)$. The Virasoro algebra is spanned by $\{L_m : m\in\mathbb{Z}\}\cup\{c\}$ such that $c$ is central, i.e., $[c,L_m]=0$ for all $m\in\mathbb{Z}$, and $[L_m,L_n]=(m-n)L_{m+n}+\delta_{n+m,0}\dfrac{1}{12}(m^3-m)c$ (see [10, 12]).

The *Neveu – Schwarz superalgebra* [5] can be presented as the $\mathbb{Z}_2$-graded vector space with $\{L_n,D\}_{n\in\mathbb{Z}}$ as a basis of the even homogeneous part and $\{F_n\}_{n\in\frac{1}{2}+\mathbb{Z}}$ as a basis of the odd homogeneous part. It is equipped with the commutator

$$[L_n,L_m]=(m-n)L_{n+m}+\frac{(n+1)n(n-1)}{8}\delta_{n+m,0}D,$$

$$[L_n,F_m]=\left(\frac{1}{2}m-n\right)F_{n+m},$$

$$[F_n,F_m]=2L_{n+m}+\frac{1}{2}\left(n^2-\frac{1}{4}\right)\delta_{n+m,0}D,$$

and the property that $D$ is central. The *Ramond superalgebra* (see [5]) has $\{L_n,D\}_{n\in\mathbb{Z}}$ as a basis of the even homogeneous part and $\{F_n\}_{n\in\mathbb{Z}}$ as a basis of the odd homogeneous part. It is equipped with the commutator

$$[L_n,L_m]=(m-n)L_{n+m}+\frac{n^3}{8}\delta_{n+m,0}D,$$

$$[L_n,F_m]=\left(\frac{1}{2}m-n\right)F_{n+m},$$

$$[F_n, F_m] = 2L_{n+m} + \frac{1}{2}n^2\delta_{n+m,0}D,$$

where $D$ is defined central.

The $q$-deformed Heisenberg – Virasoro algebra $\widehat{\mathcal{H}}$ of the Hom type [6] is a complex Hom – Lie algebra with basis $\{C_L, C_I, C_{LI}, L_n, I_n \in \mathbb{Z}\}$ with the following relations:

$$[L_n, L_m] = \big(\{n\} - \{m\}\big)L_{n+m} + \delta_{n+m,0}\frac{q^{-n}}{6(1+q^n)}\{n+1\}\{n\}\{n-1\}C_L,$$

$$[L_n, I_m] = -\{m\}I_{n+m} + \delta_{n+m,0}\frac{2q^{-n}}{(1+q^n)}\{n+1\}\{n\}C_{LI},$$

$$[I_n, I_m] = \delta_{n+m,0}\frac{2q^{-n}}{(1+q^{-n})}\{n\}C_I,$$

$$[L_n, C_L] = [I_n, C_L] = 0, \qquad [L_n, C_{LI}] = [I_n, C_{LI}] = 0, \qquad [L_n, C_I] = [I_n, C_I] = 0,$$

$$\alpha(L_n) = (1+q^n)L_n, \qquad \alpha(I_n) = (1+q^n)I_n.$$

Now, we describe central extensions of $q$-deformed Witt superalgebra $\mathcal{W}^q$ (see subsection 2.1). We recall first the following result describing its scalar cohomology.

**Theorem 5.1** [2]. *We have*

$$H^2(\mathcal{W}^q, \mathbb{C}) = \mathbb{C}[\varphi_0] \oplus \mathbb{C}[\varphi_1],$$

*where*

$$\varphi_0(L_n, L_m) = \delta_{n+m,0}\frac{1}{q^{n-2}}\frac{1+q^2}{1+q^n}\frac{\{n+1\}\{n\}\{n-1\}}{\{3\}\{2\}},$$

$$\varphi_0(L_n, G_m) = 0, \qquad \varphi_0(G_n, G_m) = 0,$$

*and*

$$\varphi_1(L_n, G_m) = \begin{cases} \delta_{n+m,-1}\frac{1}{q^{n-2}}\frac{1+q^2}{1+q^n}\frac{\{n+1\}\{n\}\{n-1\}}{\{3\}\{2\}}, & \textit{if} \quad n \geq 0, \\ -\delta_{n+m,-1}\varphi_1(L_{-n}, G_m), & \textit{if} \quad n < 0, \end{cases}$$

$$\varphi_1(L_n, L_m) = 0, \qquad \varphi_1(G_n, G_m) = 0.$$

Let

$$0 \longrightarrow (V, \beta) \longrightarrow (\mathcal{W}^q \oplus V, \gamma) \longrightarrow (\mathcal{G}, \alpha) \longrightarrow 0$$

be a central extension of $(\mathcal{W}^q, [.,.], \alpha)$ by a representation $(V, [\cdot,\cdot]_V, \beta)$, where $\gamma$ is defined by $\gamma(x, u) = \big(\alpha(x), \beta(u)\big)$, for all $x \in \mathcal{W}^q$ and $u \in V$.

As a central extension, we have $[\cdot,\cdot]_V = 0$ and

$$d\big((x,u);(y,v)\big) = \Big([x,y], [x,v]_V - (-1)^{|u||y|}[y,u]_V + \varphi(x,y)\Big) = \big([x,y], \varphi(x,y)\big),$$

where the bracket $[\cdot,\cdot]$ is defined in (2.1) and $\varphi$ is an even 2-cocycle in $Z^2(\mathcal{W}^q, V)$.

Suppose that $V$ is finite dimensional. Let $(u_1, \ldots, u_n)$ be a basis for $V_0$ and $(w_1, \ldots, w_m)$ be a basis for $V_1$. Since $\varphi(x,y) \in V_0$ for all $(x,y) \in V_i \times V_i$ and $\varphi(z,t) \in V_1$ for all $(z,t) \in V_0 \times V_1$ or $(z,t) \in V_1 \times V_0$, one can assume $\varphi(x,y) = \sum_{k=1}^{n} e_k(x,y) u_k$ and $\varphi(z,t) = \sum_{k=1}^{m} f_k(z,t) w_k$. It easy to check that $e_k$ and $f_k$ are 2-cocycles with value in $\mathbb{C}$. Hence, by Theorem 5.1, there exist $c_0 \in V_0$ and $c_1 \in V_1$ such that $\varphi(L_n, L_m) = \varphi_0(L_n, L_m) c_0$ and $\varphi(L_n, G_m) = \varphi_1(L_n, G_m) c_1$. Consequently, we obtain a $q$-deformed Witt–Virasoro superalgebra $\widehat{\mathcal{W}^q}$ of the Hom type defined by the following bracket and linear map:

$$d(L_n, L_m) = \big(\{m\} - \{n\}\big) L_{n+m} + \delta_{n+m,0} \frac{1}{q^{n-2}} \frac{1+q^2}{1+q^n} \frac{\{n+1\}\{n\}\{n-1\}}{\{3\}\{2\}} c_0,$$

$$d(L_n, G_m) = \big(\{m+1\} - \{n\}\big) G_{n+m} + \delta_{n+m,-1} \frac{1}{q^{n-2}} \frac{1+q^2}{1+q^n} \frac{\{n+1\}\{n\}\{n-1\}}{\{3\}\{2\}} c_1, \quad \text{if} \quad n \geq 0,$$

$$d(L_n, G_m) =$$

$$= \big(\{m+1\} - \{n\}\big) G_{n+m} - \delta_{n+m,-1} \frac{1}{q^{-n-2}} \frac{1+q^2}{1+q^{-n}} \frac{\{-n+1\}\{-n\}\{-n-1\}}{\{3\}\{2\}} c_1, \quad \text{if} \quad n < 0,$$

$$d(G_n, G_m) = [G_n, c_0] = [G_n, c_1] = [L_n, c_0] = [L_n, c_1] = 0,$$

$$\gamma(L_n) = (1+q^n) L_n, \qquad \gamma(G_n) = (1+q^{n+1}) L_n, \qquad \gamma(c_0) = c_0, \qquad \gamma(c_1) = c_1.$$

***Remark* 5.1.** The even 2-cocycles $\varphi$ may be written in the form $\varphi = \varphi_0 c_0 + \varphi_1 c_1$.

***Remark* 5.2.** The set of even 2-cocycles on $V$ is

$$Z^2(\mathcal{W}^q, V)_0 = \big\{\lambda_0 \varphi_0 c_0 + \lambda_1 \varphi_1 c_1 : \lambda_0, \lambda_1 \in \mathbb{C}\big\}.$$

***Remark* 5.3.** The set of odd 2-cocycles on $V$ is

$$Z^2(\mathcal{W}^q, V)_1 = \big\{\lambda_0 \varphi_1 c_1 + \lambda_1 \varphi_0 c_1 : \lambda_0, \lambda_1 \in \mathbb{C}\big\}.$$

**Proposition 5.1.**

$$Z^2(\mathcal{W}^q, V)_i \subset B^1(\widehat{\mathcal{W}^q}). \tag{5.1}$$

***Proof.*** Let $w_i \in Z^2(\mathcal{W}^q, V)_i$, $i \in \mathbb{Z}_2$. Then there exist some number $\lambda_0, \lambda_1 \in \mathbb{C}$ such that $w_i = \lambda_0 \varphi_i c_0 + \lambda_1 \varphi_{i+1} c_1$.

We define the linear map $\widehat{v} : V \to V$ by $\widehat{v}(c_0) = \lambda_0 c_0$ and $\widehat{v}(c_1) = \lambda_1 c_1$. Then

$$\delta^1_{\widehat{\mathcal{W}^q}}(\widehat{v})(x,y) = -\widehat{v}(\varphi(x,y)) = -\lambda_0 \varphi_0(x,y) c_0 - \lambda_1 \varphi_1(x,y) c_1 = -w_0(x,y).$$

We define the linear map $\widehat{v}_1 : V \to V$ by $\widehat{v}_1(c_0) = \lambda_1 c_1$ and $\widehat{v}_1(c_1) = \lambda_0 c_0$. Then

$$\delta^1_{\widehat{\mathcal{W}^q}}(\widehat{v}_1)(x,y) = -\widehat{v}_1(\varphi(x,y)) = -\lambda_1 \varphi_0(x,y) d - \lambda_0 \varphi_1(x,y) c_0 = -w_1(x,y).$$

***Remark* 5.4.** The Hom–Lie superalgebra $\widehat{\mathcal{W}^q}$ is $\mathbb{Z}$-graded by

$$\deg(L_n) = \deg(G_n) = n, \qquad \deg(c_0) = 0, \qquad \deg(c_1) = -1.$$

**6. Derivations of $q$-deformed Witt – Virasoro superalgebras of the Hom type.** Recall that an $\alpha^r$-derivation $D$ of a Hom – Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ is a linear homogeneous map in $\mathcal{G}$ satisfying $D \circ \alpha = \alpha \circ D$ and

$$D([x,y]) = \big[D(x), \alpha^r(y)\big] + (-1)^{|x||D|}\big[\alpha^r(x), D(y)\big].$$

Notice that $D$ is an $\alpha^0$-derivation if and only if $D$ is a 1-cocycle associated to the adjoint representation.

Let $0 \longrightarrow (V,\beta) \longrightarrow (K,d,\gamma) \longrightarrow (\mathcal{G},\alpha) \longrightarrow 0$ be an extension of $(\mathcal{G},\delta,\alpha)$ by a representation $(V,\lambda,\beta)$, where $K = \mathcal{G} \oplus V$ and $d = \delta + \lambda + \varphi$ $(\varphi \in Z^2(\mathcal{G},V)_0)$.

***6.1. Derivations of K.*** Let $f \in \mathcal{C}^1(K,K)$ and set $f = \widetilde{f} + \widehat{f} + v + \widehat{v}$ where $\widetilde{f} \in \mathcal{C}^1(\mathcal{G},\mathcal{G})$, $\widehat{f} \in \mathcal{C}^1(V,\mathcal{G})$, $v \in \mathcal{C}^1(\mathcal{G},V)$ and $\widehat{v} \in \mathcal{C}^1(V,V)$. For $f \in \mathcal{C}^1(K,K)$, by Proposition 3.2, we have $\delta^1_K(f) = [d,f]^*$. Then

$$\big(f \text{ is a derivation of } K\big) \Leftrightarrow \big([d,f]^* \equiv 0\big). \tag{6.1}$$

Let $f$ be a derivation. For all $x,y \in \mathcal{G}$, we obtain

$$0 = [d,f]^*(x,y) = \Big(\big[\delta,\widetilde{f}\big]^*(x,y) + \big[\varphi,\widehat{f}\big]^*(x,y), [\delta+\lambda,v]^*(x,y) + \big[\varphi,\widetilde{f}+\widehat{v}\big]^*(x,y)\Big).$$

So, $\big[\delta,\widetilde{f}\big]^*(x,y) + \big[\varphi,\widehat{f}\big]^*(x,y) = 0$ and $[\delta+\lambda,v]^*(x,y) + \big[\varphi,\widetilde{f}+\widehat{v}\big]^*(x,y) = 0$.

For all $x \in \mathcal{G},\ u \in V$, we get

$$0 = [d,f](x,v) = \Big(\big[\delta+\lambda,\widehat{f}\big]^*(x,v), \big[\lambda,\widetilde{f}+\widehat{v}\big]^*(x,v) + \big[\varphi,\widehat{f}\big]^*(x,v)\Big).$$

So, $\big[\delta+\lambda,\widehat{f}\big]^*(x,v) = 0$ and $\big[\lambda,\widetilde{f}+\widehat{v}\big]^*(x,v) + \big[\varphi,\widehat{f}\big]^*(x,v) = 0$.

For all $u \in V$, we obtain

$$0 = [d,f](u,v) = \big[\lambda,\widehat{f}\big]^*(u,v).$$

Then, under previous assumptions, we have the following theorem.

**Theorem 6.1.** *Let $\mathcal{D}\colon K \to K$ be a 1-Hom-cochain then, $\mathcal{D}$ is a $\alpha^0$-derivation of $K$, if and only if $\mathcal{D}$ satisfies*

$$\Big(\big[\delta,\widetilde{f}\big]^* + \big[\varphi,\widehat{f}\big]^*\Big)(x,y) = 0, \tag{6.2}$$

$$\Big(\big[\varphi,\widetilde{f}+\widehat{v}\big]^* + \big[\delta+\lambda,v\big]^*\Big)(x,y) = 0, \tag{6.3}$$

$$\big[\delta+\lambda,\widehat{f}\big]^*(x,v) = 0, \tag{6.4}$$

$$\Big(\big[\lambda,\widetilde{f}+\widehat{v}\big]^* + \big[\varphi,\widehat{f}\big]^*\Big)(x,v) = 0, \tag{6.5}$$

$$\Big(\big[\lambda,\widehat{f}\big]^*\Big)(u,v) = 0. \tag{6.6}$$

***6.2. Derivations algebra of $\widehat{\mathcal{W}^q}$.*** First, we recall a result about $\alpha^0$-derivations of the Hom – Lie superalgebra $\mathcal{W}^q$ (see [2] for the proof).

**Lemma 6.1.** *The set of $\alpha^0$-derivations of the Hom – Lie superalgebra $\mathcal{W}^q$ is*

$$\mathrm{Der}_{\alpha^0}(\mathcal{W}^q) = \langle \mathcal{D}_1 \rangle \oplus \langle \mathcal{D}_2 \rangle \oplus \langle \mathcal{D}_3 \rangle \oplus \langle \mathcal{D}_4 \rangle,$$

*where $\mathcal{D}_1$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are defined, with respect to the basis, as*

$$\mathcal{D}_1(L_n) = nL_n, \qquad \mathcal{D}_1(G_n) = nG_n,$$

$$\mathcal{D}_2(L_n) = 0, \qquad \mathcal{D}_2(G_n) = G_n,$$

$$\mathcal{D}_3(L_n) = nG_{n-1}, \qquad \mathcal{D}_3(G_n) = 0,$$

$$\mathcal{D}_4(L_n) = 0, \qquad \mathcal{D}_4(G_n) = L_{n+1}.$$

**Lemma 6.2.**

$$\big(f \ \text{ is an } \ \alpha^0\text{-derivation of } \ \widehat{\mathcal{W}^q}\big) \Rightarrow \big(\widehat{f} \equiv 0\big).$$

***Proof.*** Let $f$ be an $\alpha^0$-derivation of $\widehat{\mathcal{W}^q}$.
By using (6.4) and $\lambda \equiv 0$, for all $u \in V$, we have

$$\big[\delta, \widehat{f}\big]^*(L_n, u) = 0 \quad \forall n \in \mathbb{Z} \Rightarrow$$

$$\Rightarrow -(-1)^{|x||z|}\delta(\widehat{f}(u), \alpha(L_n)) = 0 \quad \forall n \in \mathbb{Z} \Rightarrow$$

$$\Rightarrow [\widehat{f}(u), L_n] = 0 \quad \forall n \in \mathbb{Z} \Rightarrow$$

$$\Rightarrow \widehat{f}(u) = 0.$$

In the following, we provide the $\alpha^0$-derivations of $\widehat{\mathcal{W}^q}$ explicitly.
**Proposition 6.1.** *The set of $\alpha^0$-derivations of $\widehat{\mathcal{W}^q}$ is*

$$\mathrm{Der}_{\alpha^0}\widehat{\mathcal{W}^q}) = \langle \overline{\mathcal{D}}_1 \rangle \oplus \langle \overline{\mathcal{D}}_2 \rangle \oplus \langle \overline{\mathcal{D}}_3 \rangle \oplus \langle \overline{\mathcal{D}}_4 \rangle,$$

*where $\overline{\mathcal{D}}_1$, $\overline{\mathcal{D}}_1$, $\overline{\mathcal{D}}_2$, $\overline{\mathcal{D}}_3$, and $\overline{\mathcal{D}}_4$ are defined, with respect to the basis, as*

$$\overline{\mathcal{D}}_1(L_n) = nL_n, \qquad \overline{\mathcal{D}}_1(G_n) = nG_n, \qquad \overline{\mathcal{D}}_1(c_i) = 0,$$

$$\overline{\mathcal{D}}_2(L_n) = 0, \qquad \overline{\mathcal{D}}_2(G_n) = G_n, \qquad \overline{\mathcal{D}}_2(c_i) = 0,$$

$$\overline{\mathcal{D}}_3(L_n) = nG_{n-1}, \qquad \overline{\mathcal{D}}_3(G_n) = 0, \qquad \overline{\mathcal{D}}_3(c_i) = 0,$$

$$\overline{\mathcal{D}}_4(L_n) = 0, \qquad \overline{\mathcal{D}}_4(G_n) = L_{n+1}, \qquad \overline{\mathcal{D}}_4(c_i) = 0,$$

*for $i = 0, 1$.*

***Proof.*** By $\widehat{f} = 0$ and (6.2), we obtain $\big[\delta, \widetilde{f}\big]^*(x, y) = 0$. Therefore, $\widetilde{f} \in \mathrm{Der}_{\alpha^0}(\mathcal{W}^q)$. So, by using Lemma 6.1, we get

$$\widetilde{f} \in \langle D_1 \rangle \oplus \langle D_2 \rangle \oplus \langle D_3 \rangle \oplus \langle D_4 \rangle.$$

If $\widetilde{f}$ is even, there exist $\lambda_1$ and $\lambda_2$ satisfying $\widetilde{f} = \lambda_1 D_1 + \lambda_2 D_2$.
If $\widetilde{f}$ is odd, there exist $\lambda_3$ and $\lambda_4$ satisfying $\widetilde{f} = \lambda_3 D_3 + \lambda_4 D_4$.
By using (6.3) and $\lambda \equiv 0$, we get

$$\Big([\varphi, \widetilde{f}+\widehat{v}] + [\delta, v]\Big)(x,y) = 0 \Rightarrow$$

$$\Rightarrow \varphi(\widetilde{f}(x),y) - (-1)^{|x||y|}\varphi(\widetilde{f}(y),x)) - \widehat{v}(\varphi(x,y)) - v(\delta(x,y)) = 0. \tag{6.7}$$

Taking $(x,y)$ to be $(L_n, L_k)$ and $(L_n, G_k)$ in (6.7), respectively, we have

$$\varphi\big(\widetilde{f}(L_n), L_k\big) - \varphi\big(\widetilde{f}(L_k), L_n\big) - \widehat{v}\big(\varphi(L_n, L_k)\big) - v\big(\delta(L_n, L_k)\big) = 0, \tag{6.8}$$

$$\varphi\big(\widetilde{f}(L_k), G_n\big) - \varphi\big(\widetilde{f}(G_n), L_k\big) - \widehat{v}\big(\varphi(L_k, G_n)\big) - v\big(\delta(L_k, G_n)\big) = 0. \tag{6.9}$$

Since $\varphi = \varphi_0 c_0 + \varphi_1 c_1$, it is easy to verify the following properties:

$$\varphi(L_1, x) = 0 \quad \forall x \in \mathcal{W}^q,$$

$$\varphi(L_n, L_k) = 0 \quad \forall n+k \neq 0,$$

$$\varphi(L_n, G_k) = 0 \quad \forall n+k \neq -1.$$

Setting $k = 1$ in (6.8), we obtain $v(L_{n+1}) = 0,\ n \neq 1$. Taking $(n,k) = (1,0)$ in (6.8), we have $v(L_1) = 0$. Setting $k = 1$ in (6.9), we get $v(G_{n+1}) = 0\ \forall n \neq 0$. Taking $(n,k) = (2,-1)$ in (6.9), we obtain $v(G_1) = 0$. Consequently, $v = 0$. Taking $k = -n$ in (6.8), by using $v = 0$ and $\widetilde{f} = \lambda_1 D_1 + \lambda_2 D_2$, we have $\widehat{v}(D) = 0$. Taking $k = -n-1$ in (6.8), by using $v = 0$ and $\widetilde{f} = \lambda_1 D_1 + \lambda_2 D_2$, we obtain $\widehat{v}(d) = 0$. Taking $k = -n$ in (6.8), by using $v = 0$ and $\widetilde{f} = \lambda_3 D_3 + \lambda_4 D_4$, we get $\widehat{v}(D) = 0$. Taking $k = -n-1$ in (6.9), by using $v = 0$ and $\widetilde{f} = \lambda_3 D_3 + \lambda_4 D_4$, we have $\widehat{v}(d) = 0$. Consequently, $\widehat{v} = 0$. Hence, one can deduce that $f = \widetilde{f}$.

***Remark* 6.1.** We have $\delta^0_{\mathcal{G}}(a) = [\alpha^{-1}(a), \cdot] = 0$ for all $a \in \mathcal{G}$. Then

$$H^1(\mathcal{G}) = Z^1(\mathcal{G})/B^0(\mathcal{G}) = \mathrm{Der}_{\alpha^0}(\mathcal{G}).$$

**7. Second cohomology of $q$-deformed Witt – Virasoro superalgebra of the Hom type with value in the adjoint module.** Let $f \in \mathcal{C}^2(K)$ be a homogeneous 2-cochain. By (2.6) and Proposition 3.2, we obtain $[d,f]^* = \delta_K(f)$. We set $f = \widetilde{f} + \widehat{f} + \overline{f} + v + \widehat{v} + \overline{v}$ where $\widetilde{f} \in \mathcal{C}^2(\mathcal{G},\mathcal{G})$, $\widehat{f} \in \mathcal{C}^{1,1}(\mathcal{G}V,\mathcal{G})$, $\overline{f} \in \mathcal{C}^2(V,\mathcal{G})$ $v \in \mathcal{C}^2(\mathcal{G},V)$, $\widehat{v} \in \mathcal{C}^{1,1}(\mathcal{G}V,V)$, and $\overline{v} \in \mathcal{C}^2(V,V)$.

In this case, for all $x,y,z \in \mathcal{G},\ u,v,w \in V$, we have

$$[d,f]^*(x,y,z) = \Big([\delta,\widetilde{f}]^* + [\varphi,\widehat{f}]^* + [\delta+\lambda, v]^* + [\varphi, \widetilde{f}+\widehat{v}]^*\Big)(x,y,z),$$

$$\Big([\delta,\widetilde{f}]^* + [\varphi,\widehat{f}]^*\Big)(x,y,z) \in \mathcal{G},$$

$$\Big([\delta+\lambda, v]^* + [\varphi,\widetilde{f}+\widehat{v}]^*\Big)(x,y,z) \in V,$$

$$[d,f]^*(x,y,w) = \Big([\delta,\widehat{f}+\widehat{v}]^* + [\varphi,\overline{f}]^* + [\lambda,\widetilde{f}+\widehat{f}+\widehat{v}]^* + [\varphi,\widehat{f}+\widehat{v}+\overline{v}]^*\Big)(x,y,w),$$

$$\Big([\delta,\overline{f}]^* + \Big([\lambda,\widehat{f}+\overline{f}+\overline{v}]^* + [\varphi,\overline{f}+\overline{v}]^*\Big)(x,u,v) = 0,$$

$$[d,f]^*(u,v,w) = 0.$$

**Proposition 7.1.** *$f \in Z^2(K)$ if and only $f$ satisfies the following conditions:*

$$\Big([\delta, \widetilde{f}]^* + [\varphi, \widehat{f}\,]^*\Big)(x, y, z) = 0, \tag{7.1}$$

$$\Big([\delta + \lambda, v]^* + [\varphi, \widetilde{f} + \widehat{v}]^*\Big)(x, y, z) = 0, \tag{7.2}$$

$$\Big([\delta + \lambda, \widehat{f}\,]^* + [\varphi, \overline{f}]^*\Big)(x, y, v) = 0, \tag{7.3}$$

$$\Big([\delta + \lambda, \widehat{v}]^* + [\lambda, \widetilde{f}] + [\varphi, \widehat{f} + \overline{v}]^*\Big)(x, y, v) = 0, \tag{7.4}$$

$$[\delta, \overline{f}]^*(x, u, v) = 0, \tag{7.5}$$

$$\Big([\lambda, \widehat{f} + \overline{f} + \overline{v}]^* + [\varphi, \overline{f} + \overline{v}]^*\Big)(x, u, v) = 0, \tag{7.6}$$

$$[\lambda, \overline{f}]^*(u, v, w) = 0. \tag{7.7}$$

***Corollary* 7.1.** 1. *If $f = \widetilde{f}$, then*

$$f \in Z^2(K) \Leftrightarrow \begin{cases} [\delta, \widetilde{f}]^*(x, y, z) = 0, \\ [\varphi, \widetilde{f}]^*(x, y, z) = 0, \\ [\lambda, \widetilde{f}]^*(x, y, v) = 0. \end{cases} \tag{7.8}$$

2. *If $f = v$, then*

$$\big(f \in Z^2(K)\big) \Leftrightarrow \big(f \in Z^2(\mathcal{G}, V)\big). \tag{7.9}$$

3. *If $\widehat{f} \equiv 0$, then*

$$\big(f \in Z^2(K)\big) \Rightarrow \big(\widetilde{f} \in Z^2(\mathcal{G}, V)\big). \tag{7.10}$$

**Lemma 7.1.**

$$\big(f \in Z^2(K)\big) \Rightarrow \big(\widehat{f} \equiv 0 \ \textit{ and } \ \overline{f} \equiv 0\big).$$

***Proof.*** Let $f \in Z^2(K)$ be a homogeneous 2-cocycle of degree $s$. Since $\lambda \equiv 0$, by using (7.3), we obtain $\big([\delta, \widehat{f}]^* + [\varphi, \overline{f}]^*\big)(x_n, x_m, v) = 0$, where $\deg(x_n) = n$, $\deg(x_m) = m$, $v \in \{c_0, c_1\}$. Thus,

$$-(-1)^{|x_m||v|}[\widehat{f}(x_n, v), \alpha(x_m)] + [\widehat{f}(x_m, v), \alpha(x_n)] + \widehat{f}([x_n, x_m], v) + \overline{f}(\varphi(x_n, x_m), v) = 0. \tag{7.11}$$

Let $g = \hat{f}(\cdot, c_0)$ and $h = \hat{f}(\cdot, c_1)$. Given a super-skew-symmetric bilinear map $f$ and $c_0$ an even element of $V$, we have $f(c_0, c_0) = 0$. Then, if we take $(x_n, x_m, v) = (L_n, L_m, c)$ in (7.11), we obtain $g([L_n, L_m]) = [g(L_n), \alpha(L_m)] + [\alpha(L_n), g(L_m)]$. Hence, $g$ is an $\alpha$-derivation. Since the set of $\alpha$-derivations is trivial (see [2]), then $g(L_n) = 0$.

We assume first that $f$ is an even 2-cocycle of degree $s$. We can assume that $\tilde{f}(L_n, G_p) = b_{s,n,p}G_{s+n+p}$, $\hat{f}(c_0, G_p) = b_{s,p}G_{s+p}$, $\hat{f}(c_1, L_p) = a'_{s,p}L_{s+p}$, $\hat{f}(d, G_p) = b'_{s,p}G_{s+p}$ and $\overline{f}(c_1, c_1) = c'_s L_{s-2}$. Then, by (2.1), (2.3) and Remark 5.1 and by taking the triple $(x_n, x_m, v)$ to be $(L_n, G_m, c_0)$, and $(L_n, G_m, c_1)$ in (7.11), respectively, we obtain

$$-b_{s,m}(1+q^n)\big(\{s+m+1\}-\{n\}\big)G_{s+m+n}+\big(\{m+1\}-\{n\}\big)b_{s,n+m}G_{n+m+s}+$$

$$+\delta_{n+m,-1}b_n\overline{f}(c_1,c_0)=0, \tag{7.12}$$

$$a'_{s,n}(1+q^m)(\{m\}-\{n+s\})G_{n+m+s}-b'_{s,m}(1+q^n)(\{s+m+1\}-\{n\})G_{s+m+n}+$$

$$\big(\{m+1\}-\{n\}\big)b'_{s,n+m}G_{n+m+s}+\delta_{n+m,-1}b_n\overline{f}(c_1,c_1)=0, \tag{7.13}$$

where

$$b_n=\begin{cases}\dfrac{1}{q^{n-2}}\dfrac{1+q^2}{1+q^n}\dfrac{\{n+1\}\{n\}\{n-1\}}{\{3\}\{2\}}, & \text{if} \quad n\geq 0,\\ -b_{-n}, & \text{if} \quad n<0.\end{cases}$$

Letting, in (7.12), respectively, $m=-1$; $m=-1$, $n=0$, $s\neq 0$; $s=0$, $n=0$; $s=0$, $n=-1$, $m=-1$, we obtain $b_{s,n-1}=(1+q^n)\dfrac{q^n-q^s}{1-q^n}b_{s,-1}$ $(n\neq 0)$; $b'_{s,-1}=0$ hence $b_{s,n-1}=0$ $\forall s\neq 0$; $b_{0,m}=0$ $\forall m\neq -1$; $b_{0,-1}=0$. Consequently, $\widehat{f}(c_0,G_n)=0$ and $\overline{f}(c_1,c_0)=0$.

Setting $(x_n,x_m,d)=(L_n,L_m,d)$ in (7.11), and since $\overline{f}(c_1,c_0)=0$, we have

$$h\big([L_n,L_m]\big)=\big[h(L_n),\alpha(L_m)\big]+\big[\alpha(L_n),h(L_m)\big].$$

Then $h$ is a $\alpha$-derivation of $\mathcal{W}_q$. Therefore, $h(L_n)=0$, i.e., $a'_{s,n}=0$ $\forall s,n\in\mathbb{Z}$. Therefore, we can rewrite (7.13) as follows:

$$-b'_{s,m}(1+q^n)\big(\{s+m+1\}-\{n\}\big)G_{s+m+n}+\big(\{m+1\}-\{n\}\big)b'_{s,n+m}G_{n+m+s}+b_nc_sL_{s-2}=0. \tag{7.14}$$

Hence, we deduce that $c'_s=0$ and

$$-b'_{s,m}(1+q^n)\big(\{s+m+1\}-\{n\}\big)+\big(\{m+1\}-\{n\}\big)b'_{s,n+m}=0. \tag{7.15}$$

Letting, in (7.15), respectively, $m=-1$, $n=0$; $m=-1$; $s=0$, $n=0$; $s=0$, $n=-2$, $m=1$, we obtain $b'_{s,-1}=0$ $(s\neq 0)$; $b'_{s,n-1}=\dfrac{(1+q^n)(q^n-q^s)}{1-q^n}b'_{s,-1}=0$ $(s\neq 0,\ n\neq 0)$; $b'_{0,m}=0$ $(m\neq -1)$; $b'_{0,-1}=(1+q^{-2})b_{0,1}=0$.

Similar computations are done if the 2-cocycle $f$ is odd, which ends the proof.

**Lemma 7.2.**

$$\big(f\in Z^2(\widehat{\mathcal{W}^q})\big)\Rightarrow(\widehat{v}\equiv 0 \ \ \textit{and} \ \ \overline{v}\equiv 0).$$

***Proof.*** By Lemma 7.1 and (7.4), we obtain $\big([\delta,\widehat{v}]^*+[\varphi,\overline{v}]^*\big)(x_n,x_m,v)=0$. Then

$$\widehat{v}([x_n,x_m],v)+\overline{v}(\varphi(x_n,x_m),v)=0. \tag{7.16}$$

By taking various triples $(x_n,x_m,v)$ with elements of the basis, we get the conclusion.

**Theorem 7.1.** *The second group of $\widehat{\mathcal{W}^q}$ with coefficients in the adjoint representation is equal to zero:*

$$H^2(\widehat{\mathcal{W}^q})=\{0\}.$$

***Proof.*** Let us recall that second cohomology group of $\mathcal{W}^q$ with coefficients in the adjoint representation is equal to zero: $H^2(\mathcal{W}^q)=\{0\}$ (see [3]).

By (7.1) and Lemma 7.1, we obtain $\big[\delta,\widehat{f}\big]^*=0$.

So, $\widetilde{f} \in Z^2(\mathcal{W}^q)$. With $H^2(\mathcal{W}^q) = \{0\}$, the 2-cocycle $\widetilde{f}$ is trivial. Then there exists a linear map $\widetilde{h}\colon \mathcal{W}^q \to \mathcal{W}^q$ such that $\widetilde{f} = \delta^1_{\mathcal{W}^q}(\widetilde{h})$.

By Lemmas 7.1 and 7.2, we obtain $f = \widetilde{f} + v$. We deduce

$$f = \delta^1_{\mathcal{W}^q}(\widetilde{h}) + v.$$

Therefore, $f = \delta^1_{\widehat{\mathcal{W}^q}}(\widetilde{h}) + w$, where $w(\mathcal{W}^q, \mathcal{W}q) \subset V$. So

$$f \in Z^2(\mathcal{W}^q) \Rightarrow \delta^2_{\widehat{\mathcal{W}^q}}(f) \equiv 0 \Rightarrow$$

$$\Rightarrow \delta^2_{\widehat{\mathcal{W}^q}}(\delta^1_{\widehat{\mathcal{W}^q}}(\widetilde{h})) + \delta^2_{\widehat{\mathcal{W}^q}}(w) \equiv 0 \Rightarrow$$

$$\Rightarrow \delta^2_{\widehat{\mathcal{W}^q}}(w) \equiv 0 \Rightarrow$$

$$\Rightarrow w \in Z^2(\widehat{\mathcal{W}^q}) \overset{\text{by (7.10)}}{\Longrightarrow}$$

$$\overset{\text{by (7.10)}}{\Longrightarrow} w \in Z^2(\mathcal{W}^q, V) \overset{\text{by (5.1)}}{\Longrightarrow}$$

$$\overset{\text{by (5.1)}}{\Longrightarrow} w \in B^1(\widehat{\mathcal{W}^q}).$$

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Received 26.12.16,
after revision — 16.09.17