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MECHANICAL SYSTEMS WITH SINGULAR EQUILIBRIA AND THE COULOMB DYNAMICS OF THREE CHARGES

МЕХАНІЧНІ СИСТЕМИ З СИНГУЛЯРНИМИ РІВНОВАГАМИ ТА КУЛОНІВСЬКА ДИНАМІКА ТРЬОХ ЗАРЯДІВ

We consider mechanical systems for which the matrices of second partial derivatives of the potential energies at equilibria have zero eigenvalues. It is assumed that their potential energies are holomorphic functions in these singular equilibrium states. For these systems, we prove the existence of proper bounded (for positive time) solutions of the Newton equation of motion convergent to the equilibria in the infinite-time limit. These results are applied to the Coulomb systems of three point charges with singular equilibrium in a line.

Розглядаються механічні системи, матриці других похідних потенціальних енергій яких у рівновазі мають нульові власні значення. Припускається, що їхні потенціальні енергії є голоморфними функціями в цих сингулярних рівновагах. Для таких систем доведено існування власних обмежених для додатного часу розв'язків ньютонівських рівнянь руху, які збігаються до рівноваги в границі нескінченного часу. Ці результати застосовуються до кулонівських систем трьох зарядів із сингулярною рівновагою на прямій.

1. Introduction and main result. We consider n-dimensional systems with a potential energy U which is singular at least on a set where some coordinates coincide and has a singular equilibrium configuration meaning that the symmetric matrix U^0 of partial second derivatives of the potential energy has zero eigenvalues at the equilibrium x^0. Such systems can be derived from mechanical systems of N d-dimensional particles (charges) interacting via singular pair or manybody potentials after a re-numeration of variables and masses with $n = dN$. The Newton equation of motion of the systems looks like

$$\mu_j \frac{d^2 x_j}{dt^2} = -\frac{\partial U(x_{(n)})}{\partial x_j}, \quad j = 1, \ldots, n, \qquad x_{(n)} = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{1.1}$$

The diagonal n-dimensional matrix with the elements (effective masses) μ_j, $j = 1, \ldots, n$, will be called by us the mass matrix and denoted by M. We assume that U is a holomorphic function in an equilibrium neighborhood.

The motivation to consider such the systems comes from the Coulomb system of three charges $e_1 = e_2 = -e_0 < 0$, $e_3 = \dfrac{e_0}{4}$ which has a singular equilibrium on a line with an equal distance a of the positive charge to the negative ones. We show this in the last section of this paper.

Our aim is to find solutions of the Newton equation for the considered systems on the infinite time interval. Not much is known about solutions on the infinite time interval for three-dimensional Coulomb systems except the systems of two opposite charges and a charge in the field of many attractive centers. Such the solutions were found for the simplest line Coulomb systems with equilibria [1] and a planar system of $n-1$ equal negative charges and a positive charge [2]. The existence of the Coulomb dynamics without collisions of charges on a finite time interval has been proven in [3] (see also [4]).

Instability of equilibrium in Coulomb systems is known from the Earnshaw theorem [5, 6]. This fact and the inverse Lagrange – Dirichlet theorem imply that the Coulomb potential energy does not attain an absolute minimum at it and U^0 does not have only positive eigenvalues.

Existence of the zero eigenvalue of U^0 does not allow one to apply the results concerning the existence of periodic solutions constructed in the Lyapunov (resonance) center theorem and its (nonresonance) generalizations proposed in [7, 8]. Singularity of U does not allow one also to apply the results concerning the existence of (proper) bounded at positive time solutions converging to the equilibrium in the limit of infinite time [9].

It is known from the celestial mechanics [10] that the zero eigenvalue of a linear part of a vector field of an ordinary differential equation is generated by integral of motions. There is a procedure of lowering of its degeneracy degree by a separation of cyclic variables with the help of a canonical transformation (see the paragraph Application to Lagrange solutions in [10]). But it is not known whether the degeneracy of the zero eigenvalue of U^0 is generated exclusively by the integrals of motion. Besides it is difficult to find them all.

In this paper we find the proper bounded solutions relying on a modification of the Siegel semi-linearization technique (see the paragraph Lyapunov theorem in [10]). This technique is applied to obtain partial solutions of an ordinary differential equation represented in a simple standard form in which a linear part f_0 of its vector field f is given by a diagonal matrix as in the case of the Poincare linearization theorem [11]. The Siegel technique allows one to linearize in new variables (at a linear invariant manifold) only a part of the many-component equation demanding a resonance condition between eigenvalues of f_0 with negative real parts to be satisfied. If the linear part of the second order equation has the zero eigenvalue then one can not reduce it to the simple standard form. In our version of the Siegel technique we start from another standard form of the Newton equation which allows some variables satisfy second order equations. Then we introduce new variables with the help of an unknown function φ such that the invariant manifold of the equation is given by the zero values of some of the new variables and at it the remaining variables satisfy the new equation in which the diagonal linear part of the vector field have negative eigenvalues. A resonance condition is not needed since it is solved on the infinite time interval with the help of the Lyapunov theorem [12, 13]. Finally we prove with the help the majorant method that φ, which satisfies a resolvent type equation, is a vector valued holomorphic function at a neighborhood of the origin.

Our main results are formulated in Theorems 1.1 and 1.2. The first theorem was utilized by us in [1] in a weaker version demanding eigenvalues of $M^{-1}U^0$ not to be zero and its negative eigenvalues satisfy a resonance condition.

Theorem 1.1. *Let M be the mass matrix and U^0 be the symmetric matrix of second derivatives at an equilibrium x^0 of a potential energy U of an n-dimensional mechanical system. Let also U be a holomorphic function in a neighborhood of x^0 and the matrix $M^{-1}U^0$ have p negative eigenvalues $\sigma_j,\ j = 1, \ldots, p$. Then the Newton equation of motion of this mechanical system admits a bounded at positive time solution depending on p real parameters which is real analytic function in them in a neighborhood of the origin and $\|x - x^0\|_\lambda < \infty,\ \|\dot{x}\|_\lambda < \infty$, where $\dot{x}$ is the velocity and*

$$\|x\|_\lambda = \sup_{t \geq 0} \max_{s \in (1,\ldots,n)} e^{\lambda t} |x_s(t)|, \qquad \lambda < \lambda_0 = \min_{j=1,\ldots,p} \sqrt{-\sigma_j}.$$

We show in the last section that for the mentioned system of three charges the eigenvalues of the matrix $M^{-1}U^0$ are determined explicitly. In the planar (three-dimensional) systems this matrix has four (six) times degenerate zero, negative and positive(doubly degenerate) eigenvalues. For the line system it has only one negative and doubly degenerate zero eigenvalues. Such the eigenvalues and Theorem 1.1 imply the following result.

Theorem 1.2. *The Newton – Coulomb equation of motion of the three point charges* $e_1 = -e_0$, $e_2 = -e_0$, $e_3 = \frac{e_0}{4} > 0$ *with masses* m_j, $j = 1, 2, 3$, *admits in the line, planar and three-dimensional systems a bounded at positive time solution which is a real analytic function in a neighborhood of the origin in one real parameter such that* $\|x - x^0\|_\lambda < \infty$, $\|\dot{x}\|_\lambda < \infty$ *and* $\lambda < \lambda_0$, $\lambda_0^2 = e_0^2(4a^3)^{-1}(m_1^{-1} + m_2^{-1} + 4m_3^{-1})$, *where* x^0 *is an equilibrium* $x_1^{01} = -a$, $x_2^{01} = a$, $x_3^{01} = 0$, $x_j^{0\alpha} = 0$, $\alpha = 2, 3$.

Note that due to the equality $\sqrt{M}M^{-1}U^0(\sqrt{M}\,)^{-1} = (\sqrt{M}\,)^{-1}U^0(\sqrt{M}\,)^{-1}$ the matrix $M^{-1}U^0$ has the same spectrum as the matrix $(\sqrt{M}\,)^{-1}U^0(\sqrt{M}\,)^{-1}$ and is similar to the diagonal matrix with real elements.

Our paper is organized as follows. In the second section we transform (1.1) into a standard form (Proposition 2.1) and formulate Theorem 2.1 which substantially diminish the number of variables in the transformed equation and permits to find its proper bounded at positive time solutions (Corollary 2.1). We prove Theorem 2.1 in the third section. In the fourth section we find eigenvalues of $M^{-1}U^0$ (Theorem 4.1 describes them) for our system of three charges proving Theorem 1.2.

2. Standard form of Newton equation and its projection. If U^0 has the zero eigenvalue, then one can transform equation (1.1) into the standard form given in the following proposition (the star in x^* will mean the complex conjugation).

Proposition 2.1. *Let* σ_j, $j = 1, \ldots, n$, *be the real eigenvalues of* $M^{-1}U^0$ *such that* $\sigma_j = 0$, $j = n_0 + 1, \ldots, n$. *Then the Newton equation of equation* (1.1) *can be mapped by a linear invertible transformation* S *into the following standard form*:

$$\frac{dx_j}{dt} = f_j(x_{(l)}) = \lambda_j x_j + X_j(x_{(l)}), \qquad j = 1, \ldots, l_0, \quad t \geq 0, \tag{2.1}$$

$$\frac{d^2x_j}{dt^2} = X'_j(x_{(l)}), \quad j = l_0 + 1, \ldots, l, \tag{2.2}$$

where $l = n + n_0$, $l_0 = 2n_0$,

$$\lambda_j = -\sqrt{-\sigma_j}, \quad j = 1, \ldots, n_0, \qquad \lambda_j = \sqrt{-\sigma_j}, \quad j = n_0 + 1, \ldots, 2n_0,$$

X_j, $X'^*_j = X'_j$ *are holomorphic in the neighborhood of the origin such that in their power expansions the sum of powers of* x_j *is not less than two and* $X_{j+n_0} = -X_j = X_j^*$, $x_j^* = x_{j+n_0}$, *if* $\sigma_j > 0$, *and* $X_j = X_j^*$, $x_j^* = x_j$, *if* $\sigma_j < 0$.

Partial solutions of (2.1), (2.2) can be found with the help of the following theorem.

Theorem 2.1. *Let real* λ_j, $j = 1, \ldots, p < l_0$, *be negative, real parts of* λ_j, $j = p + 1, \ldots, l_0$, *be nonnegative and* X_j, X'_j *be the same as in Proposition* 2.1. *Then there exist functions* $\varphi_j(x_{(p)})$, $j = p + 1, \ldots, l$, *which are holomorphic in a neighborhood of the origin and zero at it such that a partial solution of* (2.1), (2.2) *is given for* $j = p + 1, \ldots, l$ *by*

$$x_j(t) = \varphi_j\big(x_{(p)}(t)\big),$$

and $x_j(t)$ *for* $j = 1, \ldots, p$, *satisfy the projected evolution equation*

$$\frac{dx_j}{dt} = f_j^0(x_{(p)}) = \lambda_j x_j + X_j^0(x_{(p)}), \tag{2.3}$$

where

$$X_j^0(x_{(p)}) = X_j\big(x_{(p)}, \varphi_{(l\backslash p)}(x_{(p)})\big), \qquad (l\backslash p) = p+1,\ldots,l,$$

are real functions and φ_j *have the properties of* X_j, X'_j *if* (2.1), (2.2) *corresponds to* (1.1).

The solution of the projected evolution equation is obtained with the help of the well-known first global Lyapunov theorem [12, 13] a well known generalization of which is formulated in [1] (Theorem 6.2). Hence the following result is valid.

***Corollary* 2.1.** *Let the conditions of Theorem* 2.1 *be satisfied. Then there exists a partial solution of* (2.1), (2.2) *depending on* p *parameters which coincide with the initial values of the variables in* (2.3). *This solution is a holomorphic function in these parameters in a neighborhood of the origin and* $\|x\|_\lambda < \infty$, *where*

$$\|x\|_\lambda = \sup_{t\geq 0} \max_{s\in(1,\ldots,l)} e^{\lambda t}|x_s(t)|, \quad \lambda < \lambda_0 = \min_{j=1,\ldots,p} |\lambda_j|,$$

and determines real solutions of (1.1).

The reality of the solutions follows from the fact that they are expressed as real linear combinations of the variables $x'_{j+n_0} + x'_j$, $\sqrt{-\sigma_j}\big(x'_{j+n_0} - x'_j\big)$ which are real and x'_j coincides with the solution of (2.1), (2.2) corresponding to (1.1). Here one have to take into account the equality $S^{-1} = \tilde{S}^{-1}(S^0)^{-1}$ determined below. This corollary and Proposition 2.1 prove Theorem 1.1.

***Proof of Proposition* 2.1.** We assume that the potential energy U has the equilibrium at the point $x^0 = \big(x_j^0\big)$, $j = 1,\ldots,n$, at a neighborhood of which it is holomorphic, that is

$$\left(\frac{\partial U}{\partial x_j}\right)(x^0) = 0.$$

Then in the new variables $x_j - x_j^0$ the dynamic equation is rewritten as

$$\mu_j \frac{d^2 x_j}{dt^2} = -\frac{\partial U'(x_{(n)})}{\partial x_j}, \tag{2.4}$$

where

$$U'(x_{(n)}) = U\big(x_1 + x_1^0, \ldots, x_n + x_n^0\big), \qquad \left(\frac{\partial U'}{\partial x_j}\right)(0) = 0.$$

By an invertible linear transformation $\tilde{x}_j = \sum_{k=1}^n \tilde{S}_{j,k} x_k$ one diagonalizes $M^{-1}U^0$, which has eigenvalues σ_j, that is $\delta_{j,k}\sigma_j = \big(\tilde{S}M^{-1}U^0\tilde{S}^{-1}\big)_{j,k}$ and transforms (2.4) into (we omit tilde in variables)

$$\frac{d^2 x_j}{dt^2} = -\sigma_j x_j + F_j(x_{(n)}), \tag{2.5}$$

where

$$F_j(x_{(n)}) = -\sum_{k=1}^{n} \tilde{S}_{j,k} \mu_k^{-1} \left(\frac{\partial U''}{\partial x_k} \right) \big((\tilde{S}^{-1} x)_{(n)} \big),$$

$$U''(x_{(n)}) = U'(x_{(n)}) - \frac{1}{2} \sum_{j,l=1}^{n} U^0_{j,l} x_j x_l.$$

That is

$$\frac{dx_j}{dt} = v_j, \qquad \frac{dv_j}{dt} = -\sigma_j x_j + F_j(x_{(n)}). \tag{2.6}$$

Then by the linear two-dimensional transformation produced by the matrix S^0_j the last equation is mapped into (2.1), (2.2) with $l_0 = 2n_0$ and $-\lambda_{2j-1} = \lambda_{2j} = \sqrt{-\sigma_j}$, $j = 1, \ldots, n_0$. The matrix S^0_j diagonalizes the two dimensional matrix A_j, which determines the linear part of (2.6), with the zero diagonal elements and nondiagonal elements $A_{j;1,2} = 1$, $A_{j;2,1} = -\sigma_j$. That is $S^0_j A_j = \hat{\sigma_j} S^0_j$, where $\hat{\sigma_j}$ is a diagonal matrix with the eigenvalues $-\lambda_{2j-1} = \lambda_{2j} = \sqrt{-\sigma_j}$. It is not difficult to check that

$$S^0_{j;1,1} = S^0_{j;2,1} = \frac{1}{2}, \qquad -S^0_{j;1,2} = S^0_{j;2,2} = \frac{1}{2\kappa_j}, \quad \kappa_j = \sqrt{-\sigma_j}.$$

The new variables look like

$$x'_{2j-1} = \frac{1}{2}\left(x_j - \frac{1}{\kappa_j} v_j \right), \qquad x'_{2j} = \frac{1}{2}\left(x_j + \frac{1}{\kappa_j} v_j \right), \quad j = 1, \ldots, n_0,$$

$$x'_j = x_{j-n_0}, \quad j = 2n_0 + 1, \ldots, n + n_0.$$

The inverse transformation is given by

$$x_j = x'_{2j} + x'_{2j-1}, \qquad v_j = \kappa_j (x'_{2j} - x'_{2j-1}), \quad j = 1, \ldots, n_0,$$

which implies that the functions X_j, X'_j in (2.1), (2.2) are given by (we omit primes in variables)

$$X_{2j}(x_{(n+n_0)}) = -X_{2j-1}(x_{(n+n_0)}) = \frac{1}{2\kappa_j} F_j(x_2 + x_1, \ldots, x_{2n_0} + x_{2n_0-1}, x_{2n_0+1}, \ldots, x_{n+n_0}),$$

where $j = 1, \ldots, n_0$ and

$$X'_j(x_{(n+n_0)}) = F_j(x_2 + x_1, \ldots, x_{2n_0} + x_{2n_0-1}, x_{2n_0+1}, \ldots, x_{n+n_0}), \quad j = 2n_0 + 1, \ldots, n + n_0.$$

That is $X_j = -X^*_j$, $x^*_{2j} = x_{2j-1}$, if $\sigma_j > 0$, and $X_j = X^*_j$, $x^*_j = x_j$, if $\sigma_j < 0$. Let us use another numeration of variables: $(x_1, x_2, x_3, \ldots, x_{2n_0}) \to (x_1, x_3, \ldots, x_{2n_0-1},\ x_2,\ x_4, \ldots, x_{2n_0})$. In such a way (2.6) is mapped into (2.1), (2.2) with

$$\lambda_j = -\sqrt{-\sigma_j}, \quad j = 1, \ldots, n_0, \qquad \lambda_j = \sqrt{-\sigma_j}, \quad j = n_0 + 1, \ldots, 2n_0.$$

As a result $X_{j+n_0} = -X_j = X_j^*$, $x_j^* = x_{j+n_0}$, if $\sigma_j > 0$, and $X_j = X_j^*$, $x_j^* = x_j$, if $\sigma_j < 0$ and $S = S^0\tilde{S}$, where

$$(S^0 x)_j = x_j, \quad j = 2n_0 + 1, \ldots, n + n_0,$$
$$\left((S^0)^{-1}x\right)_j = x_j + x_{j+n_0}, \quad j = 1, \ldots, n_0,$$
$$\left((S^0)^{-1}x\right)_{j+n_0} = \sqrt{-\sigma_j}(x_{j+n_0} - x_j), \quad j = 1, \ldots, n_0.$$

Proposition 2.1 is proved.

3. Proof of Theorem 2.1. To prove Theorem 2.1 we introduce at first the new variables u_k inspired by [10]

$$u_j = x_j - \varphi_j(x_{(p)}), \quad j = 1, \ldots, l,$$

and

$$u_j = x_j, \qquad \varphi_j(x_{(p)}) = 0, \quad j \le p.$$

Here the functions φ_j are given by a power expansion in $x_1^{n_1} \ldots x_1^{n_p}$ with $n_1 + \ldots + n_p > 1$ and coefficients $P_{j;n_1,\ldots,n_p}$, $j = p+1, \ldots, l_0$, $P'_{j;n_1,\ldots,n_p}$, $j = l_0 + 1, \ldots, l$. They will be real if j correspond to real λ_j. The former variables are expressed in terms of the new ones as follows:

$$x_j = u_j + \varphi_j(u_{(p)}).$$

The new variables obey the following equations:

$$\frac{du_j}{dt} = \lambda_j u_j + G_j(u_{(l)}), \quad j = 1, \ldots, l_0, \tag{3.1}$$

$$\frac{d^2 u_j}{dt^2} = G'_j(u_{(l)}), \quad j = l_0 + 1, \ldots, l. \tag{3.2}$$

If one shows that the equalities

$$G_j(u_{(p)}, 0, \ldots, 0) = 0, \quad j = p+1, \ldots, l_0, \qquad G'_j(u_{(p)}, 0, \ldots, 0) = 0, \quad j = l_0 + 1, \ldots, l, \tag{3.3}$$

are true then a partial solution of (2.1), (2.2) is given by (2.3) since

$$u_j = 0, \quad j = p+1, \ldots, l,$$

is a partial solution of (3.1), (3.2). This will prove the theorem if φ_j is a holomorphic function at the origin. Now we shall prove this character of φ_j. Let

$$\varphi_{jx_k} = \frac{\partial \varphi_j}{\partial x_k}, \qquad \varphi_{jx_k x_r} = \frac{\partial^2 \varphi_j}{\partial x_k \partial x_r}.$$

Then

$$G_j = X_j + \lambda_j \varphi_j - \sum_{k=1}^{p} f_k \varphi_{jx_k},$$

$$G'_j = X'_j - \sum_{r,k=1}^{p} (f_k \varphi_{j x_k x_r} + \varphi_{j x_k} f_{k x_r}) f_r.$$

The first and second equations in (3.3) will be called the first and second structure equations. The second structure equation is rewritten as follows:

$$\sum_{r,k=1}^{p} x_r x_k \lambda_r \lambda_k \varphi_{j x_k x_r} + \sum_{k=0}^{p} \varphi_{j x_k} \lambda_k^2 x_k = X'_j - \sum_{r,k=1}^{p} \big[X_r (X_k + 2\lambda_k x_k) \varphi_{j x_k x_r} +$$

$$+ \varphi_{j x_k} (X_{k x_r} X_r + X_{k x_r} x_r \lambda_r + \lambda_k \delta_{k,r} X_r) \big],$$

where

$$x_k = \varphi_k(x_{(p)}), \quad k = p+1, \ldots, l, \tag{3.4}$$

and $\delta_{k,r}$ has the (unit) non-zero value only for $k = r$. This equation is reduced to the following recursion relation for the coefficients in the expansion of powers of variables (the sum of their powers exceeds unity):

$$\left[\left(\sum_{k=1}^{p} n_k \lambda_k \right)^2 + \sum_{k=1}^{p} n_k \lambda_k^2 \right] P'_{j;n_1,\ldots,n_p} = \Gamma'_{j;n_1,\ldots,n_p}$$

that is $\Gamma'_{j;n_1,\ldots,n_p}$ is expressed in terms of $P'_{j;n'_1,\ldots,n'_p}$ with $n'_1 + \ldots + n'_p < n_1 + \ldots + n_p$. It is easily solved since the real parts of both terms in the square brackets are not zero due to the condition

$$\left(\sum_{k=1}^{p} n_k \lambda_k \right)^2 + \sum_{k=1}^{p} n_k \lambda_k^2 \geq \lambda_- \left(\left(\sum_{k=1}^{p} n_k \right)^2 + \sum_{k=1}^{p} n_k) \right), \quad \lambda_- = \min_j \lambda_j^2, \tag{3.4a}$$

and the expansion for $\varphi_j,\ j = l_0 + 1, \ldots, l,$ is found. Now we have to prove its convergence with the help of the majorant technique.

We will use the majorant inequality $f << g$ which means that in the power expansion for g the coefficients are nonnegative and exceed absolute values of the coefficients in the power expansion for f. Let φ_j^+ be the power expansion with the coefficients $\left| P'_{j;n_1,\ldots,n_p} \right|$, that is

$$\varphi_j << \varphi_j^+.$$

Let

$$X_j << \frac{c_3 X^2}{1 - c_1 X} = \chi, \quad X'_j << \chi, \quad X = x_1 + \ldots + x_l. \tag{3.5}$$

Then the rewritten second structure equation yields

$$\sum_{r,k=1}^{p} x_r x_k \lambda_r \lambda_k \varphi^+_{j x_k x_r} + \sum_{k=1}^{p} \varphi^+_{j x_k} \lambda_k^2 x_k << \chi + \sum_{r,k=1}^{p} \big[\chi (\chi + 2\lambda_+ x_k) \varphi^+_{j x_k x_r} +$$

$$+ \varphi^+_{j x_k} (\chi_{x_r} \chi + \chi_{x_r} x_r \lambda_+ + \lambda_+ \delta_{k,r} \chi) \big].$$

From (3.4a) we obtain

$$\lambda_- \left(\sum_{r,k=1}^{p} x_r x_k \varphi^+_{jx_k x_r} + \sum_{k=1}^{p} \varphi^+_{jx_k} x_k \right) << \sum_{r,k=1}^{p} x_r x_k \lambda_r \lambda_k \varphi^+_{jx_k x_r} + \sum_{k=1}^{p} \varphi^+_{jx_k} \lambda_k^2 x_k .$$

The last two inequalities yield

$$\lambda_- \left(\sum_{r,k=1}^{p} x_r x_k \varphi^+_{jx_k x_r} + \sum_{k=1}^{p} \varphi^+_{jx_k} x_k \right) << \chi + \sum_{r,k=1}^{p} \big[\chi(\chi + 2\lambda_+ x_k) \varphi^+_{jx_k x_r} +$$
$$+ \varphi^+_{jx_k} (\chi_{x_r} \chi + \chi_{x_r} x_r \lambda_+ + \lambda_+ \delta_{k,r} \chi) \big].$$

We have also

$$\varphi^+_j << \varphi_{*j},$$

where

$$\lambda_- \left(\sum_{r,k=1}^{p} x_r x_k \varphi_{*jx_k x_r} + \sum_{k=1}^{p} \varphi_{*jx_k} x_k \right) = \chi + \sum_{r,k=1}^{p} \big[\chi(\chi + 2\lambda_+ x_k) \varphi_{*jx_k x_r} +$$
$$+ \varphi_{*jx_k} (\chi_{x_r} \chi + \chi_{x_r} x_r \lambda_+ + \lambda_+ \delta_{k,r} \chi) \big]. \tag{3.6}$$

Now we have to prove that the solutions of this majorized second structure equation is a holomorphic function. We seek the solutions of the last equation in the form

$$\varphi_{*j} = \psi(x), \qquad \varphi_{*jx_k x_r} = \psi_{xx}, \qquad \varphi_{*jx_k} = \psi_x, \quad x = x_1 + \ldots + x_p.$$

The right-hand side of the majorized second structure equation is given by

$$\chi + p^2 \chi(\chi \psi_{xx} + \psi_x \chi_x) + p\lambda_+ \big[(x\chi_x + \chi)\psi_x + 2x\chi\psi_{xx} \big].$$

Taking into account that

$$\chi_x = (1 + p'\psi_x)\chi', \qquad \chi'(y) = \partial\chi(y) = \frac{2c_3 y}{1 - c_1 y} + \frac{c_1 c_3 y^2}{(1 - c_1 y)^2}, \quad p' = l - p,$$

we see that the one-variable majorized second structure equation is derived from (3.6) and given by

$$\lambda_- x(x\psi_{xx} + \psi_x) =$$
$$= \chi + p^2 \chi \big[\chi\psi_{xx} + \psi_x (1 + p'\psi_x)\chi' \big] + p\lambda_+ \Big\{ \big[x(1 + p'\psi_x)\chi' + \chi \big] \psi_x + 2x\chi\psi_{xx} \Big\}, \tag{3.7}$$

where χ, χ' depend on $x + p'\psi$. This equation is equivalent to the recursion relation for the coefficients in the power expansion for ψ (its powers exceed unity) whose coefficients are nonnegative. Let us put $x^{-1}\psi = \Psi$. The function

$$\Phi(x) = \Psi(x) + 3x\Psi_x(x) + x^2 \Psi_{xx}(x) = \psi_x + x\psi_{xx}$$

has power expansion with nonnegative coefficients. Now we majorize the right-hand side of (3.7) in such a way that it should depend only on Φ and x. In order to do this one has to substitute $\psi_x + x\psi_{xx}$ instead of ψ_x and $x\psi_{xx}$ in (3.7). For the term in the first square bracket one obtaines

$$\chi\psi_{xx} + \psi_x(1 + p'\psi_x)\chi' << (\psi_x + x\psi_{xx})(1 + p'(\psi_x + x\psi_{xx}))\chi' + (\psi_x + x\psi_{xx})x^{-1}\chi =$$
$$= \Phi\big[(1 + p'\Phi)\chi' + x^{-1}\chi\big]$$

and the expression in the figure bracket is majorized by

$$\big[x(1 + p'\Phi)\chi' + \chi\big]\Phi + 2x\chi\Phi.$$

The right-hand side of (3.7) contains x as a multiplier since

$$\chi = x^2 \frac{c_3(1 + p'\Psi)^2}{1 - c_1x(1 + p'\Psi)} << x^2 \frac{c_3(1 + p'\Phi)^2}{1 - c_1x(1 + p'\Phi)},$$
$$\chi' = x\frac{2c_3(1 + p'\Psi)}{1 - c_1x(1 + p'\Psi)} + x^2 \frac{c_1c_3(1 + p'\Psi)^2}{(1 - c_1x(1 + p'\Psi))^2} <<$$
$$<< x\frac{2c_3(1 + p'\Phi)}{1 - c_1x(1 + 2p'\Phi)} + x^2 \frac{c_1c_3(1 + p'\Phi)^2}{1 - 2c_1x(1 + p'\Phi)}.$$

Due to the fact that $\chi,\ \chi'$ are proportional to x^2, x, respectively, (3.7) is majorized by the following rational equation for Φ_*:

$$\Phi << \frac{xP(x,\Phi)}{1 - 3c_1x(1 + p'\Phi)}, \qquad \Phi_* = \frac{xP(x,\Phi_*)}{1 - 3c_1x(1 + p'\Phi_*)}, \quad \Phi << \Phi_*,$$

where P is a polynomial of two complex variables. Here we used the relation

$$\prod_{j=1}^{k}(1 - x_j)^{-1} << \left(1 - \sum_{j=1}^{k} x_j\right)^{-1}.$$

The last equation can be rewritten as

$$F(x,\Phi_*) = \Phi - xP'(x,\Phi_*) = 0,$$

where P' is a polynomial with positive coefficients. That is $\partial_* F(0,0) \neq 0$, where ∂_* is the derivative in Φ_*. From the holomorphic implicit function theorem [13, 14] it follows that $\Phi_*(x)$ is a holomorphic function at the origin with nonnegative coefficients in its power expansion. The same is true for ψ since it is majorized by $x\Phi$. Hence the power expansion for $\varphi_j,\ j = l_0 + 1, \ldots, l$, is a holomorphic function at the origin in all the variables.

Now we have to show that the solution of the first structure equation is also a holomorphic function. This equation is given by

$$-\lambda_j\varphi_j + \sum_{k=1}^{p} \varphi_{jx_k}\lambda_k x_k = X_j - \sum_{k=1}^{p} X_k\varphi_{jx_k}$$

with (3.4). This equation is reduced to the recursion relation

$$\left(-\lambda_j + \sum_{k=0}^{p} n_k \lambda_k\right) P_{j;n_1,\ldots,n_p} = \Gamma_{j;n_1,\ldots,n_p}$$

that is $\Gamma_{j;n_1,\ldots,n_p}$ is expressed in terms of $P_{j;n'_1,\ldots,n'_p}$ with $n'_1 + \ldots + n'_p < n_1 + \ldots + n_p$. It is easily solved and the expansion for $\varphi_j,\ j = p+1, \ldots, l_0$, is found. Now we have to prove its convergence.

The inequality

$$1 < n_1 + \ldots + n_p \leq c_2 \left| -\lambda_j + \sum_{k=1}^{p} n_k \lambda_k \right|,$$

the first structure equation and (3.5) lead to

$$\sum_{k=1}^{p} \varphi^{+}_{jx_k} x_k << c_2 \left(1 + \sum_{k=1}^{p} \varphi^{+}_{jx_k}\right) \chi,$$

and the majorized first structure equation

$$\sum_{k=1}^{p} \varphi_{*jx_k} x_k = c_2 \left(1 + \sum_{k=1}^{p} \varphi_{*jx_k}\right) \chi, \quad \varphi^{+}_j << \varphi_{*j},$$

with (3.4) added, where φ^{+}_j is the power expansion with the coefficients $|P_{j;n_1,\ldots,n_p}|$. Taking into account the previous notation we derive the one-variable majorized first structure equation

$$x\psi_x = c_2(1 + \psi_x)\chi$$

which determines the recursion relation for the coefficients of the power expansion for ψ. Here

$$\chi = \frac{(x + p'\psi)^2}{1 - c_1(x + p'\psi)}.$$

The power expansion for ψ has nonnegative coefficients. Let us put $x^{-1}\psi = \Psi$. Then

$$\Phi(x) = \Psi(x) + x\Psi_x(x) = \psi_x.$$

That is the power expansion for Φ has nonnegative coefficients and

$$\Phi << \frac{c_2 x(1 + p'\Phi)^3}{1 - c_1 x(1 + p'\Phi)}, \quad \Phi << \Phi_*.$$

The final majorized first structure equation is given by

$$\Phi_* = \frac{c_2 x(1 + p'\Phi_*)^3}{1 - c_1 x(1 + p'\Phi_*)}, \quad \Phi << \Phi_*.$$

From the holomorphic implicit function theorem it follows that $\Phi_*(x)$ is a holomorphic function at the origin with nonnegative coefficients in its power expansion. The same is true for ψ since it is majorized by $x\Phi_*$. Hence the power expansion for $\varphi_j,\ j = p+1, \ldots, l_0$, is a holomorphic function at the origin in all the variables. It follows from the first equation in (3.3) that φ_j has the same properties as X_j described in the Proposition 2.1 if $\lambda_j,\ X_j,\ X'_j$ correspond to (2.1). The reality of X^0 follows from the dependence of $X_j,\ X'_j$ on $\varphi_j + \varphi_{j+n_0},\ j = 1, \ldots, n_0$, and reality of the latter since $\varphi^*_j = \varphi_{j+n_0}$ for a positive σ_j,which follows from the first equation in (3.3), and reality of both functions for a nonpositive σ_j.

Theorem 2.1 is proved.

4. Proof of Theorem 1.2. The simplest example of a mechanical system with an equilibrium is the d-dimensional system of the three point charges $e_1 = -e_0,\ e_2 = -e_0,\ e_3 = \frac{e_0}{4} > 0$ with masses m_1, m_2, m_3 and the potential energy

$$U(x_{(3)}) = \frac{1}{2} \sum_{j \neq k=1}^{3} \frac{e_j e_k}{|x_j - x_k|}, \tag{4.1}$$

where $x_{(3)} \in \mathbb{R}^{3d},\ x_j = \left(x_j^1, \ldots, x_j^d\right),\ |x|^2 = (x^1)^2 + \ldots + (x^d)^2$. Its equilibrium is determined by $x_1^{01} = -a,\ x_2^{01} = a,\ x_3^{01} = 0,\ x_j^{0\alpha} = 0,\ \alpha = 2$. This potential energy is a holomorphic function at a neighborhood of the equilibrium. The case of equal masses of the one-dimensional three charges was considered in [1], where eigenvalues of U^0 were calculated.

Theorem 1.2 is proved with the help of Theorem 1.1 and following theorem.

Theorem 4.1. *In the one-dimensional system $M^{-1}U^0$ has the doubly degenerate zero eigenvalue and the eigenvalue $-\left(m_1^{-1} + m_2^{-1} + 4m_3^{-1}\right)u',\ u' = \frac{e_0^2}{4a^3}$. In the two-dimensional and three-dimensional systems $M^{-1}U^0$ has the zero eigenvalue, which is four and six times degenerate, respectively, and the eigenvalues $-\left(m_1^{-1} + m_2^{-1} + 4m_3^{-1}\right)u',\ 2^{-1}\left(m_1^{-1} + m_2^{-1} + 4m_3^{-1}\right)u'$ the latter of which is doubly degenerate in the three-dimensional system.*

Proof. We find eigenvalues of U^0 at first for the one-dimensional case. In our calculations of partial derivatives of the potential energy we will use the two equalities for $x \in \mathbb{R}$ and $x \in \mathbb{R}^d$, respectively,

$$\frac{\partial}{\partial x_1}|x_1 - x_2|^{-k} = -k\frac{x_1 - x_2}{|x_1 - x_2|^{k+2}}, \qquad \frac{\partial}{\partial x^\alpha}\left(\sqrt{|x|^2 + b^2}\,\right)^{-k} = -k\frac{x^\alpha}{\left(\sqrt{|x|^2 + b^2}\,\right)^{k+2}}$$

which gives

$$\frac{\partial}{\partial x_j}U(x_{(3)}) = -e_j \sum_{k=1, k\neq j}^{3} e_k \frac{x_j - x_k}{|x_j - x_k|^3},$$

that is

$$\frac{\partial}{\partial x_1}U(x_{(3)}) = -e_0^2\frac{x_1 - x_2}{|x_1 - x_2|^3} + e_0 e_3 \frac{x_1 - x_3}{|x_1 - x_3|^3},$$

$$\frac{\partial}{\partial x_2}U(x_{(3)}) = -e_0^2\frac{x_2 - x_1}{|x_1 - x_2|^3} + e_0 e_3 \frac{x_2 - x_3}{|x_2 - x_3|^3},$$

$$\frac{\partial}{\partial x_3}U(x_{(3)}) = e_0 e_3 [\frac{x_3 - x_1}{|x_1 - x_3|^3} + \frac{x_3 - x_2}{|x_2 - x_3|^3}].$$

The equality $\frac{\partial}{\partial x_3}U(x_{(3)}) = 0$ holds for $x_1 = x_1^0 = -a,\ x_2 = x_2^0 = a,\ x_3^0 = 0$. This configuration is an equilibrium. This follows also from the equalities $\frac{\partial}{\partial x_j}U(x_{(3)}) = 0,\ j = 1, 2$.

The second derivatives of the potential energy are calculated as follows:

$$\frac{\partial U(x_{(3)})}{\partial x_j \partial x_k} = \frac{\partial U(x_{(3)})}{\partial x_k \partial x_j} = -2e_j e_k |x_j - x_k|^{-3}, \quad k \neq j,$$

$$\frac{\partial^2}{\partial x_j^2}U(x_{(3)}) = 2e_j \sum_{k=1,k\neq j}^{3} e_k|x_j - x_k|^{-3}.$$

Hence the second derivatives of the potential energy at the equilibrium $U^0_{j,k}$ are given by

$$U^0_{1,2} = U^0_{2,1} = -\frac{e_0^2}{4a^3} = -u', \qquad U^0_{3,1} = U^0_{1,3} = U^0_{2,3} = U^0_{3,2} = 2u',$$
$$U_{1,1} = U_{2,2} = -u', \qquad U_{3,3} = -4u'.$$

That is

$$U^0 = -u'\begin{pmatrix} 1 & 1 & q \\ 1 & 1 & q \\ q & q & q^2 \end{pmatrix} = u'U', \quad q = -2. \tag{4.2}$$

Let us put

$$M'_0(\lambda, q) = \begin{pmatrix} k_1 - \lambda & k_1 & qk_1 \\ k_2 & k_2 - \lambda & qk_2 \\ qk_3 & qk_3 & q^2k_3 - \lambda \end{pmatrix}, \quad k_j = m_j^{-1}.$$

Then

$$M^{-1}U^0 - \lambda I = -u'M'_0\left(-\frac{\lambda}{u'}, q\right), \qquad -\mathrm{Det}(M^{-1}U^0 - \lambda I) = u'^3\,\mathrm{Det}\,M'_0\left(-\frac{\lambda}{u'}, q\right), \quad q = -2,$$

and making the expansion of the determinant in the elements of the first row of M'_0 we obtain

$$\mathrm{Det}\,M'_0(\lambda, q) =$$
$$= (k_1 - \lambda)\big[(k_2 - \lambda)(q^2k_3 - \lambda) - q^2k_2k_3\big] -$$
$$-k_1\big[k_2(q^2k_3 - \lambda) - q^2k_2k_3\big] + qk_1\big[qk_2k_3 - qk_3(k_2 - \lambda)\big] =$$
$$= (k_1 - \lambda)\big[\lambda^2 - \lambda(k_2 + q^2k_3)\big] + \lambda k_1k_2 + \lambda q^2k_1k_3 =$$
$$= \lambda\big[(k_1 - \lambda)(\lambda - q^2k_3 - k_2) + k_1k_2 + q^2k_1k_3\big].$$

Hence

$$\mathrm{Det}\,M'_0(\lambda, q) = \lambda^2(-\lambda + k_1 + k_2 + q^2k_3)$$

and

$$\mathrm{Det}(M^{-1}U^0 - \lambda I) = -\lambda^2\big[\lambda + (m_1^{-1} + m_2^{-1} + 4m_3^{-1})u'\big].$$

The last formula proves the theorem for the one-dimensional case.

Let us consider the two-dimensional case. For the first partial derivatives of the planar potential energy we have

$$\frac{\partial}{\partial x_1^\alpha}U(x_{(3)}) = -e_0^2\frac{x_1^\alpha - x_2^\alpha}{|x_1 - x_2|^3} + e_0e_3\frac{x_1^\alpha - x_3^\alpha}{|x_1 - x_3|^3},$$

$$\frac{\partial}{\partial x_2^\alpha}U(x_{(3)}) = -e_0^2\frac{x_2^\alpha - x_1^\alpha}{|x_1 - x_2|^3} + e_0e_3\frac{x_2^\alpha - x_3^\alpha}{|x_2 - x_3|^3},$$

$$\frac{\partial}{\partial x_3^\alpha}U(x_{(3)}) = e_0e_3\left[\frac{x_3^\alpha - x_1^\alpha}{|x_1 - x_3|^3} + \frac{x_3^\alpha - x_2^\alpha}{|x_2 - x_3|^3}\right].$$

The last equality is zero at the equilibrium $-x_1^1 = x_2^1 = a,\ x_3^2 = x_3^1 = x_1^2 = x_2^2 = 0$. The first two give the equilibrium relation $e_3 = \frac{e_0}{4}$. The second derivatives of the potential energy are given by

$$\frac{\partial U(x_{(3)})}{\partial x_1^\alpha \partial x_2^\beta} = \frac{\partial U(x_{(3)})}{\partial x_2^\beta \partial x_1^\alpha} = e_0^2\left[\frac{\delta_{\alpha,\beta}}{|x_1 - x_2|^3} - 3\frac{(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta)}{|x_1 - x_2|^5}\right], \quad \alpha,\beta = 1,2,$$

$$\frac{\partial U(x_{(3)})}{\partial x_k^\alpha \partial x_3^\beta} = \frac{\partial U(x_{(3)})}{\partial x_3^\beta \partial x_k^\alpha} = -e_0e_3\left[\frac{\delta_{\alpha,\beta}}{|x_k - x_3|^3} - 3\frac{(x_k^\alpha - x_3^\alpha)(x_k^\beta - x_3^\beta)}{|x_k - x_3|^5}\right], \quad k,\alpha,\beta = 1,2,$$

$$\frac{\partial^2 U(x_{(3)})}{\partial x_j^\beta \partial x_j^\alpha} = e_0^2\left[-\frac{\delta_{\alpha,\beta}}{|x_1 - x_2|^3} + 3\frac{(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta)}{|x_1 - x_2|^5}\right] +$$

$$+e_0e_3\left[\frac{\delta_{\alpha,\beta}}{|x_j - x_3|^3} - 3\frac{(x_j^\alpha - x_3^\alpha)(x_j^\beta - x_3^\beta)}{|x_j - x_3|^5}\right], \quad j,\alpha,\beta = 1,2,$$

$$\frac{\partial^2 U(x_{(3)})}{\partial x_3^\beta \partial x_3^\alpha} = e_0e_3\left[\frac{\delta_{\alpha,\beta}}{|x_1 - x_3|^3} - 3\frac{(x_1^\alpha - x_3^\alpha)(x_1^\beta - x_3^\beta)}{|x_1 - x_3|^5} + \right.$$

$$\left. + \frac{\delta_{\alpha,\beta}}{|x_2 - x_3|^3} - 3\frac{(x_2^\alpha - x_3^\alpha)(x_2^\beta - x_3^\beta)}{|x_2 - x_3|^5}\right].$$

For the matrix of the second derivatives at the equilibrium we derive

$$U^0_{1,\alpha;1,\beta} = U^0_{2,\alpha;2,\beta} = e_0^2\left[\delta_{\alpha,\beta}\left(-\frac{1}{(2a)^3} + \frac{1}{4a^3}\right) + 3\delta_{\alpha,1}\delta_{\beta,1}\left(\frac{1}{(2a)^3} - \frac{1}{4a^3}\right)\right] =$$

$$= \frac{e_0^2}{(2a)^3}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}\delta_{\beta,1}) = 4^{-1}U^0_{3,\alpha;3,\beta},$$

$$U^0_{1,\alpha;2,\beta} = U^0_{2,\beta;1,\alpha} = \frac{e_0^2}{(2a)^3}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}\delta_{\beta,1}),$$

$$U^0_{k,\alpha;3,\beta} = U^0_{3,\beta;k,\alpha} = -\frac{e_0^2}{4a^3}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}\delta_{\beta,1}), \quad k,\alpha,\beta = 1,2.$$

Let's introduce the numeration

$$(1,1) = 1, \qquad (2,1) = 2, \qquad (3,1) = 3, \qquad (1,2) = 4, \qquad (2,2) = 5, \qquad (3,2) = 6,$$

$$m_4 = m_1, \qquad m_5 = m_2, \qquad m_6 = m_3,$$

where the first and second numbers in the round brackets correspond to the lower and upper indices of variables. Then $U^0_{j,k} = U^0_{k,j} = 0,\ j \leq 3,\ k \geq 4$, and

$$U^0_{1,1} = U^0_{2,2} = 4^{-1}U^0_{3,3} = -2c, \qquad U^0_{1,2} = -2c, \qquad U^0_{1,3} = U^0_{2,3} = 4c, \qquad c = \frac{u'}{2} = \frac{e_0^2}{(2a)^3},$$
$$U^0_{4,4} = U^0_{5,5} = 4^{-1}U^0_{6,6} = c, \qquad U^0_{4,5} = c, \qquad U^0_{4,6} = U^0_{5,6} = -2c.$$

This means

$$U^0 = 2cU' \oplus -cU',$$

where U' is given by (4.2).

Let $M'' = M' \oplus M'$ and M' be the 3×3 diagonal matrix with the elements $m_1,\ m_2,\ m_3$. Then

$$M''^{-1}U^0 - \lambda I = -2cM_0'\left(-\frac{\lambda}{2c}, -2\right) \oplus cM_0'\left(\frac{\lambda}{c}, -2\right),$$
$$\mathrm{Det}(M''^{-1}U^0 - \lambda I) = -2^3c^6\,\mathrm{Det}\,M_0'\left(-\frac{\lambda}{2c}, -2\right)\mathrm{Det}\,M_0'\left(\frac{\lambda}{c}, -2\right).$$

From this equality and (4.3) we derive

$$-\,\mathrm{Det}(M^{-1}U^0 - \lambda I) = \lambda^4\left[-\lambda + \left(m_1^{-1} + m_2^{-1} + 4m_3^{-1}\right)\frac{u'}{2}\right]\left[\lambda + (m_1^{-1} + m_2^{-1} + 4m_3^{-1})u'\right].$$

This concludes the proof for the two-dimensional case.

Let's consider the 3-dimensional case. Then all the formulas concerning partial derivatives of the potential energy of this sections will be true adding the condition $\alpha, \beta = 1, 2, 3$. Let's use the following numeration of the variables indices:

$$(1,3) = 7, \qquad (2,3) = 8, \qquad (3,3) = 9, \qquad m_7 = m_1, \qquad m_8 = m_2, \qquad m_9 = m_3.$$

It is not difficult to see that $U^0_{j,k} = U^0_{k,j} = 0,\ j \leq 6,\ k \geq 7$, and $U^0_{7,7} = c_1,\ U^0_{8,8} = c_1,\ U^0_{9,9} = 4c_1$, $U^0_{7,8} = c_1,\ U^0_{7,9} = U^0_{8,9} = -2c_1$. Hence

$$U^0 = U'' \oplus -c_1U',$$

where U'' coincides with the planar U^0. Moreover

$$M = M'' \oplus M', \qquad M^{-1}U^0 = M''^{-1}U'' \oplus -c_1M'^{-1}U'.$$

As a result

$$-\,\mathrm{Det}(M^{-1}U^0 - \lambda I) = \lambda^6\left[-\lambda + (m_1^{-1} + m_2^{-1} + 4m_3^{-1})\frac{u'}{2}\right]^2\left[\lambda + (m_1^{-1} + m_2^{-1} + 4m_3^{-1})u'\right].$$

Theorem 4.1 is proved.

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