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R. R. Zhang (Guangdong Univ. Education, China)

VALUE DISTRIBUTION OF DIFFERENTIAL-DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS*

РОЗПОДІЛ ЗНАЧЕНЬ ДЛЯ ДИФЕРЕНЦІАЛЬНО-РІЗНИЦЕВИХ ПОЛІНОМІВ МЕРОМОРФНИХ ФУНКЦІЙ

We obtain the results on the deficiencies of differential-difference polynomials. These results can be regarded as differential-difference analogs of some classical theorems on differential polynomials. In particular, an exact estimate of the deficiency of some differential-difference polynomials is presented. We also give examples showing that these results are best possible in a certain sense.

Отримано результати щодо дефектів диференціально-різницевих поліномів. Ці результати можна розглядати як диференціально-різницеві аналоги деяких класичних теорем для диференціальних поліномів. Зокрема, наведено точну оцінку для дефектів деяких диференціально-різницевих поліномів. Також наведено приклади, які показують, що ці результати є, в певному сенсі, найкращими.

1. Introduction and results. Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the basic notions of Nevanlinna's theory (see [7]). We use $\sigma(f)$ to denote the order of growth of $f(z)$; $\lambda(f)$ and $\lambda(1/f)$ to denote, respectively, the exponents of convergence of zero and pole sequences of $f(z)$. The hyper order of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

For $a \in \mathbb{C}$, we use $\delta(a,f)$ to denote the Nevanlinna deficiency of a respect to $f(z)$. Moreover, we denote by $S(r,f)$ any real function of growth $o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$, if $T(r,\alpha) = S(r,f)$.

The value distribution of differential polynomials has been discussed extensively and deeply. For example, Doeringer [4] investigated the differential polynomial

$$\Psi = f(z)^n Q(f) + P(f),$$

where $f(z)$ is a transcendental entire function and $Q(f)$, $P(f)$ are differential polynomials in $f(z)$ with small meromorphic coefficients such that $Q(f) \not\equiv 0$, $P(f) \not\equiv 0$. He proved that $\overline{\lim}_{r\to\infty} \dfrac{\overline{N}(r,1/\Psi)}{T(r,\Psi)} > 0$ holds for $n \geq 2+\gamma$, where γ denotes the degree of $P(f)$.

In the past decade, difference polynomials have been discussed extensively and many results have been obtained (see, e.g., [2, 8 – 10, 16, 17]). Lately, authors began to investigate differential-difference polynomials (see, e.g., [12, 14, 18]) and differential-difference equations (see, e.g., [3, 13]). The study of the value distribution of differential-difference polynomials plays an important role in the further study of differential-difference equations.

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A differential-difference polynomial is a polynomial in $f(z)$, its shifts, its derivatives and derivatives of its shifts, that is an expression of the form

$$P(z,f)=\sum_{\lambda\in J}a_{\lambda}(z)\prod_{j=1}^{\tau_{\lambda}}\{f^{(\alpha_{\lambda,j})}(z+\delta_{\lambda,j})\}^{\beta_{\lambda,j}}, \tag{1.1}$$

where J is an index set, $\delta_{\lambda,j}$ are complex constants, $\alpha_{\lambda,j}$ and $\beta_{\lambda,j}$ are nonnegative integers, and the coefficients $a_{\lambda}(z)(\not\equiv 0)$ are small meromorphic functions of $f(z)$. The maximal total degree of $P(z,f)$ is defined by

$$\deg_f P=\max_{\lambda\in J}\sum_{j=1}^{\tau_{\lambda}}\beta_{\lambda,j}.$$

Zheng and Xu [18] investigated differential-difference polynomials

$$Q_1(z,f)=F(f)P(z,f)$$

and

$$Q_2(z,f)=F(f)+P(z,f),$$

where $f(z)$ is a transcendental meromorphic function satisfying $\sigma_2(f)<1$ and $N(r,f)=S(r,f)$, $P(z,f)$ is a differential-difference polynomial of $f(z)$, and $F(f)=(f^v+a_{v-1}(z)f^{v-1}+\ldots$ $\ldots+a_0(z))^u$ is a polynomial of $f(z)$. For a small meromorphic function $\alpha(z)(\not\equiv 0)$, they proved the following results:

(i) If $uv>\deg_f P$ and $u\neq 1$, then $\delta(\alpha,Q_1)\leq 1-\dfrac{(u-1)(uv-\deg_f P)}{u(uv+\deg_f P)}<1$.

(ii) If $\dfrac{(u-1)uv}{2u-1}>\deg_f P$ and $u\neq 1$, then $\delta(\alpha,Q_2)\leq \dfrac{1}{u}+\dfrac{2u-1}{u^2v}\deg_f P<1$.

In this paper, we prove a differential-difference counterpart of the result of [4] and obtain Theorem 1.1 below, which improves the results of [18].

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f)<1$ and $N(r,f)=S(r,f)$. Let $F(z,f)=(a_v(z)f^v+a_{v-1}(z)f^{v-1}+\ldots+a_0(z))^u$, where u,v are positive integers, and $a_j(z)$, $j=0,1,\ldots,v$, are small meromorphic functions of $f(z)$ with $a_v(z)\not\equiv 0$. Suppose that $P_1(z,f)$ and $P_2(z,f)$ $(P_1(z,f)P_2(z,f)\not\equiv 0)$ are differential-difference polynomials with small meromorphic coefficients. If $(u-1)v>\deg_f P_2$, then*

$$\psi(z)=F(z,f)P_1(z,f)+P_2(z,f)$$

satisfies $\delta(0,\psi)\leq\dfrac{\deg_f P_1+v+\deg_f P_2}{uv+\deg_f P_1}<1.$

***Remark* 1.1.** Comparing Theorem 1.1 with the results of [18], we see that the condition of Theorem 1.1 is weaker and the conclusion is stronger. Examples 1 and 2 below show that both $\delta(0,\psi)=\dfrac{\deg_f P_1+v+\deg_f P_2}{uv+\deg_f P_1}$ and $\delta(0,\psi)<\dfrac{\deg_f P_1+v+\deg_f P_2}{uv+\deg_f P_1}$ in Theorem 1.1 may hold, and Example 3 below shows that Theorem 1.1 is false, if $(u-1)v=\deg_f P_2$. So the result of Theorem 1.1 is best possible in this sense.

***Example* 1.1.** Let $f(z) = e^z$, $F(z,f) = (f(z)+1)^3$, $P_1(z,f) \equiv 1$ and $P_2(z,f) = 3f'(z+\pi i) - 1$. Then $u = 3, v = 1, \deg_f P_1 = 0, \deg_f P_2 = 1$ and $\psi(z) = F(z,f)P_1(z,f) + P_2(z,f) = e^{2z}(e^z+3)$ satisfies $\delta(0,\psi) = \dfrac{2}{3} = \dfrac{\deg_f P_1 + v + \deg_f P_2}{uv + \deg_f P_1}$.

***Example* 1.2.** Let $f(z) = e^z$, $F(z,f) = (f(z)^2+1)^2$, $P_1(z,f) = f'(z)$ and $P_2(z,f) = f''(z+\pi i)$. Then $u = 2, v = 2, \deg_f P_1 = 1, \deg_f P_2 = 1$ and $\psi(z) = F(z,f)P_1(z,f) + P_2(z,f) = e^{3z}(e^{2z}+2)$ satisfies $\delta(0,\psi) = \dfrac{3}{5} < \dfrac{\deg_f P_1 + v + \deg_f P_2}{uv + \deg_f P_1} = \dfrac{4}{5}$.

***Example* 1.3.** Take $f(z) = e^z + 1$, $F(z,f) = (f(z)+1)^2$, $P_1(z,f) \equiv 1$ and $P_2(z,f) = 4f'(z+\pi i) - 4$. Then $(u-1)v = (2-1)\times 1 = \deg_f P_2$ and $\psi(z) = F(z,f)P_1(z,f) + P_2(z,f) = e^{2z}$ satisfies $\delta(0,\psi) = 1$.

In the case where $f(z)$ has two Borel exceptional values, we get the following theorem.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) < \infty$. Let $F(z,f) = (a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^u$, where u, v are positive integers, and $a_j(z)$, $j = 0, 1, \ldots, v$, satisfy $\sigma(a_j) < \sigma(f)$ and $a_v(z) \not\equiv 0$. Suppose that $P_1(z,f) (\not\equiv 0)$ and $P_2(z,f)$ are differential-difference polynomials, and the growth orders of their coefficients are less than $\sigma(f)$. Suppose further that a, ∞ are Borel exceptional values of $f(z)$ such that $F(z,a)P_1(z,a) + P_2(z,a) \not\equiv 0$. If $uv > \deg_f P_2$ or $P_2(z,f) \equiv 0$, then*

$$\psi(z) = F(z,f)P_1(z,f) + P_2(z,f)$$

satisfies $\delta(0,\psi) \le \dfrac{\deg_f P_1}{uv + \deg_f P_1} < 1$.

***Remark* 1.2.** (1) Theorem 1.2 dose not remain valid, if we replace the Borel exceptional values "a, ∞" with two finite Borel exceptional values. Indeed, take $f(z) = \tan z$, $F(z,f) = f(z)^3$, $P_1(z,f) = zf\left(z+\dfrac{\pi}{2}\right)^3$ and $P_2(z,f) = -2f(z+\pi)f\left(z+\dfrac{\pi}{2}\right) + z + \dfrac{1}{z} - 2$. We have $\psi(z) = F(z,f)P_1(z,f) + P_2(z,f) = \dfrac{1}{z}$. Obviously, $uv = 3\times 1 > \deg_f P_2 = 2$, i and $-i$ are two Borel exceptional values of $f(z)$, and $F(z,i)P_1(z,i) + P_2(z,i) = \dfrac{1}{z} \not\equiv 0$, $F(z,-i)P_1(z,-i) + P_2(z,-i) = \dfrac{1}{z} \not\equiv 0$. But $\psi(z) = \dfrac{1}{z}$ satisfies $\delta(0,\psi) = 1$.

(2) Theorem 1.2 is false, if $uv = \deg_f P_2$. Indeed, take $f(z) = e^{z^2}$, $F(z,f) = f(z)^2$, $P_1(z,f) \equiv 1$ and $P_2(z,f) = -\dfrac{e^{-2z-1}}{2z} f(z+1)f'(z) + e^z$. We have $\psi(z) = F(z,f)P_1(z,f) + P_2(z,f) = e^z$. Obviously, $uv = 2\times 1 = \deg_f P_2$, and the Borel exceptional value 0 satisfies $F(z,0)P_1(z,0) + P_2(z,0) = e^z \not\equiv 0$. But $\psi(z) = e^z$ satisfies $\delta(0,\psi) = 1$.

At last, we give an exact estimate of the deficiency $\delta(0,P)$ of differential-difference polynomial (1.1). In order to collect together its all monomials having the same degree, we introduce the notation

$$J_l = \left\{ \lambda \in J \,\middle|\, \sum_{j=1}^{\tau_\lambda} \beta_{\lambda,j} = l \right\}, \tag{1.2}$$

where $l = 0, 1, \ldots, \deg_f P$.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and*

$$N(r, f) + N(r, 1/f) = S(r, f). \tag{1.3}$$

Suppose that $P(z, f)$ is a differential-difference polynomial of the form (1.1). *Denote*

$$\max\left\{ l \Big| \sum_{\lambda\in J_l} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} \{f^{(\alpha_{\lambda,j})}(z+\delta_{\lambda,j})\}^{\beta_{\lambda,j}} \not\equiv 0, l \in \{0, 1, \ldots, \deg_f P\} \right\} = m, \tag{1.4}$$

$$\min\left\{ l \Big| \sum_{\lambda\in J_l} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} \{f^{(\alpha_{\lambda,j})}(z+\delta_{\lambda,j})\}^{\beta_{\lambda,j}} \not\equiv 0, l \in \{0, 1, \ldots, \deg_f P\} \right\} = k, \tag{1.5}$$

where J_l is defined by (1.2). *If $m > k$ or $m = k \geq 1$, then $\delta(0, P) = \dfrac{k}{m}$.*

The following example illustrates Theorem 1.3.

***Example* 1.4.** Let $f(z) = e^z$ and $P(z, f) = f(z)^4 + f'(z)f(z+\pi i)^2 + f'(z)^2$, then $m = 4, k = 2$ satisfy (1.4) and (1.5) respectively. We have $P(z, f) = e^{2z}(e^{2z} + e^z + 1)$ and $\delta(0, P) = \dfrac{2}{4} = \dfrac{1}{2}$.

By Theorem 1.3, we can easily get the following two corollaries.

***Corollary* 1.1.** *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and* (1.3). *Suppose that $P(z, f)$ is a differential-difference polynomial of the form* (1.1) *with $\deg_f P \geq 1$. If P(z, f) contains just one term of maximal total degree and $P(z, 0) \not\equiv 0$, then $\delta(0, P) = 0$.*

***Corollary* 1.2.** *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and* (1.3). *Suppose that $P(z, f)$ is a homogeneous differential-difference polynomial of the form* (1.1) *with $\deg_f P \geq 1$ and $P(z, f) \not\equiv 0$. Then, for any small meromorphic function $\alpha(z) \not\equiv 0$ of $f(z)$, we have $\delta(\alpha, P) = 0$.*

2. Proof of Theorem 1.1. We need the following lemmas.

Lemma 2.1 [15]. *Let $f(z)$ be a transcendental meromorphic function. Let $P(f)$ be a polynomial in $f(z)$ of the form*

$$P(f) = a_n(z)f(z)^n + a_{n-1}(z)f(z)^{n-1} + \ldots + a_1(z)f(z) + a_0(z),$$

where all coefficients $a_j(z)$ are small functions of $f(z)$ and $a_n(z) \not\equiv 0$. Then

$$T\big(r, P(f)\big) = nT(r, f) + S(r, f).$$

Lemma 2.2 [6]. *Let $T : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function, and let $s \in (0, \infty)$. If the hyper order of T is strictly less than one and $\delta \in (0, 1 - \sigma_2(T))$, then*

$$T(r + s) = T(r) + o(T(r)/r^\delta),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Let $f(z)$ be a meromorphic function. It is shown in [1] (Lemma 1) and [5, p. 66], that for an arbitrary $c \neq 0$, the inequalities

$$\big(1 + o(1)\big)T\big(r - |c|, f(z)\big) \leq T\big(r, f(z + c)\big) \leq \big(1 + o(1)\big)T\big(r + |c|, f(z)\big)$$

hold as $r \to \infty$. From its proof we see that the above relations are also true for counting functions. So by these relations and Lemma 2.2, we get the following lemma.

Lemma 2.3. *Let $f(z)$ be a nonconstant meromorphic function of $\sigma_2(f) < 1$, and let $c \neq 0$ be an arbitrary complex number. Then*

$$T\big(r, f(z+c)\big) = T(r, f) + S(r, f),$$

$$N\big(r, f(z+c)\big) = N(r, f) + S(r, f),$$

$$N\big(r, 1/f(z+c)\big) = N(r, 1/f) + S(r, f).$$

Applying [6] (Theorem 5.1) to [9] (Theorem 2.3), we get the following lemma.

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic solution of hyper order $\sigma_2(f) < 1$ of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f),\ P(z, f),\ Q(z, f)$ are difference polynomials in $f(z)$ with small meromorphic coefficients, $\deg_f U = n$ and $\deg_f Q \leq n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree. Then

$$m\big(r, P(z, f)\big) = S(r, f).$$

Remark **2.1.** By a careful inspection of the proof of Lemma 2.4, we see that the same conclusion holds, if $P(z, f),\ Q(z, f)$ are differential-difference polynomials in $f(z)$ and the coefficients $a_\lambda(z)$ of $P(z, f)$ and $Q(z, f)$ satisfy $m(r, a_\lambda) = S(r, f)$ instead of $T(r, a_\lambda) = S(r, f)$.

Lemma 2.5 [18]. *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_2(f) < 1$ and $N(r, f) = S(r, f)$. Let $P(z, f)$ be a differential-difference polynomial. Then*

$$T(r, P) \leq (\deg_f P)T(r, f) + S(r, f).$$

Proof of Theorem **1.1.** To prove Theorem 1.1, we follow the main idea in the proof of [17] (Theorem 1.2). First observe that $\psi(z) \not\equiv 0$. Indeed, if $\psi(z) \equiv 0$, then

$$F(z, f)P_1(z, f) \equiv -P_2(z, f). \tag{2.1}$$

Since $\deg_f P_2 < uv = \deg_f F$, it follows from Lemma 2.4 and Remark 2.1 that

$$m(r, P_1) = S(r, f).$$

Moreover, Lemma 2.3 and the assumption that $N(r, f) = S(r, f)$ give $N(r, P_1) = S(r, f)$. So $T(r, P_1) = S(r, f)$. Therefore, we have from Lemma 2.1 that

$$T(r, FP_1) = uvT(r, f) + S(r, f).$$

On the other hand, we get from Lemma 2.5 that

$$T(r, P_2) \leq (\deg_f P_2)T(r, f) + S(r, f). \tag{2.2}$$

Since $\deg_f P_2 < uv$, comparing the characteristic functions of both sides of (2.1), we obtain a contradiction. So, $\psi(z) \not\equiv 0$.

Differentiating both sides of

$$\psi(z) = F(z, f)P_1(z, f) + P_2(z, f), \tag{2.3}$$

we obtain

$$\psi'(z) = F'(z,f)P_1(z,f) + F(z,f)P_1'(z,f) + P_2'(z,f). \tag{2.4}$$

Since $\psi(z) \not\equiv 0$, multiplying both sides of (2.3) by $\dfrac{\psi'(z)}{\psi(z)}$, we have

$$\psi'(z) = \frac{\psi'(z)}{\psi(z)} F(z,f)P_1(z,f) + \frac{\psi'(z)}{\psi(z)} P_2(z,f). \tag{2.5}$$

Subtracting (2.5) from (2.4), we get

$$F'(z,f)P_1(z,f) + F(z,f)P_1'(z,f) - \frac{\psi'(z)}{\psi(z)} F(z,f)P_1(z,f) = P_2(z,f)\left(-\frac{P_2'(z,f)}{P_2(z,f)} + \frac{\psi'(z)}{\psi(z)}\right). \tag{2.6}$$

Substituting $F(z,f) = (a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^u$ and $F'(z,f) = u(a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^{u-1}(a_v'(z)f^v + va_v(z)f^{v-1}f' + \ldots + a_1'(z)f + a_1(z)f' + a_0'(z))$ into (2.6), we obtain

$$(a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^{u-1}\omega(z) = P_2(z,f)\left(-\frac{P_2'(z,f)}{P_2(z,f)} + \frac{\psi'(z)}{\psi(z)}\right), \tag{2.7}$$

where

$$\omega(z) = uP_1(z,f)(a_v'(z)f^v + va_v(z)f^{v-1}f' + \ldots + a_1'(z)f + a_1(z)f' + a_0'(z)) +$$
$$+(a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))\left(P_1'(z,f) - \frac{\psi'(z)}{\psi(z)} P_1(z,f)\right). \tag{2.8}$$

We affirm that $\omega(z) \not\equiv 0$. Otherwise, since $P_2(z,f) \not\equiv 0$, it follows from (2.7) that

$$\frac{\psi'(z)}{\psi(z)} = \frac{P_2'(z,f)}{P_2(z,f)}.$$

Integrating this equation, we have

$$\psi(z) = C_1 P_2(z,f),$$

where C_1 is a nonzero constant. Substituting $\psi(z) = C_1 P_2(z,f)$ into (2.3), we get

$$F(z,f)P_1(z,f) = (C_1 - 1)P_2(z,f). \tag{2.9}$$

From (2.9) and $F(z,f)P_1(z,f) \not\equiv 0$, we obtain $C_1 \neq 1$. Using (2.9) and following steps analogous to (2.1), (2.2), we have a contradiction. Thus, $\omega(z) \not\equiv 0$.

By (2.7) we get

$$m(r, (a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^{u-1}) \leq$$
$$\leq m\left(r, \frac{1}{\omega}\right) + m(r, P_2) + m\left(r, \frac{P_2'}{P_2}\right) + m\left(r, \frac{\psi'}{\psi}\right) + O(1). \tag{2.10}$$

Next we estimate every term in (2.10). Since $\deg_f \psi = uv + \deg_f P_1$, by Lemma 2.5, we obtain

$$T(r,\psi) \leq (uv + \deg_f P_1)T(r,f) + S(r,f). \tag{2.11}$$

By (2.2) and (2.11), we see that $S(r,\psi) = S(r,f)$ and $S(r,P_2) = S(r,f)$. So from $m\left(r,\dfrac{\psi'}{\psi}\right) = S(r,\psi)$ and $m\left(r,\dfrac{P_2'}{P_2}\right) = S(r,P_2)$, we get

$$m\left(r,\frac{\psi'}{\psi}\right) = S(r,f), \quad m\left(r,\frac{P_2'}{P_2}\right) = S(r,f). \tag{2.12}$$

By $(u-1)v > \deg_f P_2$ and (2.12), we see that Lemma 2.4 and Remark 2.1 can be applied to equation (2.7). So we have

$$m(r,\omega) = S(r,f).$$

By (2.8) and $N(r,f) = S(r,f)$, we get

$$N(r,\omega) \leq \overline{N}(r,1/\psi) + S(r,f).$$

Thus,

$$T(r,\omega) \leq \overline{N}(r,1/\psi) + S(r,f).$$

From this inequality and the first main theorem, we obtain

$$m(r,1/\omega) \leq T(r,\omega) + O(1) \leq \overline{N}(r,1/\psi) + S(r,f). \tag{2.13}$$

By Lemma 2.1, we have

$$T(r,(a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^{u-1}) = (u-1)vT(r,f) + S(r,f).$$

Noting that $N(r,f) = S(r,f)$, so

$$m(r,(a_v(z)f^v + a_{v-1}(z)f^{v-1} + \ldots + a_0(z))^{u-1}) = (u-1)vT(r,f) + S(r,f). \tag{2.14}$$

By (2.2), (2.10), (2.12)–(2.14), we get

$$(u-1)vT(r,f) \leq \overline{N}(r,1/\psi) + (\deg_f P_2)T(r,f) + S(r,f).$$

Thus,

$$((u-1)v - \deg_f P_2)T(r,f) \leq \overline{N}(r,1/\psi) + S(r,f).$$

Combining this inequality with (2.11) and noting that $(u-1)v > \deg_f P_2$, we have

$$\delta(0,\psi) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r,1/\psi)}{T(r,\psi)} \leq 1 - \frac{(u-1)v - \deg_f P_2}{uv + \deg_f P_1} = \frac{\deg_f P_1 + v + \deg_f P_2}{uv + \deg_f P_1} < 1.$$

3. Proof of Theorem 1.2. We need the following lemma.

Lemma 3.1 [11]. *Suppose that h is a nonconstant meromorphic function satisfying*

$$\overline{N}(r,h) + \overline{N}(r,1/h) = S(r,h).$$

Let $f = a_0h^p + a_1h^{p-1} + \ldots + a_p$, and $g = b_0h^q + b_1h^{q-1} + \ldots + b_q$ be polynomials in h with coefficients $a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_q$ being small functions of h and $a_0b_0a_p \not\equiv 0$. If $q \leq p$, then $m(r,g/f) = S(r,h)$.

***Proof of Theorem* 1.2.** We only discuss the case $uv > \deg_f P_2$ as the case $P_2(z, f) \equiv 0$ can be treated similarly. We first fix some notations for the proof as follows:

$$P_1(z,f) = \sum_{\lambda \in J} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} \{f^{(\alpha_{\lambda,j})}(z+\delta_{\lambda,j})\}^{\beta_{\lambda,j}} + A(z),$$

$$P_2(z,f) = \sum_{\mu \in I} b_\mu(z) \prod_{j=1}^{\sigma_\mu} \{f^{(m_{\mu,j})}(z+\eta_{\mu,j})\}^{n_{\mu,j}} + B(z),$$

where J and I are index sets, $\delta_{\lambda,j}$ and $\eta_{\mu,j}$ are complex constants, $\alpha_{\lambda,j}$ and $m_{\mu,j}$ are nonnegative integers, $\beta_{\lambda,j}$ and $n_{\mu,j}$ are positive integers, and the growth orders of $a_\lambda(z)$, $b_\mu(z)$, $A(z)$ and $B(z)$ are all less than $\sigma(f)$. Set $d(P_1) = \deg_f P_1$ and $d(P_2) = \deg_f P_2$.

Since a and ∞ are Borel exceptional values of $f(z)$, by Hadamard's factorization theorem, we get

$$f(z) = H(z)e^{h(z)} + a, \tag{3.1}$$

where $h(z)$ is a polynomial and $H(z)$ is a meromorphic function such that $\sigma(H) < \sigma(f)$. So $\sigma(f) = \deg h(z)$ and $f(z)$ is of regular growth. Therefore, $a_\lambda(z)(\lambda \in J)$, $b_\mu(z)(\mu \in I)$, $a_j(z)$, $j = 0, \ldots, v$, $A(z)$, $B(z)$, $H(z)$, $H(z+c)$ and $e^{h(z+c)-h(z)}$ are all small functions of $e^{h(z)}$ and $N(r, f) = S(r, e^h)$.

Substituting (3.1) into $F(z, f)$, $P_1(z, f)$ and $P_2(z, f)$, we obtain

$$F(z,f) = a_v(z)^u H(z)^{uv} e^{uvh(z)} + s_{uv-1}(z)e^{(uv-1)h(z)} + \ldots + s_1(z)e^{h(z)} +$$
$$+ (a_v(z)a^v + a_{v-1}(z)a^{v-1} + \ldots + a_0(z))^u, \tag{3.2}$$

$$P_1(z,f) = l_{d(P_1)}(z)e^{d(P_1)h(z)} + l_{d(P_1)-1}(z)e^{(d(P_1)-1)h(z)} + \ldots + l_1(z)e^{h(z)} +$$
$$+ \sum_{\lambda \in J} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} (a^{(\alpha_{\lambda,j})})^{\beta_{\lambda,j}} + A(z), \tag{3.3}$$

$$P_2(z,f) = r_{d(P_2)}(z)e^{d(P_2)h(z)} + r_{d(P_2)-1}(z)e^{(d(P_2)-1)h(z)} + \ldots + r_1(z)e^{h(z)} +$$
$$+ \sum_{\mu \in I} b_\mu(z) \prod_{j=1}^{\sigma_\mu} (a^{(m_{\mu,j})})^{n_{\mu,j}} + B(z), \tag{3.4}$$

where $F(z, f)$, $P_1(z, f)$ and $P_2(z, f)$ are all polynomials of $e^{h(z)}$ and their coefficients are either small functions of $e^{h(z)}$ or identically zero. Since $uv > d(P_2)$, $a_v(z)^u H(z)^{uv} \not\equiv 0$ and $P_1(z, f) \not\equiv 0$, by (3.2)–(3.4), we have

$$\psi(z) = F(z,f)P_1(z,f) + P_2(z,f) =$$
$$= w_q(z)e^{qh(z)} + w_{q-1}(z)e^{(q-1)h(z)} + \ldots + w_1(z)e^{h(z)} + F(z,a)P_1(z,a) + P_2(z,a), \tag{3.5}$$

where $w_q(z)(\not\equiv 0)$ is a small function of $e^{h(z)}$, q satisfies $uv \leq q \leq uv + d(P_1)$, and $w_{q-1}(z), \ldots$ $\ldots, w_1(z)$, $F(z, a)$, $P_1(z, a)$ and $P_2(z, a)$ are either small functions of $e^{h(z)}$ or identically zero. Since $F(z, a)P_1(z, a) + P_2(z, a) \not\equiv 0$, by Lemma 3.1, we obtain

$$m(r, 1/\psi) = S(r, e^h).$$

Noting that $uv \le q \le uv + d(P_1)$, we get from (3.5), the first main theorem and Lemma 2.1 that

$$N(r, 1/\psi) = T(r, \psi) + S(r, e^h) \ge uvT(r, e^h) + S(r, e^h),$$
$$T(r, \psi) \le (uv + d(P_1))T(r, e^h) + S(r, e^h).$$

Thus,

$$\delta(0, \psi) \le \frac{\deg_f P_1}{uv + \deg_f P_1} < 1.$$

4. Proof of Theorem 1.3. We rearrange the expression of $P(z, f)$ in the form

$$P(z, f) = \sum_{l=0}^{\deg_f P} b_l(z) f(z)^l,$$

where for $l = 0, \ldots, \deg_f P$,

$$b_l(z) = \sum_{\lambda \in J_l} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} \left(\frac{f^{(\alpha_{\lambda,j})}(z + \delta_{\lambda,j})}{f(z)} \right)^{\beta_{\lambda,j}},$$

and J_l is defined by (1.2). By (1.4) and (1.5), we see that $P(z, f)$ takes the form

$$P(z, f) = \sum_{l=k}^{m} b_l(z) f(z)^l,$$

where $b_m(z) \not\equiv 0$, $b_k(z) \not\equiv 0$. We see from (1.3), logarithmic derivative lemma and [6] (Theorem 5.1) that the coefficients $b_l(z)$, $l = k, k+1, \ldots, m$, are all small functions of $f(z)$.

If $m > k$, then $P(z, f) = f(z)^k Q(z, f)$, where $Q(z, f) = b_m(z) f(z)^{m-k} + \ldots + b_k(z)$. Lemma 2.1, Lemma 3.1, (1.3) and the first main theorem give

$$N(r, 1/P) = N(r, 1/Q) + S(r, f) = (m - k)T(r, f) + S(r, f),$$
$$T(r, P) = mT(r, f) + S(r, f).$$

Therefore, $\delta(0, P) = \dfrac{k}{m}$.

If $m = k \ge 1$, then $P(z, f) = b_m(z) f(z)^m$, where $b_m(z) \not\equiv 0$. By (1.3), we easily see that $\delta(0, P) = 1 = \dfrac{k}{m}$.

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