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LINEAR AND NONLINEAR HEAT EQUATIONS ON A p-ADIC BALL*

ЛІНІЙНЕ ТА НЕЛІНІЙНЕ РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ НА p-АДИЧНІЙ КУЛІ

We study the Vladimirov fractional differentiation operator D^α_N, $\alpha > 0$, $N \in \mathbb{Z}$, on a p-adic ball $B_N = \{x \in \mathbb{Q}_p : |x|_p \le p^N\}$. To its known interpretations via the restriction of a similar operator to $\mathbb{Q}_p$ and via a certain stochastic process on B_N, we add an interpretation as a pseudodifferential operator in terms of the Pontryagin duality on the additive group of B_N. We investigate the Green function of D^α_N and a nonlinear equation on B_N, an analog of the classical equation of porous medium.

Вивчається оператор Владимирова диференціювання дробового порядку D^α_N, $\alpha > 0$, $N \in \mathbb{Z}$, на p-адичній кулі $B_N = \{x \in \mathbb{Q}_p : |x|_p \le p^N\}$. До його відомих інтерпретацій у термінах звуження подібного оператора, визначеного на $\mathbb{Q}_p$ та через деякий випадковий процес на B_N, ми додаємо інтерпретацію у вигляді псевдодиференціального оператора в термінах дуальності Понтрягіна на адитивній групі B_N. Вивчено функцію Гріна на D^α_N та нелінійне рівняння на B_N, що є аналогом класичного рівняння пористого середовища.

1. Introduction. The theory of linear parabolic equations for real- or complex-valued functions on the field $\mathbb{Q}_p$ of p-adic numbers including the construction of a fundamental solution, investigation of the Cauchy problem, the parametrix method, is well-developed; see, for example, the monographs [16, 21]. In such equations, the time variable is real and nonnegative while the spatial variables are p-adic. There are no differential operators acting on complex-valued functions on $\mathbb{Q}_p$, but there is a lot of pseudodifferential operators. A typical example is Vladimirov's fractional differentiation operator D^α, $\alpha > 0$; see the details below. This operator (as well as its multidimensional generalization, the so-called Taibleson operator) is a p-adic counterpart of the fractional Laplacian $(-\Delta)^{\alpha/2}$ of real analysis.

Already in real analysis, an interpretation of nonlocal operators on bounded domains is not straightforward; see [3] for a survey of various possibilities. In the p-adic case, Vladimirov (see [19]) defined a version D^α_N of the fractional differentiation on a ball $B_N = \{x \in \mathbb{Q}_p : |x|_p \le p^N\}$ as follows. One takes a test function on B_N, extends it onto $\mathbb{Q}_p$ by zero, applies D^α, and restricts the resulting function to B_N. Then it is possible to consider a closure of the obtained operator, for example, on $L^2(B_N)$.

In [16] (Section 4.6), a probabilistic interpretation of this operator was given. Let $\xi_\alpha(t)$ be the Markov process with the generator D^α on $\mathbb{Q}_p$. Suppose that $\xi_\alpha(0) \in B_N$ and denote by $\xi^{(N)}_\alpha(t)$ the sum of all jumps of the process $\xi_\alpha(\tau)$, $\tau \in [0, t]$ whose p-adic absolute values exceed p^N. Consider the process $\eta_\alpha(t) = \xi_\alpha(t) - \xi^{(N)}_\alpha(t)$. Due to the ultrametric inequality, the jumps of η_α never exceed p^N by absolute value, so that the process remains almost surely in B_N. It is proved in [16] that the generator of the Markov process η_α on B_N equals (on test function) $D^\alpha_N - \lambda I$, where

$$\lambda = \frac{p-1}{p^{\alpha+1}-1} p^{\alpha(1-N)}.$$

In [16] (Theorem 4.9) the corresponding heat kernel is given explicitly.

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In this paper we find an analytic interpretation of the latter operator using harmonic analysis on B_N as an (additive) compact Abelian group (this group property, just as the above probabilistic construction, is of purely non-Archimedean nature and has no analogs in the classical theory of partial differential equations). We give an interpretation of $D_N^\alpha - \lambda I$ as a pseudodifferential operator on B_N, then consider it as an operator on $L^1(B_N)$ and study its Green function, the integral kernel of its resolvent. The choice of $L^1(B_N)$ as the basic space is motivated by applications to nonlinear equations.

The first model example of a nonlinear parabolic equation over $\mathbb{Q}_p$ is the p-adic analog of the classical porous medium equation:

$$\frac{\partial u}{\partial t} + D^\alpha(\Phi(u)) = 0, \qquad u = u(t,x), \quad t > 0, x \in \mathbb{Q}_p, \tag{1.1}$$

where Φ is a strictly monotone increasing continuous real function on $\mathbb{R}$. Its study was initiated in [13]. Here we consider this equation on B_N, taking the operator D_N^α instead of D^α:

$$\frac{\partial u}{\partial t} + D_N^\alpha(\Phi(u)) = 0. \tag{1.2}$$

As in [13], our study of Eq. (1.2) is based on general results by Crandall – Pierre [10] and Brézis – Strauss [6] enabling us to consider this equation in the framework of nonlinear semigroups of operators. Following [3] we consider Eq. (1.2) also in $L^\gamma(B_N)$, $1 < \gamma \le \infty$.

An important motivation of the present work is provided by the p-adic model of a poropus medium introduced in [14, 15].

2. Preliminaries. ***2.1. p-Adic numbers*** [19]. Let p be a prime number. The field of p-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^\nu \frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and m, n are prime to p. $\mathbb{Q}_p$ is a locally compact topological field. By Ostrowski's theorem there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$.

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0,$$

$$|xy|_p = |x|_p \cdot |y|_p,$$

$$|x + y|_p \le \max(|x|_p, |y|_p).$$

The latter property called the ultrametric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ in the topology determined by the metric $|x - y|_p$, as well as many unusual geometric properties. Note also the following consequence of the ultrametric inequality: $|x + y|_p = \max(|x|_p, |y|_p)$, if $|x|_p \ne |y|_p$.

The absolute value $|x|_p$ takes the discrete set of non-zero values p^N, $N \in \mathbb{Z}$. If $|x|_p = p^N$, then x admits a (unique) canonical representation

$$x = p^{-N}\left(x_0 + x_1 p + x_2 p^2 + \ldots\right), \tag{2.1}$$

where $x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p-1\}$, $x_0 \ne 0$. The series converges in the topology of $\mathbb{Q}_p$. For example,

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \ldots, \qquad |-1|_p = 1.$$

We denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \le 1\}$. $\mathbb{Z}_p$, as well as all balls in $\mathbb{Q}_p$, is simultaneously open and closed.

Proceeding from the canonical representation (2.1) of an element $x \in \mathbb{Q}_p$, one can define the fractional part of x as the rational number

$$\{x\}_p = \begin{cases} 0, & \text{if } N \le 0 \text{ or } x = 0, \\ p^{-N}\left(x_0 + x_1 p + \ldots + x_{N-1}p^{N-1}\right), & \text{if } N > 0. \end{cases}$$

The function $\chi(x) = \exp(2\pi i \{x\}_p)$ is an additive character of the field $\mathbb{Q}_p$, that is a character of its additive group. It is clear that $\chi(x) = 1$ if and only if $|x|_p \le 1$.

Denote by dx the Haar measure on the additive group of $\mathbb{Q}_p$ normalized by the equality $\int_{\mathbb{Z}_p} dx = 1$.

The additive group of $\mathbb{Q}_p$ is self-dual, so that the Fourier transform of a complex-valued function $f \in L^1(\mathbb{Q}_p)$ is again a function on $\mathbb{Q}_p$ defined as

$$(\mathcal{F}f)(\xi) = \int\limits_{\mathbb{Q}_p} \chi(x\xi) f(x)\, dx.$$

If $\mathcal{F}f \in L^1(\mathbb{Q}_p)$, then we have the inversion formula

$$f(x) = \int\limits_{\mathbb{Q}_p} \chi(-x\xi) \widetilde{f}(\xi)\, d\xi.$$

It is possible to extend $\mathcal{F}$ from $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ to a unitary operator on $L^2(\mathbb{Q}_p)$, so that the Plancherel identity holds in this case.

In order to define distributions on $\mathbb{Q}_p$, we have to specify a class of test functions. A function $f \colon \mathbb{Q}_p \to \mathbb{C}$ is called locally constant if there exists such an integer $l \ge 0$ that for any $x \in \mathbb{Q}_p$

$$f(x + x') = f(x) \quad \text{if} \quad \|x'\| \le p^{-l}.$$

The smallest number l with this property is called the exponent of local constancy of the function f.

Typical examples of locally constant functions are additive characters, and also cutoff functions like

$$\Omega(x) = \begin{cases} 1, & \text{if} \quad \|x\| \le 1, \\ 0, & \text{if} \quad \|x\| > 1. \end{cases}$$

In particular, Ω is continuous, which is an expression of the non-Archimedean properties of $\mathbb{Q}_p$.

Denote by $\mathcal{D}(\mathbb{Q}_p)$ the vector space of all locally constant functions with compact supports. Note that $\mathcal{D}(\mathbb{Q}_p)$ is dense in $L^q(\mathbb{Q}_p)$ for each $q \in [1, \infty)$. In order to furnish $\mathcal{D}(\mathbb{Q}_p)$ with a topology, consider first the subspace $D^l_N \subset \mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in a ball

$$B_N = \{x \in \mathbb{Q}_p \colon |x|_p \le p^N\}, \qquad N \in \mathbb{Z},$$

and the exponents of local constancy $\le l$. This space is finite-dimensional and possesses a natural direct product topology. Then the topology in $\mathcal{D}(\mathbb{Q}_p)$ is defined as the double inductive limit topology, so that

$$\mathcal{D}(\mathbb{Q}_p) = \varinjlim_{N\to\infty} \varinjlim_{l\to\infty} D_N^l.$$

If $V \subset \mathbb{Q}_p$ is an open set, the space $\mathcal{D}(V)$ of test functions on V is defined as a subspace of $\mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in V. For a ball $V = B_N$, we can identify $\mathcal{D}(B_N)$ with the set of all locally constant functions on B_N.

The space $\mathcal{D}'(\mathbb{Q}_p)$ of Bruhat – Schwartz distributions on $\mathbb{Q}_p$ is defined as a strong conjugate space to $\mathcal{D}(\mathbb{Q}_p)$.

In contrast to the classical situation, the Fourier transform is a linear automorphism of the space $\mathcal{D}(\mathbb{Q}_p)$. By duality, $\mathcal{F}$ is extended to a linear automorphism of $\mathcal{D}'(\mathbb{Q}_p)$. For a detailed theory of convolutions and direct products of distributions on $\mathbb{Q}_p$ closely connected with the theory of their Fourier transforms see [1, 16, 19].

2.2. Vladimirov's operator [1, 16, 19]. The Vladimirov operator D^α, $\alpha > 0$, of fractional differentiation, is defined first as a pseudodifferential operator with the symbol $|\xi|_p^\alpha$:

$$(D^\alpha u)(x) = \mathcal{F}^{-1}_{\xi\to x}\left[|\xi|_p^\alpha \mathcal{F}_{y\to\xi} u\right], \qquad u \in \mathcal{D}(\mathbb{Q}_p), \tag{2.2}$$

where we show arguments of functions and their direct/inverse Fourier transforms. There is also a hypersingular integral representation giving the same result on $\mathcal{D}(\mathbb{Q}_p)$ but making sense on much wider classes of functions (for example, bounded locally constant functions)

$$\left(D^\alpha u\right)(x) = \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int\limits_{\mathbb{Q}_p} |y|_p^{-\alpha-1}[u(x-y)-u(x)]\,dy. \tag{2.3}$$

The Cauchy problem for the heat-like equation

$$\frac{\partial u}{\partial t} + D^\alpha u = 0, \qquad u(0,x) = \psi(x), \qquad x \in \mathbb{Q}_p, \quad t > 0,$$

is a model example for the theory of p-adic parabolic equations. If ψ is regular enough, for example, $\psi \in \mathcal{D}(\mathbb{Q}_p)$, then a classical solution is given by the formula

$$u(t,x) = \int\limits_{\mathbb{Q}_p} Z(t, x-\xi)\psi(\xi)\,d\xi,$$

where Z is, for each t, a probability density and

$$Z(t_1+t_2, x) = \int\limits_{\mathbb{Q}_p} Z(t_1, x-y) Z(t_2, y)\,dy, \qquad t_1, t_2 > 0, \quad x \in \mathbb{Q}_p.$$

The "heat kernel" Z can be written as the Fourier transform

$$Z(t,x) = \int\limits_{\mathbb{Q}_p} \chi(\xi x) e^{-t|\xi|_p^\alpha}\,d\xi. \tag{2.4}$$

See [16] for various series representations and estimates of the kernel Z.

As it was mentioned in Introduction, the natural stochastic process in B_N corresponds to the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} + \left(D_N^\alpha u\right)(t,x) - \lambda u(t,x) = 0, \qquad x \in B_N, \quad t > 0, \tag{2.5}$$

$$u(0,x) = \psi(x), \qquad x \in B_N, \tag{2.6}$$

where the operator D_N^α is defined by restricting D^α to functions u_N supported in B_N and considering the resulting function $D^\alpha u_N$ only on B_N. Note that D_N^α defines a positive definite selfadjoint operator on $L^2(B_N)$, λ is its smallest eigenvalue.

Under certain regularity assumptions, for example if $\psi \in \mathcal{D}(B_N)$, the problem (2.5), (2.6) possesses a classical solution

$$u(t,x) = \int\limits_{B_N} Z_N(t, x-y)\psi(y)\,dy, \qquad t > 0, \quad x \in B_N,$$

where

$$Z_N(t,x) = e^{\lambda t} Z(t,x) + c(t), \tag{2.7}$$

$$c(t) = p^{-N} - p^{-N}(1-p^{-1})e^{\lambda t}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n\frac{p^{-N\alpha n}}{1-p^{-\alpha n-1}}.$$

Another interpretation of the kernel Z_N was given in [8].

It was shown in [13] that the family of operators

$$(T_N(t)u)(x) = \int\limits_{B_N} Z_N(t, x-y)\psi(y)\,dy$$

is a strongly continuous contraction semigroup on $L^1(B_N)$. Its generator A_N coincides with $D_N^\alpha - \lambda I$ at least on $\mathcal{D}(B_N)$. More generally, this is true in the distribution sense on restrictions to B_N of functions from the domain of the generator of the semigroup on $L^1(\mathbb{Q}_p)$ corresponding to D^α.

3. Harmonic analysis on the additive group of a p-adic ball. Let us consider the p-adic ball B_N as a compact subgroup of $\mathbb{Q}_p$. As we know, any continuous additive character of $\mathbb{Q}_p$ has the form $x \mapsto \chi(\xi x)$, $\xi \in \mathbb{Q}_p$. The annihilator $\{\xi \in \mathbb{Q}_p : \chi(\xi x) = 1 \text{ for all } x \in B_N\}$ coincides with the ball B_{-N}. By the duality theorem (see, for example, [18], Theorem 27), the dual group $\widehat{B_N}$ to B_N is isomorphic to the discrete group $\mathbb{Q}_p/B_{-N}$ consisting of the cosets

$$p^m\left(r_0 + r_1 p + \ldots + r_{N-m-1}p^{N-m-1}\right) + B_{-N}, \qquad r_j \in \{0,1,\ldots,p-1\}, \quad m \in \mathbb{Z}, \quad m < N. \tag{3.1}$$

Analytically, this isomorphism means that any nontrivial continuous character of B_N has the form $\chi(\xi x)$, $x \in B_N$, where $|\xi|_p > p^{-N}$ and $\xi \in \mathbb{Q}_p$ is considered as a representative of the class $\xi + B_{-N}$. Note that $|\xi|_p$ does not depend on the choice of a representative of the class.

The normalized Haar measure on B_N is $p^{-N}\,dx$. The normalization of the Haar measure on $\mathbb{Q}_p/B_{-N}$ can be made in such a way (the normalized measure will be denoted $d\mu(x + B_{-N})$) that the equality

$$\int\limits_{\mathbb{Q}_p} f(x)\,dx = \int\limits_{\mathbb{Q}_p/B_{-N}} \left(p^N \int\limits_{B_{-N}} f(x+h)\,dh\right) d\mu(x + B_{-N}) \tag{3.2}$$

holds for any $f \in \mathcal{D}(\mathbb{Q}_p)$; see [4], Chapter VII, Proposition 10; [12], (28.54). With this normalization, the Plancherel identity for the corresponding Fourier transform also holds; see [12], (31.46)(c).

On the other hand, the invariant measure on the discrete group $\mathbb{Q}_p/B_{-N}$ equals the sum of δ-measures concentrated on its elements multiplied by a coefficient β. In order to find β, it suffices to compute both sides of (3.2) for the case where f is the indicator function of the set $\{x \in \mathbb{Q}_p : |x - p^{N-1}|_p \le p^{-N}\}$. Then the left-hand side equals p^{-N} while the right-hand side equals β. Therefore $\beta = p^{-N}$.

The Fourier transform on B_N is given by the formula

$$(\mathcal{F}_N f)(\xi) = p^{-N} \int\limits_{B_N} \chi(x\xi) f(x)\, dx, \quad \xi \in (\mathbb{Q}_p \setminus B_{-N}) \cup \{0\},$$

where the right-hand side, thus also $\mathcal{F}_N f$, can be understood as a function on $\mathbb{Q}_p/B_{-N}$.

The fact that $\mathcal{F}\colon \mathcal{D}(\mathbb{Q}_p) \to \mathcal{D}(\mathbb{Q}_p)$ implies that $\mathcal{F}$ maps $\mathcal{D}(B_N)$ onto the set of functions on the discrete set $\widehat{B_N}$ having only a finite number of nonzero values. This set $\mathcal{D}(\widehat{B_N})$ with a natural locally convex topology can be seen as the set of test functions on $\widehat{B_N} = \mathbb{Q}_p/B_{-N}$. The conjugate space $\mathcal{D}'(\widehat{B_N})$ consists of all functions on $\widehat{B_N}$ (see, for example, [11]). Therefore the Fourier transform is extended, via duality, to the mapping from $\mathcal{D}'(B_N)$ to $\mathcal{D}'(\widehat{B_N})$. A theory of distributions on locally compact groups covering the case of B_N was developed by Bruhat [7]. To study deeper the operator D_N^α, we need, within harmonic analysis on B_N, a construction similar to the well-known construction of a homogeneous distribution on $\mathbb{Q}_p$ [19].

Let us introduce the usual Riesz kernel on $\mathbb{Q}_p$,

$$f_\alpha^{(N)}(x) = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} |x|_p^{\alpha-1}, \quad \operatorname{Re}\alpha > 0, \quad \alpha \not\equiv 1 \left(\operatorname{mod} \frac{2\pi i}{\log p}\mathbb{Z}\right).$$

Using the formula [19]

$$\int\limits_{|x|_p \le p^N} |x|_p^{\alpha-1} dx = \frac{1-p^{-1}}{1-p^{-\alpha}} p^{\alpha N},$$

we introduce a distribution from $\mathcal{D}'(B_N)$ setting

$$\left\langle f_\alpha^{(N)}, \varphi \right\rangle = \frac{1-p^{-1}}{1-p^{\alpha-1}} p^{\alpha N} \varphi(0) + \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int\limits_{B_N} [\varphi(x) - \varphi(0)] |x|_p^{\alpha-1} dx, \quad \varphi \in \mathcal{D}(B_N). \tag{3.3}$$

For $\operatorname{Re}\alpha > 0$, this gives

$$\left\langle f_\alpha^{(N)}, \varphi \right\rangle = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int\limits_{B_N} |x|_p^{\alpha-1} \varphi(x)\, dx.$$

On the other hand, the distribution (3.3) is holomorphic in $\alpha \not\equiv 1 \left(\operatorname{mod} \frac{2\pi i}{\log p}\mathbb{Z}\right)$. Therefore $f_{-\alpha}^{(N)}$ makes sense for any $\alpha > 0$. Noticing that

$$\frac{1-p^{-1}}{1-p^{-\alpha-1}} p^{-\alpha N} = \frac{p-1}{p^{\alpha+1}-1} p^{-\alpha N + \alpha} = \lambda$$

(see Introduction), so that

$$\left\langle f_{-\alpha}^{(N)}, \varphi \right\rangle = \lambda\varphi(0) + \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \int\limits_{B_N} [\varphi(x)-\varphi(0)]|x|_p^{-\alpha-1}dx. \tag{3.4}$$

The emergence of λ in (3.4) "explains" its role in the probabilistic construction of a process on B_N ([16], Theorem 4.9).

Theorem 3.1. *The operator D_N^{α}, $\alpha > 0$, acts from $\mathcal{D}(B_N)$ to $\mathcal{D}(B_N)$ and admits, for each $\varphi \in \mathcal{D}(B_N)$, the representations*:

(i) $D_N^{\alpha}\varphi = f_{-\alpha}^{(N)} * \varphi$ *where the convolution is understood in the sense of harmonic analysis on the additive group of B_N;*

(ii) $(D_N^{\alpha}\varphi)(x) = \lambda\varphi(x) + \dfrac{1-p^{\alpha}}{1-p^{-\alpha-1}} \displaystyle\int_{B_N} |y|_p^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy, \quad \alpha > 0;$

(iii) *on $\mathcal{D}(B_N)$, $D_N^{\alpha} - \lambda I$ coincides with the pseudodifferential operator $\varphi \mapsto \mathcal{F}_N^{-1}(P_{N,\alpha}\mathcal{F}_N\varphi)$, where*

$$P_{N,\alpha}(\xi) = \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \int\limits_{B_N} |y|_p^{-\alpha-1}[\chi(y\xi)-1]\,dy. \tag{3.5}$$

This symbol is extended uniquely from $(\mathbb{Q}_p \setminus B_{-N}) \cup \{0\}$ onto $\mathbb{Q}_p/B_{-N}$.

Proof. Denote, for brevity, $a_p = \dfrac{1-p^{\alpha}}{1-p^{-\alpha-1}}$. Let $x \in B_N$. Assuming that φ is extended by zero onto $\mathbb{Q}_p$, we find

$$(D_N^{\alpha}\varphi)(x) = a_p \int\limits_{\mathbb{Q}_p} |y|_p^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy = I_1 + I_2 + I_3,$$

where

$$I_1 = a_p \int\limits_{B_N} |y|_p^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy,$$

$$I_2 = a_p \int\limits_{|y|_p > p^N} |y|_p^{-\alpha-1}\varphi(x-y)\,dy,$$

$$I_3 = -a_p\varphi(x) \int\limits_{|y|_p > p^N} |y|_p^{-\alpha-1}dy.$$

We get using properties of p-adic integrals [19]

$$I_2 = a_p \int\limits_{|x-z|_p > p^N} |x-z|_p^{-\alpha-1}\varphi(z)\,dz = a_p \int\limits_{|z|_p > p^N} |z|_p^{-\alpha-1}\varphi(z)\,dz = 0,$$

$$I_3 = -a_p\varphi(x) \sum_{j=N+1}^{\infty} \int\limits_{|y|_p = p^j} |y|_p^{-\alpha-1}dy = -a_p\varphi(x)\left(1-\frac{1}{p}\right) \sum_{j=N+1}^{\infty} p^{-\alpha j} = \lambda\varphi(x),$$

which implies (ii). Comparing with (3.4) we prove (i).

In order to prove (3.5) we note that

$$\mathcal{F}_N\left(D_N^\alpha\varphi-\lambda\varphi\right)(\xi)=a_pp^{-N}\int\limits_{B_N}\chi(x\xi)\,dx\int\limits_{B_N}|y|_p^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy=$$

$$=a_pp^{-N}\int\limits_{B_N}|y|_p^{-\alpha-1}dy\int\limits_{B_N}\chi(x\xi)[\varphi(x-y)-\varphi(x)]\,dx=P_{n,\alpha}(\xi)\left(\mathcal{F}_N\varphi\right)(\xi),$$

$$\xi\in\mathbb{Q}_p/B_{-N}.$$

Theorem 3.1 is proved.

An important consequence of the representations given in Theorem 3.1 is the fact that, in contrast to operators on $\mathbb{Q}_p$, $D_N^\alpha:\mathcal{D}(B_N)\to\mathcal{D}(B_N)$, so that we can define in a straightforward way, the action of this operator on distributions. In particular, the pseudodifferential representation remains valid on $\mathcal{D}'(B_N)$. Below (Theorem 4.2) this will be used to describe the domain of the operator A_N on $L^1(B_N)$.

4. The Green function. In Section 2 (just as in [13]) we defined the operator A_N as the generator of the semigroup T_N on $L^1(B_N)$. We can write its resolvent $(A_N+\mu I)^{-1}$, $\mu>0$, as

$$\left((A_N+\mu I)^{-1}u\right)(x)=\int\limits_0^\infty e^{-\mu t}dt\int\limits_{B_N}Z_N(t,x-\xi)u(\xi)\,d\xi,\quad u\in L^1(B_N),\tag{4.1}$$

where Z_N is given in (2.7).

Theorem 4.1. *The resolvent* (4.1) *admits the representation*

$$\left((A_N+\mu I)^{-1}u\right)(x)=\int\limits_{B_N}K_\mu(x-\xi)u(\xi)\,d\xi+\mu^{-1}p^{-N}\int\limits_{B_N}u(\xi)\,d\xi,\quad u\in L^1(B_N),\ \mu>0,\tag{4.2}$$

where for $0\ne x\in B_N$, $|x|_p=p^m$,

$$K_\mu(x)=\int\limits_{p^{-N+1}\le|\eta|_p\le p^{-m+1}}\frac{\chi(\eta x)}{|\eta|_p^\alpha-\lambda+\mu}\,d\eta.\tag{4.3}$$

If $\alpha>1$, *then, for any* $x\in B_N$,

$$K_\mu(x)=\int\limits_{|\eta|_p\ge p^{-N+1}}\frac{\chi(\eta x)}{|\eta|_p^\alpha-\lambda+\mu}\,d\eta.\tag{4.4}$$

The kernel K_μ *is continuous for* $x\ne0$ *and belongs to* $L^1(B_N)$.

If $\alpha>1$, *then* K_μ *is continuous on* B_N. *If* $\alpha=1$, *then*

$$|K_\mu(x)|\le C|\log|x|_p|,\quad x\in B_N.\tag{4.5}$$

If $\alpha<1$, *then*

$$|K_\mu(x)|\le C|x|_p^{\alpha-1},\quad x\in B_N.\tag{4.6}$$

Proof. Let us use the representation (2.7) substituting it into the equality

$$\int_{B_N} Z_N(t,x)\,dx = 1$$

(for the latter see Theorem 4.9 in [16]). We find

$$c(t) = p^{-N} - e^{\lambda t} p^{-N} \int_{B_N} Z(t,y)\,dy,$$

so that

$$Z_N(t,x) = e^{\lambda t}\left[Z(t,x) - p^{-N}\int_{B_N} Z(t,y)\,dy\right] + p^{-N}, \qquad x \in B_N.$$

Let us consider the expression in brackets proceeding from the definition (2.4) of the kernel Z. Using the integration formula from Chapter 1, § 4 of [19] we obtain

$$Z(t,x) - p^{-N}\int_{B_N} Z(t,y)\,dy = I_1(t,x) + I_2(t,x),$$

where

$$I_1(t,x) = \int_{|\xi|_p \ge p^{-N+1}} \chi(\xi x) e^{-t|\xi|_p^\alpha}\,d\xi,$$

$$I_2(t,x) = \int_{|\xi|_p \le p^{-N}} [\chi(\xi x) - 1] e^{-t|\xi|_p^\alpha}\,d\xi,$$

and $I_2(t,x) = 0$ for $x \in B_N$.

Let $|x|_p = p^m$, $m \le N$. Then there exists such an element $\xi_0 \in \mathbb{Q}_p$, $|\xi_0|_p = p^{-m+1}$, that $\chi(\xi_0 x) \ne 0$. Then making the change of variables $\xi = \eta + \xi_0$ we find using the ultrametric property

$$\int_{|\xi|_p \ge p^{-m+2}} \chi(x\xi) e^{-t|\xi|_p^\alpha}\,d\xi = \chi(x\xi_0) \int_{|\eta|_p \ge p^{-m+2}} \chi(x\eta) e^{-t|\eta|_p^\alpha}\,d\eta,$$

so that

$$\int_{|\xi|_p \ge p^{-m+2}} \chi(x\xi) e^{-t|\xi|_p^\alpha}\,d\xi = 0.$$

Therefore

$$I_1(t,x) = \int_{p^{-N+1} \le |\xi|_p \le p^{-m+1}} \chi(x\xi) e^{-t|\xi|_p^\alpha}\,d\xi,$$

thus

$$Z_N(t,x) = e^{\lambda t} \int_{p^{-N+1} \le |\xi|_p \le p^{-m+1}} \chi(x\xi) e^{-t|\xi|_p^\alpha}\,d\xi + p^{-N}, \quad |x|_p = p^m.$$

Substituting this in (4.1) and integrating in t we come to (4.2) and (4.3). Note that $|\eta|_p^\alpha > \lambda$, as $|\eta|_p \ge p^{-N+1}$.

If $\alpha > 1$, then the integral in (4.4) is convergent. For $|x|_p = p^m$ we prove repeating the above argument that

$$\int\limits_{|\eta|_p \ge p^{-m+2}} \frac{\chi(\eta x)}{|\eta|_p^\alpha - \lambda + \mu}\, d\eta = 0.$$

Therefore in this case the representation (4.3) can be written in the form (4.4).

Obviously, $K_\mu(x)$ is continuous for $x \neq 0$. If $\alpha > 1$, then there exists the limit

$$\lim_{x\to 0} K_\mu(x) = \int\limits_{|\eta|_p \ge p^{-N+1}} \frac{d\eta}{|\eta|_p^\alpha - \lambda + \mu} < \infty,$$

so that in this case K_μ is continuous on B_N.

Let $\alpha < 1$. By (4.3) and an integration formula from [19], Chapter 1, § 4,

$$K_\mu(x) = \sum_{l=-N+1}^{-m+1} \frac{1}{p^{\alpha l} - \lambda + \mu} \int\limits_{|\xi|_p = p^l} \chi(\xi x)\, d\xi =$$

$$= \left(1 - \frac{1}{p}\right) \sum_{l=-N+1}^{-m} \frac{p^l}{p^{\alpha l} - \lambda + \mu} - \frac{p^{-m}}{p^{\alpha(-m+1)} - \lambda + \mu}, \quad |x|_p = p^m.$$

For some $\gamma > 0$, $p^{\alpha l} - \lambda + \mu \ge \gamma p^{\alpha l}$. Computing the sum of a progression we obtain the estimate (4.6). Similarly, if $\alpha = 1$, then $|K_\mu(x)| \le C(-m+N)$, which gives, as $m \to -\infty$, the inequality (4.5).

Theorem 4.1 is proved.

If $\alpha > 1$, we can also give an interpretation of the resolvent $(A_N + \mu I)^{-1}$ in terms of the harmonic analysis on B_N. We have

$$(A_N + \mu I)^{-1} u = \big(K_\mu + \mu^{-1}\mathbf{1}\big) * u, \quad u \in L^1(B_N), \tag{4.7}$$

where $\mathbf{1}(x) \equiv 1$, K_μ is given by (4.4), and the convolution is taken in the sense of the additive group of B_N.

Denote by Π_N the set of all rational numbers of the form

$$p^l \left(\nu_0 + \nu_1 p + \ldots + \nu_{-l+N-1} p^{-l+N-1}\right), \quad l < N,$$

where $\nu_j \in \{0, 1, \ldots, p-1\}$, $\nu_0 \neq 0$. As a set, the quotient group $\mathbb{Q}_p/B_{-N}$ coincides with $\Pi_N \cup \{0\}$, and

$$\{\xi \in \mathbb{Q}_p : \ |\xi|_p \ge p^{-N+1}\} = \bigcup_{\eta \in \Pi_N} (\eta + B_{-N})$$

where the sets $\eta + B_{-N}$ with different $\eta \in \Pi_N$ are disjoint.

Taking into account the fact that $\chi(\rho x) = 1$ for $x \in B_N$, $\rho \in B_{-N}$, we find from (4.4) that

$$K_\mu(x) = p^{-N} \sum_{0 \neq \eta \in \mathbb{Q}_p/B_{-N}} \frac{\chi(\eta x)}{|\eta|_p^\alpha - \lambda + \mu}.$$

Let us describe the domain $DomA_N$ of the generator of our semigroup $T_N(t)$ on $L^1(B_N)$ in terms of distributions on B_N.

Theorem 4.2. *If $\alpha > 1$, then the set $\mathrm{Dom}\, A_N$ consists of those and only those $u \in L^1(B_N)$, for which $f_{-\alpha}^{(N)} * u \in L^1(B_N)$ where the convolution is understood in the sense of the distribution space $\mathcal{D}'(B_N)$. If $u \in \mathrm{Dom}\, A_N$, then $A_N u = f_{-\alpha}^{(N)} * u - \lambda u$ where the convolution is understood in the sense of the distributions from $\mathcal{D}'(B_N)$.*

Proof. Let $u = (A_N + \mu I)^{-1} f$, $f \in L^1(B_N)$, $\mu > 0$. Representing this resolvent as a pseudodifferential operator, we prove that $f_{-\alpha}^{(N)} * u - \lambda u + \mu u = f$ in the sense of $\mathcal{D}'(B_N)$.

Conversely, let $u \in L^1(B_N)$, $D_N^\alpha u = f_{-\alpha}^{(N)} * u \in L^1(B_N)$ where D_N^α is understood in the sense of $\mathcal{D}'(B_N)$. Set $f = (D_N^\alpha - \lambda I + \mu I)u$, $\mu > 0$. Denote $u' = (A_N + \mu I)^{-1} f$. Then $u' \in \mathrm{Dom}\, A_N$, and the above argument shows that

$$(D_N^\alpha - \lambda I + \mu I)(u - u') = 0.$$

Applying the pseudodifferential representation we see that

$$[P_{N,\alpha}(\xi) + \mu]\left[(\mathcal{F}_N u)(\xi) - (\mathcal{F}_N u')(\xi)\right] = 0, \quad \xi \in \mathbb{Q}_p/B_{-N}.$$

It is seen from (3.5) that the factor $P_{N,\alpha}(\xi) + \mu$ is real-valued, strictly positive and locally constant on B_N. Therefore the distribution $\mathcal{F}_N u - \mathcal{F}_N u'$ is zero. Since $\mathcal{F}_N$ is an isomorphism (see [7]), we find that $u = u'$, so that $u \in \mathrm{Dom}\, A_N$.

Theorem 4.2 is proved.

5. Nonlinear equations. Let us consider the equation (1.2) where Φ is a strictly monotone increasing continuous real function, $\Phi(0) = 0$, and the linear operator D_N^α is understood as the operator $A_N + \lambda I$ on $L^1(B_N)$. By the results from [10] and [6], the nonlinear operator $D_N^\alpha \circ \Phi$ is m-accretive, which implies the unique mild solvability of the Cauchy problem for the equation (1.2) with the initial condition $u(0, x) = u_0(x)$, $u_0 \in L^1(B_N)$; see, e.g., [2] for the definitions. As in the classical case [3], this mild solution can be interpreted also as a weak solution.

Following [3], we will show that the above construction of the L^1-mild solution gives also L^γ-solutions for $1 < \gamma \le \infty$.

Theorem 5.1. *Let $u(t, x)$, $t > 0$, $x \in B_N$, be the above mild solution. If $0 < u_0 \in L^\gamma(B_N)$, $1 \le \gamma \le \infty$, then $u(t, \cdot) \in L^\gamma(B_N)$ and*

$$\|u(t, \cdot)\|_{L^\gamma(B_N)} \le \|u_0\|_{L^\gamma(B_N)}. \tag{5.1}$$

Proof. The case $\gamma = 1$ has been considered, while the case $\gamma = \infty$ will be implied by the inequality (5.1) for finite values of γ (see Exercise 4.6 in [5]).

Thus, now we assume that $1 < \gamma < \infty$. It is sufficient to prove (5.1) for $u_0 \in \mathcal{D}(B_N)$. Indeed, if that is proved, we approximate in $L^\gamma(B_N)$ an arbitrary function $u_0 \in L^\gamma(B_N)$ by a sequence $u_{0,j} \in \mathcal{D}(B_N)$. For the corresponding solutions $u_j(t, x)$ we have

$$\|u_j(t, \cdot)\|_{L^\gamma(B_N)} \le \|u_{0,j}\|_{L^\gamma(B_N)}. \tag{5.2}$$

Since our nonlinear semigroup consists of operators continuous on $L^1(B_N)$, we see that, for each $t \ge 0$, $u_j(t, \cdot) \to u(t, \cdot)$ in $L^1(B_N)$. By (5.2), the sequence $\{u_j(t, \cdot)\}$ is bounded in $L^\gamma(B_N)$. These two properties imply the weak convergence $u_j(t, \cdot) \rightharpoonup u(t, \cdot)$ in $L^\gamma(B_N)$ (see Exercise 4.16 in [5]).

Next, we use the weak lower semicontinuity of the L^γ-norm (see Theorem 2.11 in [17]), that is the inequality

$$\liminf_j \|u_j(t,\cdot)\|_{L^\gamma(B_N)} \ge \|u(t,\cdot)\|_{L^\gamma(B_N)}.$$

Passing to the lower limit in both sides of (5.2), we come to (5.1).

Let us prove (5.1) for $u_0 \in \mathcal{D}(B_N)$, $1 < \gamma < \infty$. By the Crandall – Liggett theorem (see [2] or [9]), $u(t,x)$ is obtained as a limit in $L^1(B_N)$,

$$u(t,\cdot) = \lim_{k\to\infty} \left(I + \frac{t}{k} D_N^\alpha \circ \Phi \right)^{-k} u_0,$$

that is $u(t,\cdot) = \lim_{k\to\infty} u_k$ where u_k are found recursively from the relation

$$\frac{t}{k+1} D_N^\alpha \circ \Phi(u_{k+1}) + u_{k+1} = u_k. \tag{5.3}$$

Under our assumptions, $u(t,x) > 0$ (this follows from Theorem 4 in [10]). The nonlinear operator $\left(I + \frac{t}{k} D_N^\alpha \circ \Phi \right)^{-1}$ is also positivity preserving (Proposition 1 in [10]), so that $u_k > 0$ for all k.

Note that the operator D_N^α commutes with shifts while the equation (5.3) for u_{k+1} has a unique solution in $L^1(B_N)$. As a result, if $u_0 \in \mathcal{D}(B_N)$, then all the functions u_k belong to $\mathcal{D}(B_N)$.

Rewriting (5.3) in the form

$$\left(\frac{t}{k+1} \right)^{-1} (u_{k+1} - u_k) = -D_N^\alpha \circ \Phi(u_{k+1}), \tag{5.4}$$

multiplying both sides by $u_{k+1}^{\gamma-1}$ and integrating on B_N we find

$$\left(\frac{t}{k+1} \right)^{-1} \int\limits_{B_N} (u_{k+1} - u_k) u_{k+1}^{\gamma-1} dx = - \int\limits_{B_N} u_{k+1}^{\gamma-1} D_N^\alpha \circ \Phi(u_{k+1})\, dx. \tag{5.5}$$

Let $w = u_{k+1}^{\gamma-1}$. Then $w \in \mathcal{D}(B_N)$. It follows from (5.4) that $D_N^\alpha \Phi(u_{k+1}) \in \mathcal{D}(B_N)$. Also we have $\Phi(u_{k+1}) \in \mathcal{D}(B_N)$, so that $\Phi(u_{k+1})$ belongs to the domain of a selfadjoint realization of the operator D_N^α in $L^2(B_N)$. Therefore we can transform the integral in the right-hand side of (5.5) as follows:

$$\int\limits_{B_N} u_{k+1}^{\gamma-1} D_N^\alpha \circ \Phi(u_{k+1})\, dx = \int\limits_{B_N} \Phi(w^{\frac{1}{\gamma-1}}) D_N^\alpha(w)\, dx. \tag{5.6}$$

The right-hand side of (5.6) is nonnegative by Lemma 2 of [6]. Now it follows from (5.5) that

$$\int\limits_{B_N} u_{k+1}^\gamma dx \le \int\limits_{B_N} u_k u_{k+1}^{\gamma-1} dx.$$

Applying the Hölder inequality we find

$$\int\limits_{B_N} u_{k+1}^{\gamma} dx \leq \left(\int\limits_{B_N} u_k^{\gamma} dx \right)^{1/\gamma} \left(\int\limits_{B_N} u_{k+1}^{\gamma} dx \right)^{\frac{\gamma-1}{\gamma}},$$

which implies the inequality

$$\|u_{k+1}\|_{L^\gamma(B_N)} \leq \|u_k\|_{L^\gamma(B_N)}$$

and, by induction, the inequality

$$\|u_{k+1}\|_{L^\gamma(B_N)} \leq \|u_0\|_{L^\gamma(B_N)}.$$

Passing to the limit, we prove (5.1).

Theorem 5.1 is proved.

References

1. *Albeverio S., Khrennikov A. Yu., Shelkovich V. M.* Theory of p-adic distributions. Linear and nonlinear models. – Cambridge Univ. Press, 2010.
2. *Barbu V.* Nonlinear differential equations of monotone types in Banach spaces. – New York: Springer, 2010.
3. *Bonforte M., Vázquez J. L.* Fractional nonlinear degenerate diffusion equations on bounded domains // Nonlinear Anal. – 2016. – **131**. – P. 363 – 398.
4. *Bourbaki N.* Elements of mathematics. Integration II. – Berlin: Springer, 2004.
5. *Brézis H.* Functional analysis, Sobolev spaces and partial differential equations. – New York: Springer, 2011.
6. *Brézis H., Strauss W.* Semilinear elliptic equations in L^1 // J. Math. Soc. Jap. – 1973. – **25**. – P. 15 – 26.
7. *Bruhat F.* Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques // Bull. Soc. Math. France. – 1961. – **89**. – P. 43 – 75.
8. *Casas-Sánchez O. F., Rodríguez-Vega J. J.* Parabolic type equations on p-adic balls // Bol. Mat. – 2015. – **22**. – P. 97 – 106.
9. *Clément Ph. et al.* One-parameter semigroups. – Amsterdam: North-Holland, 1987.
10. *Crandall M., Pierre M.* Regularizing effects for $u_t + A\psi(u) = 0$ in L^1 // J. Funct. Anal. – 1982. – **45**. – P. 194 – 212.
11. *Helemskii A. Ya.* Lectures and exercises on functional analysis. – Providence: Amer. Math. Soc., 2006.
12. *Hewitt E., Ross K. A.* Abstract harmonic analysis. – Berlin: Springer, 1979. – Vol. II.
13. *Khrennikov A., Kochubei A. N.* p-Adic analogue of the porous medium equation // J. Fourier Anal. and Appl. (to appear), arXiv: 1611.08863.
14. *Khrennikov A., Oleschko K., Correa Lopez M. J.* Application of p-adic wavelets to model reaction-diffusion dynamics in random porous media // J. Fourier Anal. and Appl. – 2016. – **22**. – P. 809 – 822.
15. *Khrennikov A., Oleschko K., Correa Lopez M. J.* Modeling fluid's dynamics with master equations in ultrametric spaces representing the treelike structure of capillary networks // Entropy. – 2016. – **18**. – Art. 249. – 28 p.
16. *Kochubei A. N.* Pseudo-differential equations and stochastics over non-Archimedean fields. – New York: Marcel Dekker, 2001.
17. *Lieb E. H., Loss M.* Analysis. – Providence: Amer. Math. Soc., 2001.
18. *Morris S. A.* Pontryagin duality and the structure of locally compact Abelian groups. – Cambridge Univ. Press, 1977.
19. *Vladimirov V. S., Volovich I. V., Zelenov E. I.* p-Adic analysis and mathematical physics. – Singapore: World Sci., 1994.
20. *Vladimirov V. S.* Tables of integrals of complex-valued functions of p-adic arguments. – Moscow: Steklov Math. Inst., 2003 (in Russian). English version: ArXiv: math-ph/9911027.
21. *Zúñiga-Galindo W. A.* Pseudodifferential equations over non-Archimedean spaces // Lect. Notes Math. – 2016. – **2174**. – xvi+175 p.

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