UDC 517.9

M. Fečkan, M. Pospíšil (Comenius Univ. Bratislava and Math. Inst. Slovak Acad. Sci., Slovak Republic)

ON EQUATIONS WITH GENERALIZED PERIODIC RIGHT-HAND SIDE *

ПРО РІВНЯННЯ З УЗАГАЛЬНЕНОЮ ПЕРІОДИЧНОЮ ПРАВОЮ ЧАСТИНОЮ

Periodic solutions are studied for second-order differential equations with generalized forcing. Analytical bifurcation results are derived with application to forced harmonic and Duffing oscillators.

Вивчаються періодичні розв'язки для диференціальних рівнянь другого порядку з узагальненою примушуючою силою. Аналітичні результати для біфуркацій отримано та застосовано до вимушених гармонічних коливань та осцилятора Даффінга.

1. Introduction. In this paper we shall investigate a weakly forced second-order differential equation

$$\ddot{x}(t) + h(x(t)) = \varepsilon F(t), \tag{1.1}$$

where $h \in C(\mathbb{R}^n, \mathbb{R}^n)$ is an analytic function, $\varepsilon \in \mathbb{R}$ is a small parameter and $F\colon \mathbb{R} \to \mathbb{R}^n$ is a $4a$-periodic generalized function which can behave like Dirac δ-function at the points $\{(1+2k)a \mid k \in \mathbb{Z}\}$. We apply a method of nonsmooth transformation of time proposed by Pilipchuk in [6] but we do not use shooting and numerical computations [7]. Instead, we use the implicit function theorem and Lyapunov – Schmidt method to obtain analytical results on the existence of periodic solutions of (1.1). Later, we investigate particular functions h, concretely a linear and cubic case. So, we consider a weakly forced harmonic oscillator equation

$$\ddot{x}(t) + b^2 x(t) = \varepsilon F(t) \tag{1.2}$$

and a weakly forced Duffing equation

$$\ddot{x}(t) + b^2 x^3(t) = \varepsilon F(t). \tag{1.3}$$

Related results are also given in [8]. Finally, we note that the theory on the existence of periodic solutions in evolution equations is well developed [2] and our paper is a contribution to this nice theory.

2. General results. In this section we consider general equation (1.1) and look for a continuous solution x with possible finite jumps in $\dot{x}$ and a generalized function $\ddot{x}$. Now we recall some results of [7], for the reader's convenience. First, we suppose the transformation

$$x(t) = X\left(\tau\left(\frac{t}{a}\right)\right) + Y\left(\tau\left(\frac{t}{a}\right)\right)\tau'\left(\frac{t}{a}\right) \tag{2.1}$$

for sufficiently smooth functions X, Y and

$$\tau(s) = \frac{2}{\pi}\arcsin\left(\sin\frac{\pi s}{2}\right). \tag{2.2}$$

Note that τ is a 4-periodic piecewise-linear saw-tooth function. Moreover,

$$\tau(4k+1) = 1, \qquad \tau(4k+3) = -1, \quad k \in \mathbb{Z}. \tag{2.3}$$

The following lemma describes some properties of the derivatives of τ. Throughout the paper, we

* This paper was supported by the Slovak Grant Agency VEGA No. 2/0153/16 and No. 1/0078/17 and the Slovak Research and Development Agency under the contract No. APVV-14-0378.

shall use a so-called test function $\chi\colon \mathbb{R}\to\mathbb{R}$, which is a sufficiently smooth function with a compact support.

Lemma 2.1. *For τ given by* (2.2) *the following holds in the sense of distributions*:

$$\tau'(s) = \operatorname{sgn}\cos\left(\frac{\pi s}{2}\right),$$

$$\tau''(s) = 2\sum_{k\in\mathbb{Z}} \delta(s-4k-3) - \delta(s-4k-1)$$

with the Dirac δ-function.

Proof. Taking the test function χ,

$$\int_{-\infty}^{\infty} \tau'(s)\chi(s)\,ds = -\int_{-\infty}^{\infty} \tau(s)\chi'(s)\,ds =$$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty} \arcsin\left(\sin\frac{\pi s}{2}\right)\chi'(s)\,ds = \int_{-\infty}^{\infty} \operatorname{sgn}\cos\left(\frac{\pi s}{2}\right)\chi(s)\,ds.$$

For the second statement we have

$$\int_{-\infty}^{\infty} \tau''(s)\chi(s)\,ds = -\int_{-\infty}^{\infty} \operatorname{sgn}\cos\left(\frac{\pi s}{2}\right)\chi'(s)\,ds =$$

$$= \sum_{k\in\mathbb{Z}}\left(\int_{1+4k}^{3+4k}\chi'(s)\,ds - \int_{-1+4k}^{1+4k}\chi'(s)\,ds\right) = 2\sum_{k\in\mathbb{Z}}\big(\chi(3+4k)-\chi(1+4k)\big) =$$

$$= 2\sum_{k\in\mathbb{Z}}\left(\int_{-\infty}^{\infty}\big(\delta(s-4k-3)-\delta(s-4k-1)\big)\chi(s)\,ds\right).$$

The proof is complete by changing the order of the sum and integral due to a finite number of nonzero summands.

Lemma 2.1 is proved.

Since τ is not invertible on the whole $\mathbb{R}$, the change of coordinates is considered on subintervals and its periodicity is used. So, (2.1) means

$$x(t) = \begin{cases} X\left(\tau\left(\dfrac{t}{a}\right)\right) + Y\left(\tau\left(\dfrac{t}{a}\right)\right), & t\in\bigcup_{k\in\mathbb{Z}}((4k-1)a,(4k+1)a),\\[2ex] X\left(\tau\left(\dfrac{t}{a}\right)\right) - Y\left(\tau\left(\dfrac{t}{a}\right)\right), & t\in\bigcup_{k\in\mathbb{Z}}((4k+1)a,(4k+3)a).\end{cases}$$

In particular, for $t\in(-a,a)$, $\tau(s)=s$ and $\tau'(s)=1$, i.e.,

$$x(a\tau) = X(\tau) + Y(\tau), \quad \tau\in(-1,1). \tag{2.4}$$

On the other side, for $t \in (a, 3a),\ \tau(s) = 2 - s$ and $\tau'(s) = -1$, i.e.,

$$x(a(2-\tau)) = X(\tau) - Y(\tau), \quad \tau \in (-1, 1). \tag{2.5}$$

Summing and subtracting equations (2.4), (2.5) we obtain the inverse transformation

$$\begin{aligned} X(\tau) &= \frac{1}{2}\big(x(a\tau) + x(a(2-\tau))\big), \\ Y(\tau) &= \frac{1}{2}\big(x(a\tau) - x(a(2-\tau))\big) \end{aligned} \tag{2.6}$$

for $\tau \in (-1, 1)$. Note that the values of τ' at the points of $\Lambda := \{1 + 2k \mid k \in \mathbb{Z}\}$ are not known. Hence, the continuity of x on $\mathbb{R}$ gives a necessary condition $Y(\pm 1) = 0$. The generalized derivative of (2.1) is given in the next lemma. For simplicity, we omit the argument $\dfrac{t}{a}$ of τ unless it makes confusion.

Lemma 2.2. *Let x be given by* (2.1). *Then*

$$\dot{x}(t) = a^{-1}\big(X'(\tau)\tau' + Y'(\tau)\big)$$

in the sense of distributions.

Proof. Let χ be an arbitrary test function. Then applying Lemma 2.1 we obtain

$$\int_{-\infty}^{\infty} \dot{x}(s)\chi(s)\,ds = -\int_{-\infty}^{\infty} \left(X\left(\tau\left(\frac{s}{a}\right)\right) + Y\left(\tau\left(\frac{s}{a}\right)\right)\tau'\left(\frac{s}{a}\right)\right)\chi'(s)\,ds =$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} X'\left(\tau\left(\frac{s}{a}\right)\right)\tau'\left(\frac{s}{a}\right)\chi(s)\,ds + \frac{1}{a}\int_{-\infty}^{\infty} Y'\left(\tau\left(\frac{s}{a}\right)\right)\left(\tau'\left(\frac{s}{a}\right)\right)^2\chi(s)\,ds +$$

$$+\frac{2}{a}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty} Y\left(\tau\left(\frac{s}{a}\right)\right)\left(\delta\left(\frac{s}{a} - 4k - 3\right) - \delta\left(\frac{s}{a} - 4k - 1\right)\right)\chi(s)\,ds =$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} \left(X'\left(\tau\left(\frac{s}{a}\right)\right)\tau'\left(\frac{s}{a}\right) + Y'\left(\tau\left(\frac{s}{a}\right)\right)\right)\chi(s)\,ds,$$

where the last identity follows from $\tau' \in \{\pm 1\}$ on $\mathbb{R}\backslash\Lambda$, (2.3) and $Y(\pm 1) = 0$.

Lemma 2.2 is proved.

The statement on the second derivative of x follows.

Lemma 2.3. *Let x be given by* (2.1). *Then*

$$\ddot{x}(t) = a^{-2}\big(X''(\tau) + X'(\tau)\tau'' + Y''(\tau)\tau'\big)$$

in the sense of distributions.

Proof. Taking an arbitrary test function χ we get

$$\int_{-\infty}^{\infty} \ddot{x}(s)\chi(s)\,ds = -\frac{1}{a}\int_{-\infty}^{\infty} \left(X'\left(\tau\left(\frac{s}{a}\right)\right)\tau'\left(\frac{s}{a}\right) + Y'\left(\tau\left(\frac{s}{a}\right)\right)\right)\chi'(s)\,ds =$$

$$= \frac{1}{a^2} \int\limits_{-\infty}^{\infty} \left(X''\left(\tau\left(\frac{s}{a}\right)\right)\left(\tau'\left(\frac{s}{a}\right)\right)^2 + \right.$$

$$\left. + X'\left(\tau\left(\frac{s}{a}\right)\right)\tau''\left(\frac{s}{a}\right) + Y''\left(\tau\left(\frac{s}{a}\right)\right)\tau'\left(\frac{s}{a}\right)\right)\chi(s)\, ds =$$

$$= \frac{1}{a^2} \int\limits_{-\infty}^{\infty} \left(X''\left(\tau\left(\frac{s}{a}\right)\right) + X'\left(\tau\left(\frac{s}{a}\right)\right)\tau''\left(\frac{s}{a}\right) + \right.$$

$$\left. + Y''\left(\tau\left(\frac{s}{a}\right)\right)\tau'\left(\frac{s}{a}\right)\right)\chi(s)\, ds.$$

Lemma 2.3 is proved.

We assume that the function F can be written as

$$F(t) = Q\left(\tau\left(\frac{t}{a}\right)\right) + P\left(\tau\left(\frac{t}{a}\right)\right)\tau'\left(\frac{t}{a}\right) + f\left(\tau\left(\frac{t}{a}\right)\right)\tau''\left(\frac{t}{a}\right)$$

for sufficiently smooth functions $Q,\ P,\ f : \mathbb{R} \to \mathbb{R}^n$. Therefore after transformation (2.1), if $h(x) = \sum_{k=0}^{\infty} h_k x^k$, equation (1.1) becomes

$$a^{-2}\big(X''(\tau) + X'(\tau)\tau'' + Y''(\tau)\tau'\big) + \sum_{k=0}^{\infty} h_k \big(X(\tau) + Y(\tau)\tau'\big)^k =$$

$$= \varepsilon\big(Q(\tau) + P(\tau)\tau' + f(\tau)\tau''\big). \tag{2.7}$$

Note that

$$h(x) = \sum_{k=0}^{\infty} h_k \sum_{j=0}^{k} \binom{k}{j} X^{k-j}(\tau) Y^j(\tau)(\tau')^j.$$

Since the functions $1,\ \tau',\ \tau''$ are of different smoothness on $\mathbb{R}$, and using $(\tau')^2 = 1$ on $(-1, 1)$, equation (2.7) is split into the system

$$\begin{gathered} a^{-2}X''(\tau) + \sum_{k=0}^{\infty} h_k \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} X^{k-2j}(\tau) Y^{2j}(\tau) = \varepsilon Q(\tau), \\ a^{-2}Y''(\tau) + \sum_{k=0}^{\infty} h_k \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2j+1} X^{k-2j-1}(\tau) Y^{2j+1}(\tau) = \varepsilon P(\tau) \end{gathered} \tag{2.8}$$

for $\tau \in (-1, 1)$ with the boundary conditions

$$a^{-2}X'(\pm 1) = \varepsilon f(\pm 1), \qquad Y(\pm 1) = 0.$$

Denoting $K := X + Y,\ L := X - Y$ we separate equations (2.8) to get

$$\begin{gathered} a^{-2}K''(\tau) + h(K(\tau)) = \varepsilon\big(Q(\tau) + P(\tau)\big), \\ a^{-2}L''(\tau) + h(L(\tau)) = \varepsilon\big(Q(\tau) - P(\tau)\big) \end{gathered} \tag{2.9}$$

for $\tau \in (-1, 1)$ with the mixed boundary conditions

$$K(\pm 1) - L(\pm 1) = 0, \qquad K'(\pm 1) + L'(\pm 1) = 2\varepsilon a^2 f(\pm 1). \tag{2.10}$$

Now we recall the nonlinear variation of constants formula of Alekseev [1].

Lemma 2.4. *If $\varphi(c, t)$ is a solution of*

$$\dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}, \tag{2.11}$$

such that $\varphi'(c, 0)$ is regular, then the nonautonomous problem

$$\dot{x}(t) = f(x(t)) + g(t), \quad t \in \mathbb{R}, \tag{2.12}$$

has a solution $\varphi(c(t), t)$ with $c(t)$ satisfying

$$c(t) = c + \int\limits_0^t \varphi'(c(s), s)^{-1} g(s)\, ds, \quad t \in \mathbb{R}, \tag{2.13}$$

for some $c \in \mathbb{R}^n$, where the prime and the dot denote derivatives with respect to c and t, respectively.

Proof. Variational equation corresponding to (2.11) along the solution $\varphi(c, t)$ is

$$(\dot{\varphi}(c, t))' = f'(\varphi(c, t))\varphi'(c, t), \quad t \in \mathbb{R}.$$

By Liouville theorem [4] (Theorem 1.2), $\varphi'(c, t)$ is nonsingular for any $t \in \mathbb{R}$. Now suppose that the solution of (2.12) has the form $\varphi(c(t), t)$ with $c(t)$ to be determined. Then differentiating with respect to t we obtain

$$\dot{x}(t) = \varphi'(c(t), t)\dot{c}(t) + \dot{\varphi}(c(t), t) =$$
$$= \varphi'(c(t), t)\dot{c}(t) + f(\varphi(c(t), t)) = f(\varphi(c(t), t)) + g(t).$$

Therefore

$$\dot{c}(t) = \big(\varphi'(c(t), t)\big)^{-1} g(t),$$

and we get (2.13). Then the statement follows.

Lemma 2.4 is proved.

Note that nonhomogeneous problem (2.12) does not have to possess a unique solution. Next, let us consider

$$\begin{gathered} \dot{x}(t) = f(x(t)) + \varepsilon g(t), \quad t \in [a_1, a_2], \\ Ax(a_1) = \varepsilon b_1, \\ Bx(a_2) = \varepsilon b_2 \end{gathered} \tag{2.14}$$

for $a_1 < a_2$, $\bar{b} = (b_1^*, b_2^*)^* \in \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{k \times n}$, $B \in \mathbb{R}^{(n-k) \times n}$, $0 < k < n$. Then by Lemma 2.4, a general solution of the differential equation of (2.14) is given by $\varphi(c(\varepsilon, \xi, t), t)$ with $c(\varepsilon, \xi, t)$ satisfying $c(\varepsilon, \xi, t) = \xi + \varepsilon \int_{a_1}^t \varphi'(c(\varepsilon, \xi, s), s)^{-1} g(s)\, ds$. So we need to solve

$$\beta(\varepsilon, \xi) := \begin{cases} A\varphi(\xi, a_1) = \varepsilon b_1, \\ B\varphi(c(\varepsilon, \xi, a_2), a_2) = \varepsilon b_2. \end{cases} \tag{2.15}$$

For $\varepsilon = 0$, (2.15) is reduced to

$$\beta(\xi) := \beta(0,\xi) = 0. \tag{2.16}$$

Thus we have to suppose that (2.16) has a solution ξ_0. If it is nondegenerate, i.e., $D\beta(\xi_0)$ is regular, then we can directly apply the implicit function theorem to (2.15). If $D\beta(\xi_0)$ is singular, then Lyapunov – Schmidt method should be used. When $f(x)$ in (2.14) is an affine mapping, i.e., $f(x) = Mx + m$ for a matrix M and a vector m, then $\beta(\varepsilon,\xi,a_1,a_2)$ is linear in ξ and matrix analysis is applied. In general nonlinear case of (2.15), the most simple case is, when we suppose:

(C_1) There is a nondegenerate 1-parametric family of solutions $\xi \in C^3(\mathbb{R},\mathbb{R}^n)$ of (2.16), i.e., it holds $\beta(\xi(\alpha)) = 0$, $\ker D\beta(\xi(\alpha)) = [\xi'(\alpha)]$. Moreover, $\varphi(\xi(\alpha+T),\cdot) = \varphi(\xi(\alpha),\cdot)$ for any $\alpha \in \mathbb{R}$. Then we introduce the orthogonal projection

$$P(\alpha)\colon \mathbb{R}^n \to \operatorname{im} D\beta(\xi(\alpha)) = \big(\ker D\beta(\xi(\alpha))^*\big)^\perp = [\psi(\alpha)]^\perp,$$

and we split (2.15) as follows:

$$\begin{gathered} P(\alpha)\big(\beta(\varepsilon,\xi(\alpha)+\varsigma) - \varepsilon\bar{b}\big) = 0, \quad \varsigma \in [\xi'(\alpha)]^\perp, \\ \psi^*(\alpha)\big(\beta(\varepsilon,\xi(\alpha)+\varsigma) - \varepsilon\bar{b}\big) = 0. \end{gathered} \tag{2.17}$$

By the implicit function theorem, we can solve the first equation of (2.17) to get $\varsigma = \varsigma(\alpha,\varepsilon)$ for any ε small. Clearly $\varsigma(\alpha,0) = 0$. Then inserting $\varsigma(\alpha,\varepsilon)$ into the second equation of (2.17), we get

$$\widetilde{B}(\varepsilon,\alpha) := \psi^*(\alpha)\big(\beta(\varepsilon,\xi(\alpha)+\varsigma(\alpha,\varepsilon)) - \varepsilon\bar{b}\big) = 0. \tag{2.18}$$

We compute

$$\begin{gathered} \widetilde{B}(0,\alpha) = \psi^*(\alpha)\beta(\xi(\alpha)) = 0, \\ M(\alpha) := \widetilde{B}_\varepsilon(0,\alpha) = \psi^*(\alpha)\big(\beta_\varepsilon(0,\xi(\alpha)) - \bar{b}\big). \end{gathered}$$

Hence instead of (2.18), we consider

$$B(\varepsilon,\alpha) := \begin{cases} M(\alpha), & \varepsilon = 0, \\ \dfrac{\widetilde{B}(\varepsilon,\alpha)}{\varepsilon}, & \varepsilon \neq 0. \end{cases} \tag{2.19}$$

From (2.15), we derive

$$\beta_\varepsilon(0,\xi) = \begin{cases} 0, \\ B\varphi'(\xi,a_2)\displaystyle\int_{a_1}^{a_2} \varphi'(\xi,s)^{-1} g(s)\,ds \end{cases}$$

and

$$\beta'(0,\xi) = \begin{cases} A\varphi'(\xi,a_1), \\ B\varphi'(\xi,a_2). \end{cases}$$

Consequently, we obtain

$$M(\alpha) = -\psi_1^*(\alpha)b_1 + \psi_2^*(\alpha)\left(B\varphi'(\xi(\alpha), a_2)\int\limits_{a_1}^{a_2}\varphi'(\xi(\alpha), s)^{-1}g(s)\,ds - b_2\right)$$

for

$$\varphi'(\xi(\alpha), a_1)^*A^*\psi_1(\alpha) + \varphi'(\xi(\alpha), a_2)^*B^*\psi_2(\alpha) = 0. \tag{2.20}$$

Setting

$$\theta(\alpha, s) = \left(\varphi'(\xi(\alpha), s)^{-1}\right)^*\varphi'(\xi(\alpha), a_2)^*B^*\psi_2(\alpha), \tag{2.21}$$

we get

$$M(\alpha) = \int\limits_{a_1}^{a_2}\theta^*(\alpha, s)g(s)\,ds - \psi_1^*(\alpha)b_1 - \psi_2^*(\alpha)b_2. \tag{2.22}$$

Summarizing, we arrive at the following result.

Theorem 2.1. *Assume* $f \in C^3(\mathbb{R}^n, \mathbb{R}^n)$, $g \in C^3\big([a_1, a_2], \mathbb{R}^n\big)$ *and* (C$_1$) *holds. If there is a simple zero* α_0 *of* (2.22), *i.e.,* $M(\alpha_0) = 0$ *and* $DM(\alpha_0)$ *is regular, then there is an* $\alpha \in$ $\in C^1((-\delta, \delta), \mathbb{R})$ *for some* $\delta > 0$ *such that* $\alpha(0) = \alpha_0$ *and* (2.14) *has a unique solution* $x(\varepsilon, t) =$ $= \varphi(\xi(\alpha(\varepsilon)), t) + O(\varepsilon)$ *for any* $\varepsilon \in (-\delta, \delta)$.

Proof. The result follows from the implicit function theorem applying to $B(\varepsilon, \alpha) = 0$ given by (2.19).

***Remark* 2.1.** Under assumptions of Theorem 2.1, (2.14) has a unique solution $x(\varepsilon, t) = \varphi(\xi(\alpha_0), t) + O(\varepsilon)$ for any $\varepsilon \in (-\delta, \delta)$.

***Remark* 2.2.** Clearly $M(\alpha)$ is T-periodic, so if it is changing the sign over its period, then we can apply the Brouwer degree method to solve $B(\varepsilon, \alpha) = 0$, and we get a solution of (2.14) for ε small.

Note (2.21) satisfies the adjoint variational linear equation

$$\dot{w}(t) = -Df(\varphi(\xi(\alpha), t))^*w(t), \quad t \geq 0, \tag{2.23}$$

along with

$$\begin{gathered}\theta(\alpha, a_1) = (\varphi'(\xi(\alpha), a_1)^{-1})^*\varphi'(\xi(\alpha), a_2)^*B^*\psi_2(\alpha) = \\ = -(\varphi'(\xi(\alpha), a_1)^{-1})^*\varphi'(\xi(\alpha), a_1)^*A^*\psi_1(\alpha) = \\ = -A^*\psi_1(\alpha), \\ \theta(\alpha, a_2) = B^*\psi_2(\alpha),\end{gathered} \tag{2.24}$$

where we apply (2.20). Assuming

(C$_2$) A and B are surjective,

then A^* and B^* are injective, and (2.24) is equivalent to

$$\begin{aligned}\theta(\alpha, a_1) &\in \ker A^{\perp}, \qquad \psi_1(\alpha) = -{A^*}^{-1}\theta(\alpha, a_1), \\ \theta(\alpha, a_2) &\in \ker B^{\perp}, \qquad \psi_2(\alpha) = {B^*}^{-1}\theta(\alpha, a_2),\end{aligned} \tag{2.25}$$

since $\operatorname{im} A^* = \ker A^{\perp}$ and $\operatorname{im} B^* = \ker B^{\perp}$.

Finally, we consider the unperturbed (2.9) and (2.10)

$$\begin{aligned} a^{-2}K''(\tau) + h(K(\tau)) = 0, \\ a^{-2}L''(\tau) + h(L(\tau)) = 0 \end{aligned} \tag{2.26}$$

for $\tau \in (-1, 1)$ with the mixed boundary conditions

$$K(\pm 1) - L(\pm 1) = 0, \qquad K'(\pm 1) + L'(\pm 1) = 0. \tag{2.27}$$

Then we claim that

(i) $L(\tau) = K(-\tau + 2)$,

(ii) K is 4-periodic.

To prove (i), we note that any solution of (2.26) is defined on $\mathbb{R}$. So we fix $K(\tau)$ solving $a^{-2}K''(\tau) + h(K(\tau)) = 0$ and take $\widetilde{L}(\tau) = K(-\tau + 2)$. Then $\widetilde{L}(\tau)$ satisfies the 2nd equation of (2.26), and (2.27) implies

$$\widetilde{L}(1) = K(1) = L(1), \qquad \widetilde{L}'(1) = -K'(1) = L'(1),$$

so the uniqueness of solutions gives $\widetilde{L}(\tau) = L(\tau)$. This proves (i). To prove (ii), using also (i) we compute

$$K(-1) = L(-1) = K(3), \qquad K'(-1) = -L'(-1) = K'(3),$$

which gives (ii). Reversibly, if $K(\tau)$ solves the 1st equation of (2.26) and it is 4-periodic, then taking $L(\tau)$ by (i), we get a solution of the 2nd equation of (2.26) with (2.27). We note then we have a family $K(\tau + \alpha)$ satisfying (i) and (ii).

Moreover, we consider the linearization of (2.26),

$$\begin{aligned} a^{-2}U''(\tau) + h'(K(\tau))U(\tau) = 0, \\ a^{-2}V''(\tau) + h'(L(\tau))V(\tau) = 0 \end{aligned} \tag{2.28}$$

along K and L satisfying (i), (ii) with the mixed boundary conditions

$$U(\pm 1) - V(\pm 1) = 0, \qquad U'(\pm 1) + V'(\pm 1) = 0. \tag{2.29}$$

Then again it holds

(i′) $V(\tau) = U(-\tau + 2)$,

(ii′) U is 4-periodic.

Indeed, we take $\widetilde{V}(\tau) = U(-\tau + 2)$. Then

$$\begin{aligned} a^{-2}\widetilde{V}''(\tau) + h'(L(\tau))\widetilde{V}(\tau) = \\ = a^{-2}U''(-\tau + 2) + h'(K(-\tau + 2))U(-\tau + 2) = 0, \end{aligned}$$

so $\widetilde{V}(\tau)$ satisfies the 2nd equation of (2.28), and (2.29) implies

$$\widetilde{V}(1) = U(1) = V(1), \qquad \widetilde{V}'(1) = -U'(1) = V'(1),$$

thus we get $\widetilde{V}(\tau) = V(\tau)$. This proves (i′). To prove (ii′), we take $\widetilde{U}(\tau) = U(\tau + 4)$ and derive

$$a^{-2}\widetilde{U}''(\tau) + h'(K(\tau))\widetilde{U}(\tau) =$$

$$= a^{-2}U''(\tau+4) + h'(K(\tau+4))U(\tau+4) = 0,$$

so $\widetilde{U}(\tau)$ satisfies the 1st equation of (2.28), and using (2.29) with (i$'$), we obtain

$$\widetilde{U}(-1) = U(3) = V(-1) = U(-1),$$
$$\widetilde{U}'(-1) = U'(3) = -V(-1) = U(-1),$$

which gives (ii$'$). Reversibly, if $U(\tau)$ solves the 1st equation of (2.28) under (i), (ii) and it is 4-periodic, then taking $V(\tau)$ by (i$'$), we get a solution of the 2nd equation of (2.28) with (2.29).

Furthermore, assuming (ii), $K(\tau)$ is surrounded by periodic solutions $K(r,t)$ with periods $T(r)$, i.e., $K(0,t) = K(t),\ K(r,\tau+T(r)) = K(r,\tau)$ and $T(0) = 4$. Then $W(\tau) = \partial_r K(0,\tau)$ solves

$$a^{-2}W''(\tau) + h'(K(\tau))W(\tau) = 0,$$
$$W(\tau+4) + T'(0)K'(\tau) = W(\tau).$$

Hence if

$$T'(0) \neq 0, \tag{2.30}$$

then $W(\tau)$ is a non 4-periodic solution of the first equation of (2.28). But $K'(\tau)$ is a 4-periodic solution of the first equation of (2.28). Summarizing, under (2.30), the dimension of solutions of BVP (2.28), (2.29) is 1.

3. Harmonic oscillator. In this section we consider equation (1.2) under transformation (2.1). Following the previous section we derive

$$\begin{aligned} a^{-2}K''(\tau) + b^2K(\tau) &= \varepsilon\big(Q(\tau)+P(\tau)\big), \\ a^{-2}L''(\tau) + b^2L(\tau) &= \varepsilon\big(Q(\tau)-P(\tau)\big) \end{aligned} \tag{3.1}$$

for $\tau \in (-1,1)$ with the boundary conditions

$$K(\pm1) - L(\pm1) = 0, \qquad K'(\pm1) + L'(\pm1) = 2\varepsilon a^2 f(\pm1). \tag{3.2}$$

Denoting $K_1 = K,\ L_1 = L$ we rewrite (3.1), (3.2) as

$$\begin{aligned} K_1'(\tau) &= K_2(\tau), \\ K_2'(\tau) &= -a^2b^2K_1(\tau) + \varepsilon a^2\big(Q(\tau)+P(\tau)\big), \\ L_1'(\tau) &= L_2(\tau), \\ L_2'(\tau) &= -a^2b^2L_1(\tau) + \varepsilon a^2\big(Q(\tau)-P(\tau)\big) \end{aligned} \tag{3.3}$$

along with

$$K_1(\pm1) - L_1(\pm1) = 0, \qquad K_2(\pm1) + L_2(\pm1) = 2\varepsilon a^2 f(\pm1). \tag{3.4}$$

The unperturbed problem has the general solution

$$\begin{aligned} \varphi(c,\tau) := \big(&c_1\sin ab\tau + c_2\cos ab\tau, c_1ab\cos ab\tau - c_2ab\sin ab\tau, \\ &c_3\sin ab\tau + c_4\cos ab\tau, c_3ab\cos ab\tau - c_4ab\sin ab\tau\big)^* \end{aligned} \tag{3.5}$$

with $c = (c_1, c_2, c_3, c_4)^*$. Note that $\varphi'(c,\tau) = \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}$ with

$$\Phi = \begin{pmatrix} \sin ab\tau & \cos ab\tau \\ ab\cos ab\tau & -ab\sin ab\tau \end{pmatrix}$$

does not depend on c. Hence, $\det \varphi'(c,\tau) = a^2b^2 \neq 0$. We set $a_1 = -1,\ a_2 = 1$ and denote

$$\beta_1(c) := A\varphi(c,-1) = \begin{pmatrix} -c_1S + c_2C + c_3S - c_4C \\ ab(c_1C + c_2S + c_3C + c_4S) \end{pmatrix},$$

$$\beta_2(c) := B\varphi(c,1) = \begin{pmatrix} c_1S + c_2C - c_3S - c_4C \\ ab(c_1C - c_2S + c_3C - c_4S) \end{pmatrix}$$

with $S = \sin ab,\ C = \cos ab$ and

$$A = B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

Hence,

$$\begin{pmatrix} \beta_1(c) \\ \beta_2(c) \end{pmatrix} = Wc, \qquad W := \begin{pmatrix} -S & C & S & -C \\ abC & abS & abC & abS \\ S & C & -S & -C \\ abC & -abS & abC & -abS \end{pmatrix}.$$

The following lemma describes the null space of W, $\ker W$, and image of W, $\operatorname{im} W$, with respect to a and b.

Lemma 3.1. *Let $a, b > 0$ be fixed. Then*

$$\ker W = \begin{cases} 0, & \dfrac{2ab}{\pi} \notin \mathbb{Z}, \\ \big[(1,0,-1,0)^*, (0,1,0,1)^*\big], & \dfrac{ab}{\pi} \in \mathbb{Z}, \\ \big[(1,0,1,0)^*, (0,1,0,-1)^*\big], & \dfrac{ab}{\pi} - \dfrac{1}{2} \in \mathbb{Z}, \end{cases}$$

$$\operatorname{im} W = \begin{cases} \mathbb{R}^4, & \dfrac{2ab}{\pi} \notin \mathbb{Z}, \\ \big[(0,1,0,1)^*, (1,0,1,0)^*\big], & \dfrac{ab}{\pi} \in \mathbb{Z}, \\ \big[(1,0,-1,0)^*, (0,1,0,-1)^*\big], & \dfrac{ab}{\pi} - \dfrac{1}{2} \in \mathbb{Z}, \end{cases}$$

where $[v_1, v_2]$ denotes the linear span of vectors $v_1,\ v_2$.

Proof. Note that $\det W = -16a^2b^2S^2C^2$. Thus for $S \neq 0 \neq C$, W is nonsingular. Investigating the cases $S \neq 0 = C$ and $S = 0 \neq C$ separately gives the statement.

Now we consider particular cases of a and b as distinguished in Lemma 3.1.

Theorem 3.1. *Let* $a, b > 0$ *be fixed,* $Q,\ P,\ f$ *be sufficiently smooth functions, and* φ *be given by* (3.5). *Then* $\varphi(c(\tau), \tau)$ *is a solution of* (3.3), (3.4), *if*

$$c(\tau) = c(-1) + \frac{\varepsilon a}{b} \int\limits_{-1}^{\tau} v(\sigma)\, d\sigma, \tag{3.7}$$

where

$$v(\tau) = \begin{pmatrix} (Q(\tau) + P(\tau)) \cos ab\tau \\ -(Q(\tau) + P(\tau)) \sin ab\tau \\ (Q(\tau) - P(\tau)) \cos ab\tau \\ -(Q(\tau) - P(\tau)) \sin ab\tau \end{pmatrix}$$

and

if $\dfrac{2ab}{\pi} \notin \mathbb{Z}$, *then*

$$c(-1) = 2\varepsilon a^2 W^{-1} \left(v_0 - \int\limits_{-1}^{1} v_1(\sigma) d\sigma \right), \tag{3.8}$$

if $\dfrac{ab}{\pi} \in \mathbb{Z}$ *and*

$$\begin{gathered} \int\limits_{-1}^{1} P(\sigma) \sin(ab(1 - \sigma))\, d\sigma = 0, \\ f(1) - \int\limits_{-1}^{1} Q(\sigma) \cos(ab(1 - \sigma))\, d\sigma = f(-1), \end{gathered} \tag{3.9}$$

then

$$c(-1) = 2\varepsilon a^2 W|_{(\ker W)^{\perp}}^{-1} \left(v_0 - \int\limits_{-1}^{1} v_1(\sigma) d\sigma \right) + w, \tag{3.10}$$

for some $w \in \ker W$,

if $\dfrac{ab}{\pi} - \dfrac{1}{2} \in \mathbb{Z}$ *and*

$$\int\limits_{-1}^{1} P(\sigma) \sin(ab(1 - \sigma))\, d\sigma = 0,$$

$$f(1) - \int\limits_{-1}^{1} Q(\sigma) \cos(ab(1 - \sigma))\, d\sigma = -f(-1),$$

then $c(-1)$ *is given by* (3.10) *for some* $w \in \ker W$, *where*

$$v_0 = \big(0, f(-1), 0, f(1)\big)^*,$$

$$v_1(\tau) = \left(0, 0, \frac{P(\tau)}{ab}\sin(ab(1-\tau)), Q(\tau)\cos(ab(1-\tau))\right)^*$$

and $|_{(\ker W)^\perp}$ *denotes the restriction onto the orthogonal complement to* $\ker W$.

Proof. Equation (3.7) follows from Lemma 2.4. It only remains to determine $c(-1)$.

From (3.7) and (3.4) we get that $c(-1)$ has to satisfy

$$\beta_1(c(-1)) = \begin{pmatrix} 0 \\ 2\varepsilon a^2 f(-1) \end{pmatrix}, \qquad \beta_2(c(1)) = \begin{pmatrix} 0 \\ 2\varepsilon a^2 f(1) \end{pmatrix},$$

where

$$\beta_2(c(1)) = \begin{pmatrix} S & C & -S & -C \\ abC & -abS & abC & -abS \end{pmatrix}\left(c(-1) + \frac{\varepsilon a}{b}\int\limits_{-1}^{1} v(\sigma)d\sigma\right).$$

In other words,

$$Wc(-1) + \frac{\varepsilon a}{b}\int\limits_{-1}^{1} W_1 v(\sigma)d\sigma = 2\varepsilon a^2 v_0, \tag{3.11}$$

where

$$W_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S & C & -S & -C \\ abC & -abS & abC & -abS \end{pmatrix}.$$

If $\dfrac{2ab}{\pi} \notin \mathbb{Z}$, (3.11) is equivalent to (3.8) by Lemma 3.1. If $\dfrac{ab}{\pi} \in \mathbb{Z}$, $c(-1)$ exists if and only if

$$v_0 - \int\limits_{-1}^{1} v_1(\sigma)\, d\sigma \in \operatorname{im} W.$$

Note that the first coordinate of the left-hand side of the above inclusion is zero. Hence by Lemma 3.1, first condition of (3.9) has to be satisfied. Moreover, the second coordinate has to be equal to the fourth, which gives the second condition. Then taking the inverse of $W|_{(\ker W)^\perp}$ one obtains the second statement. The third case follows analogously.

Theorem 3.1 is proved.

For the original problem (1.2) we immediately obtain the next statement.

***Corollary* 3.1.** *Let the assumptions of Theorem* 3.1 *be fulfilled. Equation* (1.2) *has the solution*

$$x(t) = \begin{cases} \varphi_1\left(c\left(\tau\left(\dfrac{t}{a}\right)\right), \tau\left(\dfrac{t}{a}\right)\right), & t \in \bigcup_{k\in\mathbb{Z}}((4k-1)a, (4k+1)a), \\ \varphi_3\left(c\left(\tau\left(\dfrac{t}{a}\right)\right), \tau\left(\dfrac{t}{a}\right)\right), & t \in \bigcup_{k\in\mathbb{Z}}((4k+1)a, (4k+3)a), \end{cases}$$

where φ *has coordinates* φ_i, $i = 1, \ldots, 4$.

Proof. Using (2.1) with $X = \frac{1}{2}(K+L)$, $Y = \frac{1}{2}(K-L)$ and Theorem 3.1 gives the statement.

***Example* 3.1.** Let us consider the equation

$$\ddot{x}(t) + b^2 x(t) = \varepsilon\left(\sin^2\frac{\pi t}{2} + (-1)^{\lfloor \frac{t+1}{2}\rfloor} + \sum_{k\in\mathbb{Z}}\delta(t-4k-1)\right). \tag{3.12}$$

In this case, the equality

$$F(t) = \sin^2\frac{\pi\tau(t)}{2} + \tau'(t) + \frac{1}{4}\left(1+\sin\frac{\pi\tau(t)}{2}\right)\tau''(t) \tag{3.13}$$

holds almost everywhere, since

$$\tau(t) = \begin{cases} t-4k, & t\in(4k-1,4k+1], \quad k\in\mathbb{Z},\\ 2-t+4k, & t\in(4k+1,4k+3], \quad k\in\mathbb{Z},\end{cases}$$

and

$$\sin\frac{\pi\tau(t)}{2} = \begin{cases} \sin\left(\frac{\pi t}{2} - 2k\pi\right) = \sin\frac{\pi t}{2}, & t\in(4k-1,4k+1], \quad k\in\mathbb{Z},\\ \sin\left(\pi(2k+1) - \frac{\pi t}{2}\right) = \sin\frac{\pi t}{2}, & t\in(4k+1,4k+3], \quad k\in\mathbb{Z}.\end{cases}$$

Hence

$$a = 1, \qquad Q(\tau) = \sin^2\frac{\pi\tau}{2}, \qquad P(\tau) = 1, \qquad f(\tau) = \frac{1}{4}\left(1+\sin\frac{\pi\tau}{2}\right). \tag{3.14}$$

Function F is sketched in Fig. 1.

For simplicity, we take $b = \frac{\pi}{4}$. Then applying Theorem 3.1,

$$c(-1) = \frac{\sqrt{2}\varepsilon}{15\pi^2}\big(15\pi - 176, -(15\pi+64), 15\pi+64, -15\pi+176\big)^*.$$

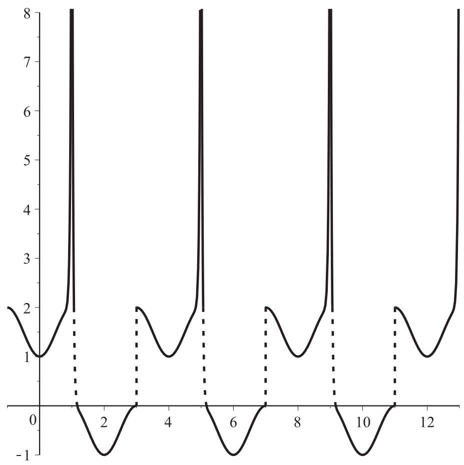
Consequently,

$$c(\tau) = \frac{\varepsilon}{15\pi^2}\begin{pmatrix} 15\sqrt{2}\pi + 360S_1 - 20S_3 - 12S_5\\ -(15\pi+240)\sqrt{2} + 360C_1 + 20C_3 - 12C_5\\ 15\sqrt{2}\pi - 120S_1 - 20S_3 - 12S_5\\ -(15\pi-240)\sqrt{2} - 120C_1 + 20C_3 - 12C_5\end{pmatrix},$$

where $S_i = \sin\frac{i\pi\tau}{4}$, $C_i = \cos\frac{i\pi\tau}{4}$ for $i = 1,3,5$. By Corollary 3.1, the solution of (3.12) has the form

$$x(t) = \begin{cases} \frac{\varepsilon}{15\pi^2}\bigg(360 + 8\cos\pi\tau(t) + 15\sqrt{2}\pi\sin\frac{\pi\tau(t)}{4} - \\ \qquad -(15\pi+240)\sqrt{2}\cos\frac{\pi\tau(t)}{4}\bigg), & t\in\bigcup_{k\in\mathbb{Z}}(4k-1,4k+1),\\ \frac{\varepsilon}{15\pi^2}\bigg(-120 + 8\cos\pi\tau(t) + 15\sqrt{2}\pi\sin\frac{\pi\tau(t)}{4} - \\ \qquad -(15\pi-240)\sqrt{2}\cos\frac{\pi\tau(t)}{4}\bigg), & t\in\bigcup_{k\in\mathbb{Z}}(4k+1,4k+3)\end{cases} \tag{3.15}$$

(see Fig. 1) with $\tau(t)$ given by (2.2).



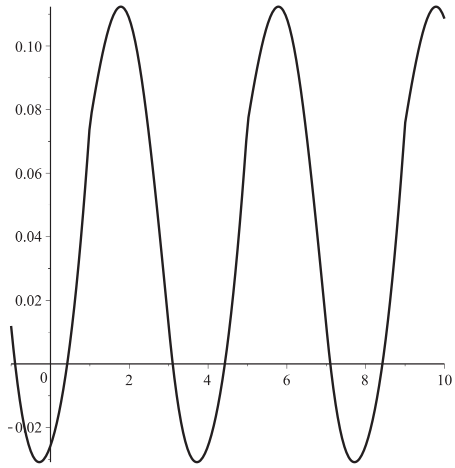


Fig. 1. Sketch of F given by (3.13), and the solution x of (1.2) given by (3.15).

4. Duffing equation. Here we consider equation (1.3) under transformation (2.1). In this case, system (2.9), (2.10) has the form

$$a^{-2}K''(\tau) + b^2K^3(\tau) = \varepsilon(Q(\tau) + P(\tau)),$$
$$a^{-2}L''(\tau) + b^2L^3(\tau) = \varepsilon(Q(\tau) - P(\tau)) \tag{4.1}$$

for $\tau \in (-1, 1)$ with the mixed boundary conditions

$$K(\pm 1) - L(\pm 1) = 0, \qquad K'(\pm 1) + L'(\pm 1) = 2\varepsilon a^2 f(\pm 1). \tag{4.2}$$

Denoting $K_1 = K,\ L_1 = L$ we rewrite (4.1), (4.2) as

$$K_1'(\tau) = K_2(\tau),$$
$$K_2'(\tau) = -a^2b^2K_1^3(\tau) + \varepsilon a^2(Q(\tau) + P(\tau)),$$
$$L_1'(\tau) = L_2(\tau), \tag{4.3}$$
$$L_2'(\tau) = -a^2b^2L_1^3(\tau) + \varepsilon a^2(Q(\tau) - P(\tau))$$

along with

$$K_1(\pm 1) - L_1(\pm 1) = 0, \qquad K_2(\pm 1) + L_2(\pm 1) = 2\varepsilon a^2 f(\pm 1). \tag{4.4}$$

Corresponding unperturbed system has the solution

$$\varphi(c,\tau) := \big(c_1 \operatorname{cn}_1(\tau), -abc_1^2 \operatorname{sn}_1(\tau)\operatorname{dn}_1(\tau), c_3 \operatorname{cn}_3(\tau), -abc_3^2 \operatorname{sn}_3(\tau)\operatorname{dn}_3(\tau)\big)^*, \tag{4.5}$$

where

$$c = (c_1, c_2, c_3, c_4),$$
$$\operatorname{cn}_i(\tau) = \operatorname{cn}\big((ab\tau + c_{i+1})c_i, 1/\sqrt{2}\,\big),$$
$$\operatorname{sn}_i(\tau) = \operatorname{sn}\big((ab\tau + c_{i+1})c_i, 1/\sqrt{2}\,\big),$$

$$\operatorname{dn}_i(\tau) = \operatorname{dn}\big((ab\tau + c_{i+1})c_i, 1/\sqrt{2}\,\big)$$

for $i = 1, 3$ and cn, sn, dn are Jacobi elliptic functions [5]. The derivative of $\varphi(c, \tau)$ satisfies $\varphi'(c,\tau) = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_3 \end{pmatrix}$, where

$$\Phi_i = \Phi_i(c,\tau) = \begin{pmatrix} \operatorname{cn}_i - c_i(ab\tau + c_{i+1}) \operatorname{sn}_i \operatorname{dn}_i & -c_i^2 \operatorname{sn}_i \operatorname{dn}_i \\ -abc_i(2 \operatorname{sn}_i \operatorname{dn}_i + c_i(ab\tau + c_{i+1}) \operatorname{cn}_i^3) & -abc_i^3 \operatorname{cn}_i^3 \end{pmatrix} \tag{4.6}$$

for $i = 1, 3$. Hence, $\det \varphi'(c,\tau) = a^2 b^2 c_1^3 c_3^3 \neq 0$ if and only if $c_1 \neq 0 \neq c_3$. Let us denote

$$\varphi'(c,\tau)^{-1} \begin{pmatrix} 0 \\ a^2(Q(\tau) + P(\tau)) \\ 0 \\ a^2(Q(\tau) - P(\tau)) \end{pmatrix} =$$

$$= \begin{pmatrix} -\dfrac{a(Q(\tau) + P(\tau)) \operatorname{sn}_1 \operatorname{dn}_1}{bc_1} \\ \dfrac{a(Q(\tau) + P(\tau))(-\operatorname{cn}_1 + c_1(ab\tau + c_2) \operatorname{sn}_1 \operatorname{dn}_1)}{bc_1^3} \\ -\dfrac{a(Q(\tau) - P(\tau)) \operatorname{sn}_3 \operatorname{dn}_3}{bc_3} \\ \dfrac{a(Q(\tau) - P(\tau))(-\operatorname{cn}_3 + c_3(ab\tau + c_4) \operatorname{sn}_3 \operatorname{dn}_3)}{bc_3^3} \end{pmatrix} =: v(c,\tau).$$

We want to apply Theorem 2.1, thus we take $a_1 = -1,\ a_2 = 1$ and consider matrices $A,\ B$ given by (3.6).

First, we look for a solution of the unperturbed problem (4.3), (4.4), i.e., of the corresponding unperturbed problem satisfying zero boundary conditions. So we only have to find $c \in \mathbb{R}^4$ such that equations

$$c_1 \operatorname{cn}_1(\pm 1) - c_3 \operatorname{cn}_3(\pm 1) = 0,$$
$$c_1^2 \operatorname{sn}_1(\pm 1) \operatorname{dn}_1(\pm 1) + c_3^2 \operatorname{sn}_3(\pm 1) \operatorname{dn}_3(\pm 1) = 0$$

are satisfied for both signs. Immediately, we obtain the trivial solution:

Lemma 4.1. *Let $a, b > 0$ be arbitrary and fixed, and φ be defined by* (4.5). *Then* $\varphi(0, c_2, 0, c_4, \tau) \equiv 0$ *for any $c_2, c_4 \in \mathbb{R}$ is a solution of* (4.3), (4.4) *with* $\varepsilon = 0$.

Let $c_1 = c_3$. Denoting $U := (\pm ab + c_2)c_1,\ V := (\pm ab + c_4)c_3$ and omitting the argument $k = \dfrac{1}{\sqrt{2}}$ of the elliptic functions we have

$$\begin{aligned} \operatorname{cn} U - \operatorname{cn} V &= 0, \\ \operatorname{sn} U \operatorname{dn} U + \operatorname{sn} V \operatorname{dn} V &= 0. \end{aligned} \tag{4.7}$$

Summing the squares of these equations we get

$$1 - k^2(1-\operatorname{cn}^2 U)^2 - (\operatorname{cn} U \operatorname{cn} V - \operatorname{sn} U \operatorname{sn} V \operatorname{dn} U \operatorname{dn} V) = 0,$$

where we used

$$\operatorname{dn}^2 U = 1 - k^2 \operatorname{sn}^2 U = 1 - k^2(1-\operatorname{cn}^2 U) = \operatorname{dn}^2 V$$

following from the first equation of (4.7). Next, applying [9] (§ 22.21),

$$\operatorname{cn}(U+V) = \frac{\operatorname{cn} U \operatorname{cn} V - \operatorname{sn} U \operatorname{sn} V \operatorname{dn} U \operatorname{dn} V}{1 - k^2 \operatorname{sn}^2 U \operatorname{sn}^2 V}, \tag{4.8}$$

we derive

$$1 - k^2(1-\operatorname{cn}^2 U)^2 - \operatorname{cn}(U+V)\big(1 - k^2 \operatorname{sn}^2 U \operatorname{sn}^2 V\big) = 0,$$

i.e.,

$$\big(1 - k^2(1-\operatorname{cn}^2 U)^2\big)\big(1 - \operatorname{cn}(U+V)\big) = 0.$$

Since the first bracket is nonzero, $\operatorname{cn}(U+V) = 1$. Now we apply the definition of cn saying that $\operatorname{cn} u = \cos\phi$, where

$$u = \int\limits_0^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}. \tag{4.9}$$

Hence, $\operatorname{cn}(U+V) = 1$ if and only if

$$U + V = \int\limits_0^{2j\pi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = 4j \int\limits_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = 4jK$$

for some $j \in \mathbb{Z}$, where $K = K(k)$ is the complete elliptic integral of the first kind. That means,

$$\begin{aligned} (2ab + c_2 + c_4)c_1 &= 4iK, \\ (-2ab + c_2 + c_4)c_1 &= 4jK \end{aligned} \tag{4.10}$$

for some integers i, j. Therefore $c_1 = \dfrac{(i-j)K}{ab}$ which has to be nonzero. Thus $c_1 = \dfrac{mK}{ab}$ for some $m \in \mathbb{Z}\backslash\{0\}$. Then, from system (4.10),

$$c_4 = 2ab\left(\frac{2i}{m} - 1\right) - c_2$$

for some $i \in \mathbb{Z}$. Concluding the above, we get the following lemma.

Lemma 4.2. *Let* $a, b > 0$, $m \in \mathbb{Z}$, $m \neq 0$ *be arbitrary and fixed, and* φ *be defined by* (4.5). *Then*

$$\varphi\left(\frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab},\ c_2,\ \frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab},\ -2ab - c_2,\ \tau\right)$$

for any $c_2 \in \mathbb{R}$ *is a solution of* (4.3), (4.4) *with* $\varepsilon = 0$.

Proof. Let $i \in \mathbb{Z}$ be arbitrary and fixed. From the above computations we get the solution

$$\varphi\left(\frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab},\ c_2, \frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab},\ 2ab\left(\frac{2i}{m}-1\right)-c_2,\ \tau\right).$$

Then using

$$\begin{gathered}\operatorname{cn}(u+2jK)=(-1)^j\operatorname{cn}(u),\\ \operatorname{sn}(u+2jK)=(-1)^j\operatorname{sn}(u),\\ \operatorname{dn}(u+2jK)=\operatorname{dn}(u)\end{gathered} \tag{4.11}$$

for each $j \in \mathbb{Z}$, $u \in \mathbb{R}$, one can prove the statement.

On the other side, let $c_1 = -c_3$. Note that by [9] (§ 22.12) functions cn, dn are even and sn is odd. Hence, from (4.8) we have

$$\operatorname{cn}(U-V)=\frac{\operatorname{cn}U\operatorname{cn}V+\operatorname{sn}U\operatorname{sn}V\operatorname{dn}U\operatorname{dn}V}{1-k^2\operatorname{sn}^2U\operatorname{sn}^2V}.$$

Following the above arguments we derive $\operatorname{cn}(U-V)=-1$ which holds if and only if $U-V = 2(1+2j)K$. Consequently, $c_1=\dfrac{mK}{ab}$ for some $m\in\mathbb{Z}\backslash\{0\}$, and

$$c_4=2ab\left(\frac{1+2i}{m}-1\right)-c_2.$$

We summarize this result to a lemma.

Lemma 4.3. *Let $a,b>0$, $m\in\mathbb{Z}$, $m\neq 0$ be arbitrary and fixed, and φ be defined by* (4.5). *Then*

$$\varphi\left(\frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab},\ c_2, -\frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab},\ 2ab\left(\frac{1}{m}-1\right)-c_2,\ \tau\right)$$

for any $c_2\in\mathbb{R}$ is a solution of (4.3), (4.4) *with $\varepsilon=0$.*

Proof. The statement can be proved as Lemma 4.2.

***Remark* 4.1.** Using the properties (4.11), for the third coordinate of φ from Lemma 4.3 we obtain

$$\begin{gathered}\varphi_3\left(\frac{mK}{ab},c_2,-\frac{mK}{ab},2ab\left(\frac{1}{m}-1\right)-c_2,\tau\right)=\\ =-\frac{mK}{ab}\operatorname{cn}\left(\left(ab\tau+2ab\left(\frac{1}{m}-1\right)-c_2\right)\left(-\frac{mK}{ab}\right)\right)=\\ =\frac{mK}{ab}\operatorname{cn}\left(\left(ab\tau-2ab-c_2\right)\frac{mK}{ab}\right)=\\ =\varphi_3\left(\frac{mK}{ab},c_2,\frac{mK}{ab},-2ab-c_2,\tau\right).\end{gathered}$$

Analogously, it can be shown that the fourth coordinates of φs from Lemmas 4.2 and 4.3 are equal. That means that Lemmas 4.2 and 4.3 give the same solutions.

Denoting $\xi(\alpha) := \left(\frac{mK}{ab}, \alpha, \frac{mK}{ab}, -2ab - \alpha\right)^*$, Lemma 4.2 implies that $\beta(\xi(\alpha)) = 0$ for all $\alpha \in \mathbb{R}$. Moreover, $\varphi(\xi(\alpha + T), \cdot) = \varphi(\xi(\alpha), \cdot)$ for all $\alpha \in \mathbb{R}$ and $T = \frac{4ab}{|m|}$. To verify (C_1) one only needs to investigate $\ker D\beta(\xi(\alpha))$. Note that since

$$\beta(\xi) = \begin{pmatrix} A \\ 0 \end{pmatrix} \varphi(\xi, -1) + \begin{pmatrix} 0 \\ A \end{pmatrix} \varphi(\xi, 1),$$

we get

$$D\beta(\xi) = \begin{pmatrix} A \\ 0 \end{pmatrix} D\varphi(\xi, -1) + \begin{pmatrix} 0 \\ A \end{pmatrix} D\varphi(\xi, 1).$$

Using that $D\varphi(\xi, \tau)$ is the fundamental matrix solution of the variational equation of unperturbed (4.3),

$$\begin{aligned} U_1'(\tau) &= U_2(\tau), \\ U_2'(\tau) &= -3a^2b^2K_1^2(\tau)U_1(\tau), \\ V_1'(\tau) &= V_2(\tau), \\ V_2'(\tau) &= -3a^2b^2L_1^2(\tau)V_1(\tau), \end{aligned} \tag{4.12}$$

which leads to (2.28), we see that $\ker D\beta(\xi(\alpha))$ is given by (2.28) with (2.29), so we can use results of Section 2. Taking

$$c_1 = \frac{mK\left(\frac{1}{\sqrt{2}}\right)}{ab} + r$$

in (4.5), we get that its minimal period is $T_{\min}(r) = \dfrac{4K\left(\frac{1}{\sqrt{2}}\right)}{mK\left(\frac{1}{\sqrt{2}}\right) + abr}$. Thus we take

$$T(r) = mT_{\min}(r) = \frac{4mK\left(\frac{1}{\sqrt{2}}\right)}{mK\left(\frac{1}{\sqrt{2}}\right) + abr}.$$

Clearly $T(0) = 4$ and $T'(0) \neq 0$, so (2.30) is satisfied, and consequently, assumption (C_1) is verified.

Now, instead of calculating $\theta(\alpha, \tau)$ from (2.21), we derive it as a solution of the adjoint variational equation. That is the adjoint system to (4.12),

$$\begin{aligned} U_1'(\tau) &= 3a^2b^2K_1^2(\tau)U_2(\tau), \\ U_2'(\tau) &= -U_1(\tau), \\ V_1'(\tau) &= 3a^2b^2L_1^2(\tau)V_2(\tau), \\ V_2'(\tau) &= -V_1(\tau), \end{aligned} \tag{4.13}$$

which leads to (2.28) of the form

$$\begin{aligned} &U_2''(\tau) + 3a^2b^2K_1^2(\tau)U_2(\tau) = 0, \\ &V_2''(\tau) + 3a^2b^2L_1^2(\tau)V_2(\tau) = 0. \end{aligned} \tag{4.14}$$

Furthermore, we derive (see (3.6))

$$\ker A = \ker B = \big[(0,-1,0,1)^*,\ (1,0,1,0)^*\big].$$

Hence (2.25) leads to

$$-U_2(\pm 1) + V_2(\pm 1) = 0, \qquad U_1(\pm 1) + V_1(\pm 1) = 0,$$

i.e.,

$$U_2(\pm 1) - V_2(\pm 1) = 0, \qquad U_2'(\pm 1) + V_2'(\pm 1) = 0,$$

which is just (2.29). So by Section 2, we can take $U_2(\tau) = K_1'(\tau)$, $V_2(\tau) = K_1'(-\tau+2) = -L_1'(\tau)$. Using (4.13), we also have $U_1(\tau) = -K_1''(\tau)$ and $V_1(\tau) = L_1''(\tau)$. We recall that in the notation of (4.5), we have

$$\varphi_1(c,\tau) = K_1(\tau), \qquad \varphi_2(c,\tau) = K_1'(\tau),$$
$$\varphi_3(c,\tau) = L_1(\tau), \qquad \varphi_4(c,\tau) = L_1'(\tau).$$

Summarizing, in the notation of Section 2, we obtain $a_1 = -1,\ a_2 = 1$ and

$$\begin{aligned} \theta(\alpha,\tau) &= \big(-\varphi_2'(\xi(\alpha),\tau), \varphi_2(\xi(\alpha),\tau), \varphi_4'(\xi(\alpha),\tau), -\varphi_4(\xi(\alpha),\tau)\big)^*, \\ b_1 &= 2a(0, f(-1))^*, \qquad b_2 = 2a(0, f(1))^*, \\ g(\tau) &= a^2(0, Q(\tau)+P(\tau), 0, Q(\tau)-P(\tau))^*, \\ A^{*-1}(x_1,x_2,x_3,x_4)^* &= (x_1,x_2)^* \qquad \text{for} \quad (x_1,x_2,x_3,x_4)^* \in \operatorname{im} A^*. \end{aligned} \tag{4.15}$$

Then by (2.25), we derive

$$\psi_1(\alpha) = \begin{pmatrix} \varphi_2'(\xi(\alpha),-1) \\ -\varphi_2(\xi(\alpha),-1) \end{pmatrix}, \qquad \psi_2(\alpha) = \begin{pmatrix} -\varphi_2'(\xi(\alpha),1) \\ \varphi_2(\xi(\alpha),1) \end{pmatrix}.$$

Thus formula (2.22) possesses the form

$$M(\alpha) = \int_{-1}^{1} \big(\varphi_2(\xi(\alpha),s)(Q(s)+P(s)) - \varphi_4(\xi(\alpha),s)(Q(s)-P(s))\big)\, ds +$$
$$+\, 2a\varphi_2(\xi(\alpha),-1)f(-1) - 2a\varphi_2(\xi(\alpha),1)f(1). \tag{4.16}$$

***Example* 4.1.** Let us consider the equation

$$\ddot{x}(t) + b^2x^3(t) = \varepsilon\bigg(\sin^2\frac{\pi t}{2} + (-1)^{\lfloor\frac{t+1}{2}\rfloor} + \sum_{k\in\mathbb{Z}} \delta(t-4k-1)\bigg). \tag{4.17}$$

Then F is given by (3.13) and sketched in Fig. 1; a, Q, P and f are given by (3.14). Since $\varphi_i(c,\tau)=\dot\varphi_{i-1}(c,\tau)$ for $i=2,4$ and any $c\in\mathbb{R}^4$, $t\in\mathbb{R}$, we obtain

$$M(\alpha)=\int\limits_{-1}^{1}\dot\varphi_1(\xi(\alpha),s)\left(\sin^2\frac{\pi s}{2}+1\right)-\dot\varphi_3(\xi(\alpha),s)\left(\sin^2\frac{\pi s}{2}-1\right)ds-\varphi_2(\xi(\alpha),1)=$$

$$=2\Big(\varphi_1(\xi(\alpha),1)-\varphi_1(\xi(\alpha),-1)\Big)-$$

$$-\frac{\pi}{2}\int\limits_{-1}^{1}\big(\varphi_1(\xi(\alpha),s)-\varphi_3(\xi(\alpha),s)\big)\sin\pi s\,ds-\varphi_2(\xi(\alpha),1). \tag{4.18}$$

Using

$$\varphi_3(\xi(\alpha),t)=\frac{mK\left(\frac{1}{\sqrt2}\right)}{b}\,\mathrm{cn}\left((bt-2b-\alpha)\frac{mK\left(\frac{1}{\sqrt2}\right)}{b},\frac{1}{\sqrt2}\right)=$$

$$=(-1)^m\frac{mK\left(\frac{1}{\sqrt2}\right)}{b}\,\mathrm{cn}\left((bt-\alpha)\frac{mK\left(\frac{1}{\sqrt2}\right)}{b},\frac{1}{\sqrt2}\right)$$

we can write M as

$$M(\alpha)=2\big(\varphi_1(\xi(\alpha),1)-\varphi_1(\xi(\alpha),-1)\big)-$$

$$-\frac{m\pi K\left(\frac{1}{\sqrt2}\right)}{2b}\big(I_+-(-1)^mI_-\big)-\varphi_2(\xi(\alpha),1)$$

for

$$I_\pm:=\int\limits_{-1}^{1}\mathrm{cn}\left((bs\pm\alpha)\frac{mK\left(\frac{1}{\sqrt2}\right)}{b},\frac{1}{\sqrt2}\right)\sin\pi s\,ds=$$

$$=\int\limits_{-1\pm\frac{\alpha}{b}}^{1\pm\frac{\alpha}{b}}\mathrm{cn}\left(zmK\left(\frac{1}{\sqrt2}\right),\frac{1}{\sqrt2}\right)\sin\left(\pi\left(z\mp\frac{\alpha}{b}\right)\right)dz.$$

Now, if $m\in\mathbb{Z}\backslash\{0\}$ is even, then cn in $I_\pm$ is integrated over an integer multiple of its period, i.e.,

$$I_\pm=\int\limits_{-1}^{1}\mathrm{cn}\left(zmK\left(\frac{1}{\sqrt2}\right),\frac{1}{\sqrt2}\right)\sin\left(\pi\left(z\mp\frac{\alpha}{b}\right)\right)dz=$$

$$= \mp \sin \frac{\pi\alpha}{b} \int\limits_{-1}^{1} \operatorname{cn}\left(zmK\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) \cos(\pi z) dz.$$

To evaluate this integral, we make use of Fourier series expansion of cn function from [3] (8.146),

$$\operatorname{cn}(zmK) = \frac{2\pi}{kK} \sum_{j=1}^{\infty} \frac{e^{-\pi(j-1/2)}}{1+e^{-\pi(2j-1)}} \cos\left(\frac{(2j-1)\pi m z}{2}\right).$$

Multiplying this identity by $\cos(\pi z)$ and integrating over $(-1,1)$, we get

$$I_{\pm} = \mp \sin\left(\frac{\pi\alpha}{b}\right) \frac{2\sqrt{2}\pi}{K\left(\frac{1}{\sqrt{2}}\right)} \sum_{j=1}^{\infty} \frac{e^{-\pi(j-1/2)}}{1+e^{-\pi(2j-1)}} \delta_{1,\frac{(2j-1)|m|}{2}} =$$

$$= \mp \sin\left(\frac{\pi\alpha}{b}\right) \frac{\sqrt{2}\pi}{K\left(\frac{1}{\sqrt{2}}\right) \cosh\left(\frac{\pi}{2}\right)} \delta_{2,|m|},$$

where $\delta_{i,j}$ is the Kronecker symbol. Furthermore,

$$\varphi_1(\xi(\alpha),1) - \varphi_1(\xi(\alpha),-1) = \frac{mK\left(\frac{1}{\sqrt{2}}\right)}{b} \left(\operatorname{cn}\left(\left(1+\frac{\alpha}{b}\right)mK\left(\frac{1}{\sqrt{2}}\right)\right) -\right.$$

$$\left. - \operatorname{cn}\left(\left(-1+\frac{\alpha}{b}\right)mK\left(\frac{1}{\sqrt{2}}\right)\right)\right) =$$

$$= \frac{mK\left(\frac{1}{\sqrt{2}}\right)}{b} (1-(-1)^m) \operatorname{cn}\left(\left(1+\frac{\alpha}{b}\right)mK\left(\frac{1}{\sqrt{2}}\right)\right) = 0.$$

Therefore,

$$M(\alpha) = \frac{\sqrt{2}m\pi^2 \sin\left(\frac{\pi\alpha}{b}\right)}{b\cosh\left(\frac{\pi}{2}\right)} \delta_{2,|m|} - \varphi_2(\xi(\alpha),1). \tag{4.19}$$

Next, since $\frac{d}{d\alpha}\varphi_i(\xi(\alpha),t) = \frac{1}{b}\dot{\varphi}_i(\xi(\alpha),t)$ for $i=1,2$ and any $\alpha, t \in \mathbb{R}$, we get

$$M'(\alpha) = \frac{\sqrt{2}m\pi^3 \cos\left(\frac{\pi\alpha}{b}\right)}{b^2\cosh\left(\frac{\pi}{2}\right)} \delta_{2,|m|} - \frac{1}{b}\dot{\varphi}_2(\xi(\alpha),1).$$

Note that the periodicity of Jacobi elliptic functions yields that M of (4.19) is $\frac{4b}{|m|}$-periodic. This means that $\alpha + \frac{4jb}{|m|}$ is a root of M whenever $j \in \mathbb{Z}$ and α is a root of M.

Now, we look for roots of M given by (4.19). If $m \neq \pm 2$, then

$$M(\alpha) = \frac{m^2 K^2\left(\frac{1}{\sqrt{2}}\right)}{b} \operatorname{sn}\left(\left(1 + \frac{\alpha}{b}\right) m K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) \times$$
$$\times \operatorname{dn}\left(\left(1 + \frac{\alpha}{b}\right) m K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right).$$

Since

$$\operatorname{dn}^2\left(t, \frac{1}{\sqrt{2}}\right) = 1 - \frac{1}{2} \operatorname{sn}^2\left(t, \frac{1}{\sqrt{2}}\right) \geq \frac{1}{2}$$

for all $t \in \mathbb{R}$, the roots of M are precisely the roots of $\operatorname{sn}\left(\left(1 + \frac{\alpha}{b}\right) m K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)$. Applying the definition of sn saying that $\operatorname{sn} u = \sin \phi$ where u is given by (4.9), one can see that $\operatorname{sn} u = 0$ if and only if $u = 2jK$ for $j \in \mathbb{Z}$. Therefore, $M\left(\alpha^0_{j,m}\right) = 0$ for

$$\alpha^0_{j,m} = \left(\frac{2j}{m} - 1\right) b, \quad j \in \mathbb{Z}, \tag{4.20}$$

and $m \in \mathbb{Z} \backslash \{0, \pm 2\}$ even. Note that all $\alpha^0_{j,m}$ with j even are just $\alpha^0_{0,m}$ shifted by an integer multiple of period $\frac{4b}{|m|}$. Analogously, $\alpha^0_{j,m}$ with j odd are shifted $\alpha^0_{1,m}$. Using $\operatorname{cn}^2 u + \operatorname{sn}^2 u = 1$ and that $\operatorname{cn}(t, k)$ solves the equation

$$\ddot{x}(t) = (2k^2 - 1) x(t) - 2k^2 x^3(t), \quad t \in \mathbb{R}, \tag{4.21}$$

we get at these points

$$M'(\alpha^0_{j,m}) = \frac{m^3 K^3\left(\frac{1}{\sqrt{2}}\right)}{b^2} \operatorname{cn}\left(\left(1 + \frac{\alpha}{b}\right) m K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) \neq 0.$$

Proposition 4.1. *For each $m \in \mathbb{Z} \backslash \{0, \pm 2\}$ even and $j = 0, 1$, there exists $\delta > 0$ such that* (4.17) *has a unique solution $x_{j,m}$ given by*

$$x_{j,m}(\varepsilon, t) = \begin{cases} \dfrac{mK\left(\frac{1}{\sqrt{2}}\right)}{b} \operatorname{cn}\left((b\tau(t) + \alpha^0_{j,m}) \dfrac{mK\left(\frac{1}{\sqrt{2}}\right)}{b}, \dfrac{1}{\sqrt{2}}\right) + O(\varepsilon), \\ \qquad t \in \bigcup_{k \in \mathbb{Z}} (4k - 1, 4k + 1), \\ \dfrac{mK\left(\frac{1}{\sqrt{2}}\right)}{b} \operatorname{cn}\left((b\tau(t) - 2b - \alpha^0_{j,m}) \dfrac{mK\left(\frac{1}{\sqrt{2}}\right)}{b}, \dfrac{1}{\sqrt{2}}\right) + O(\varepsilon), \\ \qquad t \in \bigcup_{k \in \mathbb{Z}} (4k + 1, 4k + 3), \end{cases} \tag{4.22}$$

for any $\varepsilon \in (-\delta, \delta)$, where $\alpha^0_{j,m}$ is given by (4.20).

Proof. By Theorem 2.1 we obtain a solution

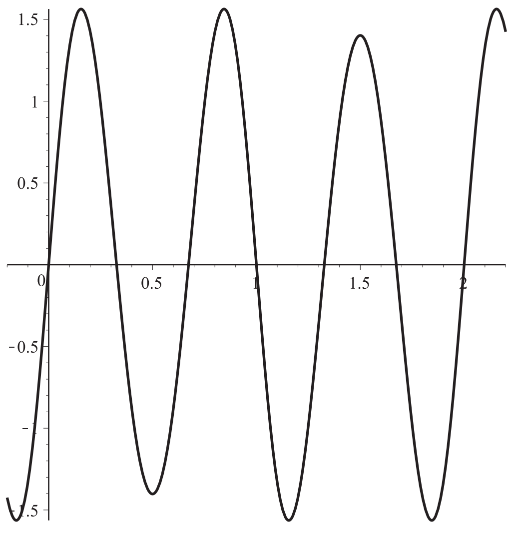
$$\varphi\left(\frac{mK\left(\frac{1}{\sqrt{2}}\right)}{b},\ \alpha_{j,m}(\varepsilon),\ \frac{mK\left(\frac{1}{\sqrt{2}}\right)}{b},\ -2b-\alpha_{j,m}(\varepsilon),\tau\right),\quad \alpha_{j,m}(0)=\alpha_{j,m}^0,$$

of the corresponding form of (4.3) for φ given by (4.5). Then the statement is proved as Corollary 3.1 along with Remark 2.1.

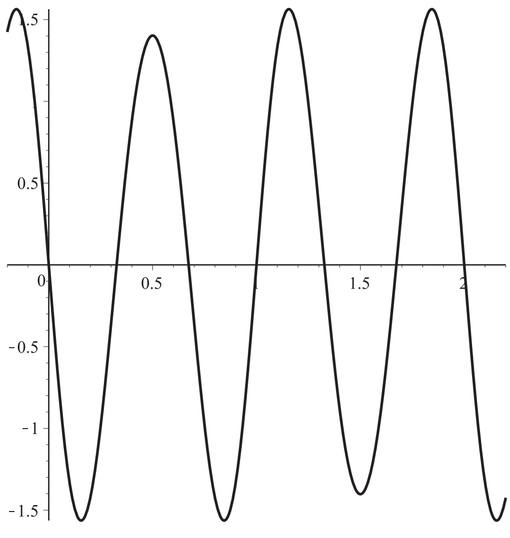
Finally, if $|m|=2$, it is not possible to find analytically the roots of

$$M(\alpha)=\frac{\sqrt{2}m\pi^2\sin\left(\frac{\pi\alpha}{b}\right)}{b\cosh\left(\frac{\pi}{2}\right)}+\frac{m^2K^2\left(\frac{1}{\sqrt{2}}\right)}{b}\times$$
$$\times\operatorname{sn}\left(\left(1+\frac{\alpha}{b}\right)mK\left(\frac{1}{\sqrt{2}}\right),\frac{1}{\sqrt{2}}\right)\operatorname{dn}\left(\left(1+\frac{\alpha}{b}\right)mK\left(\frac{1}{\sqrt{2}}\right),\frac{1}{\sqrt{2}}\right).\qquad(4.23)$$

However, we can determine the number of its simple roots. Clearly, α is a simple root of M if and only if it is a simple root of the function $\widetilde{M}(\alpha):=bM(\alpha b)$ which is 2-periodic and independent of b. The graphs of $\widetilde{M}$ for $m=\pm2$ are given in Fig. 2. One can see that there are six simple roots $\alpha_{j,m}^0$, $j=0,1,\ldots,5$, of M in $[0,2b)$.



(*a*)



(*b*)

Fig. 2. Graphs of $\widetilde{M}$ for $m=-2$ (*a*) and $m=2$ (*b*), respectively.

Proposition 4.2. *For each* $m=\pm2$ *and* $j=0,1,\ldots,5$, *there exists* $\delta>0$ *such that* (4.17) *has a unique solution* $x_{j,m}$ *given by* (4.22) *for any* $\varepsilon\in(-\delta,\delta)$ *with* $\alpha_{j,m}^0$ *numerically computed roots of* M *given by* (4.23).

Remark **4.2.** 1. The unperturbed solutions of (4.17) for $m=2,4$ are sketched in Fig. 3.

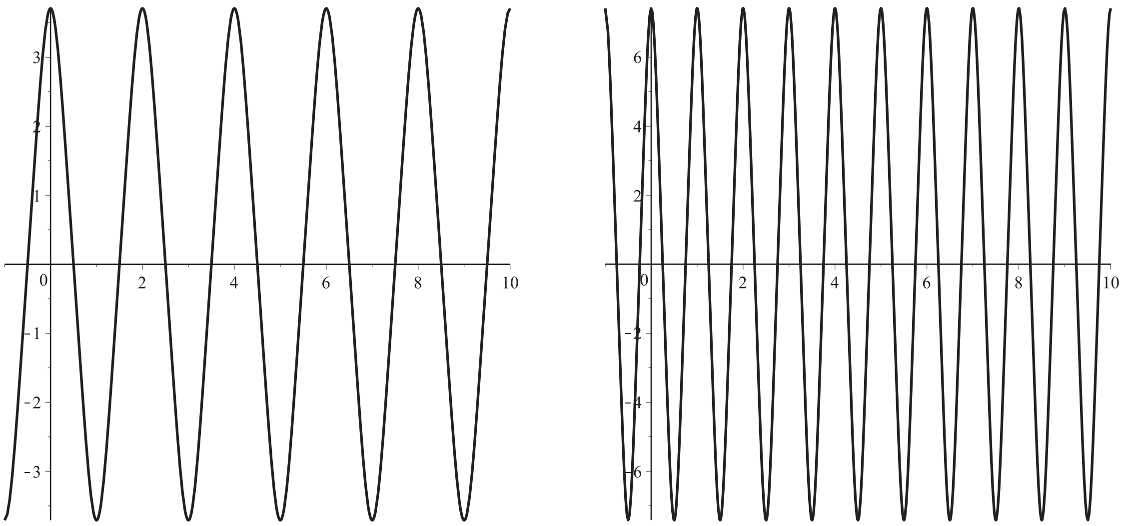


Fig. 3. Sketch of solutions $x_{0,m}(0,t)$ of (4.17) for $m = 2, 4$.

2. If $m \neq \pm 2$ is even, then we analytically showed that 0 is a simple root of M. Next, from (4.11), if $m = \pm 2$ then $\operatorname{sn}(mK) = (-1)^{m/2}\operatorname{sn}(0) = 0$. Hence by (4.23), 0 is a root of M. Moreover,

$$M'(0) = \frac{m}{b^2}\left(\frac{\sqrt{2}\pi^3}{\cosh\left(\dfrac{\pi}{2}\right)} - 4K^3\left(\frac{1}{\sqrt{2}}\right)\right) \doteq -\frac{8.01858m}{b^2} \neq 0,$$

where we used $\operatorname{cn}(mK) = (-1)^{m/2}\operatorname{cn}(0) = (-1)^{m/2}$ and (4.21). So, 0 is a simple root of M whenever m is even.

On the other hand, if $m \in \mathbb{Z}$ is odd, then directly from (4.18),

$$M(0) = -\varphi_2(\xi(0), 1) =$$

$$= \frac{m^2K^2\left(\dfrac{1}{\sqrt{2}}\right)}{b}\operatorname{sn}\left(mK\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)\operatorname{dn}\left(mK\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) =$$

$$= (-1)^{(m-1)/2}\frac{m^2K^2\left(\dfrac{1}{\sqrt{2}}\right)}{\sqrt{2}b} \neq 0,$$

where we used

$$\operatorname{sn}(mK) = \operatorname{sn}(K + (m-1)K) = (-1)^{(m-1)/2}\operatorname{sn}(K) = (-1)^{(m-1)/2},$$

$$\operatorname{dn}(K) = \sqrt{1-k^2} = \frac{1}{\sqrt{2}}$$

(see [9]). So there is no solution persisting from $\varphi(\xi(0), t)$ if $m \in \mathbb{Z}$ is odd.

3. Integrating (4.18) over its period we get

$$\int_0^{\frac{4b}{|m|}} M(\alpha)d\alpha = \int_{-1}^{1} b\left(\left(\sin^2\frac{\pi s}{2}+1\right)\int_0^{4b/|m|}\frac{d}{d\alpha}\varphi_1(\xi(\alpha),s)\,d\alpha -\right.$$

$$\left.-\left(\sin^2\frac{\pi s}{2}-1\right)\int_0^{4b/|m|}\frac{d}{d\alpha}\varphi_3(\xi(\alpha),s)d\alpha\right)ds-$$

$$-b\int_0^{4b/|m|}\frac{d}{d\alpha}\varphi_1(\xi(\alpha),1)\,d\alpha = 0.$$

Since M is not identically zero, it is changing its sign over the period interval, and Remark 2.2 can be applied. This justifies the above analytical results on the existence of at least one root of M, and also proves the existence of a solution of (4.17) for ε close to 0.

References

1. *Alekseev V. M.* An estimate for the perturbations of the solutions of ordinary differential equations // Vestnik Moskov. Univ. – 1961. – № 2 (in Russian).
2. *Farkas M.* Periodic motions. – New York: Springer-Verlag, 1994.
3. *Gradshteyn I. S., Ryzhik I. M.* Table of integrals, series, and products. – 7th ed. – Acad. Press, 2007.
4. *Hartman P.* Ordinary differential equations. – New York: John Wiley & Sons, Inc., 1964.
5. *Lawden D. F.* Elliptic functions and applications. – New York: Springer, 1989.
6. *Pilipchuk V. N.* Transformation of oscillating systems by means of a pair of nonsmooth periodic functions // Dokl. Akad. Nauk Ukr. SSR. Ser. A. – 1988. – P. 37 – 40.
7. *Pilipchuk V. N.* Nonlinear dynamics: between linear and impact limits. – Berlin: Springer-Verlag, 2010.
8. *Samoilenko A. M., Boichuk A. A., Zhuravlev V. F.* Weakly nonlinear boundary value problems for operator equations with impulse action // Ukr. Math. J. – 1997. – **49**, № 2. – P. 272 – 288.
9. *Whittaker E. T., Watson G. N.* A course of modern analysis. – 4th ed. – Cambridge: Cambridge Univ. Press, 1927.

Received 31.07.17