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SECOND-ORDER DIFFERENTIAL SUBORDINATIONS ON A CLASS OF ANALYTIC FUNCTIONS DEFINED BY RAFID-OPERATOR

ДИФЕРЕНЦІАЛЬНІ ПІДПОРЯДКУВАННЯ ДРУГОГО ПОРЯДКУ НА КЛАСІ АНАЛІТИЧНИХ ФУНКЦІЙ, ЩО ВИЗНАЧЕНІ ОПЕРАТОРОМ РАФІДА

The purpose of the present paper is to introduce a new class of analytic functions by using the Rafid-integral operator and obtain some subordination results.

За допомогою оператора Рафіда введено новий клас аналітичних функцій. Отримано деякі результати щодо підпорядкування.

1. Introduction. Let $\mathbb{C}$ be complex plane and $\mathbb{U}=\{z\in\mathbb{C}\colon\ |z|<1\}=\mathbb{U}\backslash\{0\}$, open unit disc in $\mathbb{C}$. Let $H(\mathbb{U})$ be the class of functions analytic in $\mathbb{U}$. For $n\in N=\{1,2,3,\ldots\}$ and $a\in\mathbb{C}$, let $H[a,n]$ the subclass of $H(\mathbb{U})$ consisting of the form

$$f(z)=z+a_nz^n+a_{n+1}z^{n+1}+\ldots$$

with $H_0\equiv H[0,1],\ H\equiv H[1,1]$. Let A_n be the class of all analytic functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k \tag{1.1}$$

in the open unit disk $\mathbb{U}$ with $A_1=A$. A function $f\in H(\mathbb{U})$ is univalent if it is one to one in $\mathbb{U}$. Let S denote the subclass of A consisting of functions univalent in $\mathbb{U}$. If a function $f\in A$ maps $\mathbb{U}$ onto a convex domain and f is univalent, then f is called a convex function. Let

$$K=\left\{f\in A\colon \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,\ z\in\mathbb{U}\right\}$$

denote the class of all convex functions defined in $\mathbb{U}$ and normalized by $f(0)=0,\ f'(0)=1$.

Let f and F be members of $H(\mathbb{U})$. The function f is said to be subordinate to F, if there exists a Schwartz function w analytic in $\mathbb{U}$ with

$$w(0)=0 \qquad \text{and} \qquad |w(z)|<1, \quad z\in\mathbb{U},$$

such that

$$f(z)=F(w(z)).$$

In such a case we write

$$f(z)\prec F(z) \quad \text{or} \quad f\prec F.$$

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence [6, 12]:

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \quad \text{and} \quad f(U) \subset F(U).$$

The method of differential subordinations (also known the admissible functions method) was first introduced by Miller and Mocanu in 1978 [10] and the theory started to develop in 1981 [11]. All the details captured in a book by Miller and Mocanu in 2000 [6]. Recent years, many authors investigated properties of differential subordinations (see [1, 3, 4, 8] and others).

Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \longrightarrow \mathbb{C}$ and h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and and satisfies the second-order differential subordination

$$\Psi\left(p(z), zp'(z), zp''(z); z\right) \prec h(z), \tag{1.2}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination or more simply dominant, if $p \prec q$ for all p satisfying (1.2). A dominant q_1 satisfying $q_1 \prec q$ for all dominants q of (1.2), is said to be the best dominant of (1.2).

Recently, Athsan and Buti [6] introduced Rafid-operator of $f \in R$ for $0 \leq \mu < 1,\ 0 \leq \theta < 1$ is denoted by R_μ^θ and defined as follows:

$$R_\mu^\theta f(z) = \frac{1}{(1-\mu)^{\theta+1}\Gamma(\theta+1)} \int\limits_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)\,dt. \tag{1.3}$$

Thus, if $f \in A$ is of the form (1.1), we can obtain from (1.3) that

$$R_\mu^\theta f(z) = z + \sum_{k=2}^{\infty} L(k,\mu,\theta) a_k z^k, \tag{1.4}$$

where $L(k,\mu,\theta) = (1-\mu)^{k-1}\dfrac{\Gamma(k+\theta)}{\Gamma(\theta+1)}$.

Using the equation (1.4), it is easily seen that

$$R_\mu^\theta\left(zf'(z)\right) = z\left(R_\mu^\theta f(z)\right)'.$$

We adopte methods of [5] and introduce a new class by using Rafid-operator R_μ^θ:

Definition 1.1. *Let $\Re_{\mu,\theta}(\beta)$ be the class of functions $f \in A$ satisfying*

$$\Re\left\{\left(R_\mu^\theta f(z)\right)'\right\} > \beta,$$

where $z \in \mathbb{U},\ 0 \leq \beta < 1$ and R_μ^θ is the Rafid-operator.

In order to prove our main results we should need the following lemmas:

Lemma 1.1 [5]. *Let h be convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ be a complex number with $\Re\{\gamma\} \geq 0$. If $p \in H[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \tag{1.5}$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int\limits_0^z t^{\gamma/n-1} h(t)dt, \quad z \in \mathbb{U}.$$

The function q is convex and is the best dominant of the subordination (1.5).

Lemma 1.2 [7]. *Let $\Re\{\mu\} > 0,\ n \in \mathbb{N}$ and let*

$$w = \frac{n^2 + |\mu|^2 - \left|n^2 - \mu^2\right|}{4n\Re\{\mu\}}.$$

Let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$ and suppose that

$$\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w.$$

If

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$$

is analytic in $\mathbb{U}$ and

$$p(z) + \frac{1}{\mu} zp'(z) \prec h(z), \tag{1.6}$$

then

$$p(z) \prec q(z),$$

where q is a solution of the differential equation

$$q(z) + \frac{n}{\mu} zq'(z) = h(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{\mu}{nz^{\mu/n}} \int\limits_0^z t^{\mu/n-1} h(t)dt, \quad z \in \mathbb{U}.$$

Moreover, q is the best dominant of the differential subordination (1.6).

Lemma 1.3 [9]. *Let r be a convex function in $\mathbb{U}$ and let*

$$h(z) = r(z) + n\beta zr'(z), \quad z \in \mathbb{U},$$

where $\beta > 0$ and $n \in \mathbb{N}$. If

$$p(z) = r(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots, \quad z \in \mathbb{U},$$

is holomorphic in $\mathbb{U}$ *and*

$$p(z) + \beta z p'(z) \prec h(z), \quad z \in \mathbb{U},$$

then

$$p(z) \prec r(z),$$

and this result is sharp.

In the present paper, making use of the subordination results of [5] and [7] we will prove our main results.

2. Main results.

Theorem 2.1. *The set* $\Re_{\mu,\theta}(\beta)$ *is convex.*

Proof. Let

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k, \quad z \in \mathbb{U}, \quad j = 1, \ldots, m,$$

be in the class $\Re_{\mu,\theta}(\beta)$. Then, by the Definition 1.1, we get

$$\Re\left\{\left(R_\mu^\theta f(z)\right)'\right\} = \Re\left\{1 + \sum_{k=2}^{\infty} L(k,\mu,\theta) a_{k,j} k z^{k-1}\right\} > \beta. \tag{2.1}$$

For any positive numbers $\lambda_1,\ \lambda_2, \ldots, \lambda_m$ such that

$$\sum_{j=1}^{m} \lambda_j = 1.$$

We have to show that the function

$$h(z) = \sum_{j=1}^{m} \lambda_j f_j(z)$$

member of $\Re_{\mu,\theta}(\beta)$; that is,

$$\Re\left\{\left(R_\mu^\theta h(z)\right)'\right\} > \beta. \tag{2.2}$$

Thus, we have

$$R_\mu^\theta h(z) = z + \sum_{k=2}^{\infty} L(k,\mu,\theta) \left(\sum_{j=1}^{m} \lambda_j a_{k,j}\right) z^k. \tag{2.3}$$

If we differentiate (2.3) with respect to z, then we obtain

$$\left(R_\mu^\theta h(z)\right)' = 1 + \sum_{k=2}^{\infty} k L(k,\mu,\theta) \left(\sum_{j=1}^{m} \lambda_j a_{k,j}\right) z^{k-1}.$$

Thus, we have

$$\Re\left\{\left(R_\mu^\theta h(z)\right)'\right\} = 1 + \sum_{j=1}^{m} \lambda_j \Re\left\{\sum_{k=2}^{\infty} kL(k,\mu,\theta) a_{k,j} z^{k-1}\right\} >$$

$$> 1 + \sum_{j=1}^{m} \lambda_j (\beta - 1) \text{ (by (2.2))} = \beta.$$

Thus, the inequality (2.1) holds and we have desired result.

Theorem 2.1 is proved.

Theorem 2.2. *Let q be convex function in $\mathbb{U}$ with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{1}{\gamma+1} z q'(z), \quad z \in \mathbb{U},$$

where γ is a complex number with $\Re\{\gamma\} > -1$. If $f \in \Re_{\mu,\theta}(\beta)$ and $F = \Upsilon_\gamma f$, where

$$F(z) = \Upsilon_\gamma f(z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \tag{2.4}$$

then

$$\left(R_\mu^\theta f(z)\right)' \prec h(z) \tag{2.5}$$

implies

$$\left(R_\mu^\theta F(z)\right)' \prec q(z),$$

and this result is sharp.

Proof. From the equality (2.4), we can write

$$z^\gamma F(z) = (\gamma+1) \int_0^z t^{\gamma-1} f(t) dt. \tag{2.6}$$

By differentiating (2.6) with respect to z, we obtain

$$(\gamma) F(z) + zF'(z) = (\gamma+1) f(z),$$

and by applying the operator R_μ^θ to the last equation, then we get

$$(\gamma) R_\mu^\theta F(z) + z\left(R_\mu^\theta F(z)\right)' = (\gamma+1) R_\mu^\theta f(z). \tag{2.7}$$

If we differentiate (2.7) with respect to z, we have

$$\left(R_\mu^\theta F(z)\right)' + \frac{1}{\gamma+1} z \left(R_\mu^\theta F(z)\right)'' = \left(R_\mu^\theta f(z)\right)'. \tag{2.8}$$

By using the differential subordination given by (2.5) in the equality (2.8), we obtain

$$\left(R_{\mu}^{\theta}F(z)\right)' + \frac{1}{\gamma+1}z\left(R_{\mu}^{\theta}F(z)\right)'' \prec h(z). \tag{2.9}$$

Now, we define

$$p(z) = \left(R_{\mu}^{\theta}F(z)\right)'. \tag{2.10}$$

Then by a simple computation we get

$$p(z) = \left[z + \sum_{k=2}^{\infty} L(k,\mu,\theta)\frac{\gamma+1}{\gamma+k}a_k z^k\right]' =$$

$$= 1 + p_1 z + p_2 z^2 + \ldots, \quad p \in H[1,1].$$

Using (2.10) in the subordination (2.9), we obtain

$$p(z) + \frac{1}{\gamma+1}zp'(z) \prec h(z) = q(z) + \frac{1}{\gamma+1}zq'(z), \quad z \in \mathbb{U}.$$

If we use Lemma 1.2, we write

$$p(z) \prec q(z).$$

So we obtain the desired result and q is the best dominant.

Theorem 2.2 is proved.

***Example* 2.1.** If we choose in Theorem 2.1

$$\gamma = i+1, \quad q(z) = \frac{1+z}{1-z},$$

thus we get

$$h(z) = \frac{(i+2) - ((i+2)z+2)z}{(i+2)(1-z)^2}.$$

If $f \in \Re_{\mu,\theta}(\beta)$ and F is given by

$$F(z) = \Upsilon_i f(z) = \frac{i+2}{z^{i+1}}\int_0^z t^i f(t)dt,$$

then, by Theorem 2.2, we obtain

$$\left(R_{\mu}^{\theta}f(z)\right)' \prec h(z) = \frac{(i+2) - ((i+2)z+2)z}{(i+2)(1-z)^2} \Longrightarrow$$

$$\Longrightarrow \left(R_{\mu}^{\theta}F(z)\right)' \prec \frac{1+z}{1-z}.$$

Theorem 2.3. *Let* $\Re\{\gamma\} > -1$ *and let*

$$w = \frac{1 + |\gamma + 1|^2 - \left|\gamma^2 + 2\gamma\right|}{4\Re\{\gamma + 1\}}.$$

Let h *be an analytic function in* $\mathbb{U}$ *with* $h(0) = 1$ *and suppose that*

$$\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w.$$

If $f \in \Re_{\mu,\theta}(\beta)$ *and* $F = \Upsilon_{\gamma}^{\theta} f$, *where* F *is defined by* (2.4). *Then*

$$\left(R_{\mu}^{\theta} f(z)\right)' \prec h(z) \tag{2.11}$$

implies

$$\left(R_{\mu}^{\theta} F(z)\right)' \prec q(z),$$

where q *is the solution of the differential equation*

$$h(z) = q(z) + \frac{1}{\gamma + 1} zq'(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{\gamma + 1}{z^{\gamma+1}} \int\limits_0^z t^{\gamma} f(t)dt.$$

Moreover, q *is the best dominant of the subordination* (2.11).

Proof. If we choose $n = 1$ and $\mu = \gamma + 1$ in Lemma 1.2, then the proof is hold by means of the Theorem 2.2.

Theorem 2.4. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad 0 \leq \beta < 1, \tag{2.12}$$

be convex in $\mathbb{U}$, *with* $h(0) = 1$ *and* $0 \leq \beta < 1$. *If* $f \in A$ *and verifies the differential subordination*

$$\left(R_{\mu}^{\theta} f(z)\right)' \prec h(z),$$

then

$$\left(R_{\mu}^{\theta} F(z)\right)' \prec q(z) = (2\beta - 1) + \frac{2(1 - \beta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}}.$$

Where τ *is given by*

$$\tau(\gamma) = \int\limits_0^z \frac{t^{\gamma}}{t + 1} dt \tag{2.13}$$

and F *given by equation* (2.4). *The function* q *is convex and is the best dominant.*

Proof. If

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad 0 \leq \beta < 1,$$

then h is convex and, by means of Theorem 2.3, we have

$$\left(R_{\mu}^{\theta} F(z)\right)' \prec q(z).$$

By using Lemma 1.1, we get

$$q(z) = \frac{\gamma + 1}{z^{\gamma+1}} \int\limits_0^z t^{\gamma} h(t) dt = \frac{\gamma + 1}{z^{\gamma+1}} \int\limits_0^z t^{\gamma} \left[\frac{1 + (2\beta - 1)t}{1 + t}\right] dt =$$

$$= (2\beta - 1) + \frac{2(1 - \beta)(\gamma + 1)}{z^{\gamma+1}} \tau(\gamma).$$

Where τ is given by (2.13), so we obtain

$$\left(R_{\mu}^{\theta} F(z)\right)' \prec q(z) = (2\beta - 1) + \frac{2(1 - \beta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}}.$$

The function q is convex and is the best dominant.

Theorem 2.4 is proved.

Theorem 2.5. *If* $0 \leq \beta < 1,\ 0 \leq \mu < 1,\ \delta \geq 0,\ \Re\{\gamma\} > -1$ *and* $\mathcal{F} = \Upsilon_{\gamma} f$ *is defined by* (2.4), *then we have*

$$\Upsilon_{\gamma}\left(\Re_{\mu,\theta}(\beta)\right) \subset \Re_{\mu,\theta}(\rho),$$

where

$$\rho = \min_{|z|=1} \Re\{q(z)\} = \rho(\gamma, \beta) = (2\beta - 1) + 2(1 - \beta)(\gamma + 1)\tau(\gamma) \tag{2.14}$$

and τ *is given by* (2.13).

Proof. Let h is given by the equation (2.12), $f \in \Re_{\mu,\theta}(\beta)$ and $\mathcal{F} = \Upsilon_{\gamma} f$ is defined by (2.4). Then h is convex and by Theorem 2.3, we deduce

$$\left(R_{\mu}^{\theta} F(z)\right)' \prec q(z) = (2\beta - 1) + \frac{2(1 - \beta)(\gamma + 1)\tau(\gamma)}{z^{\gamma+1}}, \tag{2.15}$$

where τ is given by (2.13). Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis and $\Re\{\gamma\} > -1$, we have

$$\Re\left\{\left(R_{\mu}^{\theta} F(z)\right)'\right\} \geq \min_{|z|=1} \Re\{q(z)\} = \Re\{q(1)\} = \rho(\gamma, \beta) =$$

$$= (2\beta - 1) + 2(1 - \beta)(\gamma + 1)(1 - \beta)\tau(\gamma).$$

From the inequality (2.15), we get

$$\Upsilon_{\gamma}\left(\Re_{\mu,\theta}(\beta)\right) \subset \Re_{\mu,\theta}(\rho),$$

where ρ is given by (2.14).

Theorem 2.5 is proved.

Theorem 2.6. *Let q be a convex function with $q(0) = 1$ and h a function such that*

$$h(z) = q(z) + zq'(z), \quad z \in \mathbb{U}.$$

If $f \in A$, then the subordination

$$(R_\mu^\theta f(z))' \prec h(z) \tag{2.16}$$

implies that

$$\frac{R_\mu^\theta f(z)}{z} \prec q(z),$$

and the result is sharp.

Proof. Let

$$p(z) = \frac{R_\mu^\theta f(z)}{z}. \tag{2.17}$$

Differentiating (2.17), we have

$$(R_\mu^\theta f(z))' = p(z) + zp'(z).$$

If we calculate $p(z)$, then we obtain

$$p(z) = \frac{R_\mu^\theta f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} L(k, \mu, \theta) a_k z^k}{z} =$$

$$= 1 + p_1 z + p_2 z^2 + \ldots, \quad p \in H[1, 1]. \tag{2.18}$$

Using (2.18) in the subordination (2.16) we get

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z).$$

Hence by applying Lemma 1.3, we conclude that

$$p(z) \prec q(z)$$

that is,

$$\frac{R_\mu^\theta f(z)}{z} \prec q(z),$$

and this result is sharp and q is the best dominant.

Theorem 2.6 is proved.

***Example* 2.2.** If we take $\mu = 0, \theta = 1$ in equality (1.4) and $q(z) = \dfrac{1}{1 - z}$ in Theorem 2.6, then

$$h(z) = \frac{1}{(1 - z)^2}$$

and

$$R_0^1 f(z) = z + \sum_{k=2}^{\infty} \Gamma(k+1) a_k z^k. \tag{2.19}$$

Differentiating (2.19) with respect to z, we get

$$\left(R_0^1 f(z)\right)' = 1 + \sum_{k=2}^{\infty} \Gamma(k+1) k a_k z^{k-1} =$$

$$= 1 + p_1 z + p_2 z^2 + \ldots, \quad p \in H[1,1].$$

By using Theorem 2.6, we have

$$\left(R_0^1 f(z)\right)' \prec h(z) = \frac{1}{(1-z)^2}$$

implies

$$\frac{R_0^1 f(z)}{z} \prec q(z) = \frac{1}{1-z}.$$

Theorem 2.7. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1+z}, \quad z \in \mathbb{U},$$

be convex in $\mathbb{U}$, *with* $h(0) = 1$ *and* $0 \leq \beta < 1$. *If* $f \in A$ *satisfies the differential subordination*

$$(R_\mu^\theta f(z))' \prec h(z), \tag{2.20}$$

then

$$\frac{R_\mu^\theta f(z)}{z} \prec q(z) = (2\beta - 1) + \frac{2(1-\beta)\ln(1+z)}{z}.$$

The function q is convex and is the best dominant.

Proof. Let

$$p(z) = \frac{R_\mu^\theta f(z)}{z} = 1 + p_1 z + p_2 z^2 + \ldots, \quad p \in H[1,1]. \tag{2.21}$$

Differentiating (2.21), we have

$$(R_\mu^\theta f(z))' = p(z) + z p'(z). \tag{2.22}$$

Using (2.22), the differential subordination (2.20) becomes

$$(R_\mu^\theta f(z))' \prec h(z) = \frac{1 + (2\beta - 1)z}{1+z}.$$

By using Lemma 1.1, we deduce

$$p(z) \prec q(z) = \frac{1}{z} \int h(t)dt = (2\beta - 1) + \frac{2(1-\beta)\ln(1+z)}{z}.$$

Using the relation (2.21) we obtain desired result.

Theorem 2.7 is proved.

Corollary. *If* $f \in \Re_{\mu,\theta}(\beta)$, *then*

$$\Re\left(\frac{R_{\mu}^{\theta}f(z)}{z}\right) > (2\beta-1)+2(1-\beta)\ln(2)\,.$$

Proof. If $f \in \Re_{\mu,\theta}(\beta)$, then from Definition 1.1

$$\Re\left\{\left(R_{\mu}^{\theta}f(z)\right)'\right\} > \beta, \quad z \in \mathbb{U},$$

which is equivalent to

$$(R_{\mu}^{\theta}f(z))' \prec h(z) = \frac{1+(2\beta-1)z}{1+z}.$$

Using Theorem 2.7, we have

$$\frac{R_{\mu}^{\theta}f(z)}{z} \prec q(z) = (2\beta-1) + \frac{2(1-\beta)\ln(1+z)}{z}.$$

Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis, we deduce that

$$\Re\left(\frac{R_{\mu}^{\theta}f(z)}{z}\right) > \Re q(1) = (2\beta-1)+2(1-\beta)\ln(2)\,.$$

Theorem 2.8. *Let* q *be a convex function such that* $q(0)=1$ *and let* h *be the function*

$$h(z) = q(z) + zq'(z), \quad z \in \mathbb{U}.$$

If $f \in A$ *and verifies the differential subordination*

$$\left(\frac{zR_{\mu}^{\theta}f(z)}{R_{\mu}^{\theta}F(z)}\right)' \prec h(z), \quad z \in \mathbb{U}, \tag{2.23}$$

then

$$\frac{R_{\mu}^{\theta}f(z)}{R_{\mu}^{\theta}F(z)} \prec q(z), \quad z \in \mathbb{U},$$

and this result is sharp.

Proof. For the function $f \in A$, given by the equation (1.1), we have

$$R_{\mu}^{\theta}F(z) = z + \sum_{k=2}^{\infty} L(k,\mu,\theta)\frac{\gamma+1}{k+\gamma}a_k b_k z^k, \quad z \in \mathbb{U}.$$

Let us consider

$$p(z) = \frac{R_{\mu}^{\theta}f(z)}{R_{\mu}^{\theta}F(z)} = \frac{z+\sum_{k=2}^{\infty}L(k,\mu,\theta)a_k b_k z^k}{z+\sum_{k=2}^{\infty}L(k,\mu,\theta)\dfrac{\gamma+1}{k+\gamma}a_k b_k z^k} =$$

$$= \frac{1 + \sum_{k=2}^{\infty} L(k, \mu, \theta) a_k b_k z^{k-1}}{1 + \sum_{k=2}^{\infty} L(k, \mu, \theta) \dfrac{\gamma + 1}{k + \gamma} a_k b_k z^{k-1}}.$$

We get

$$(p(z))' = \frac{(R_\mu^\theta f(z))'}{R_\mu^\theta F(z)} - p(z) \frac{(R_\mu^\theta F(z))'}{R_\mu^\theta F(z)}.$$

Then

$$p(z) + zp'(z) = \left(\frac{z R_\mu^\theta f(z)}{R_\mu^\theta F(z)} \right)', \quad z \in \mathbb{U}. \tag{2.24}$$

Using the relation (2.24) in the inequality (2.23), we obtain

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z)$$

and, by using Lemma 1.3,

$$p(z) \prec q(z),$$

that is,

$$\frac{R_\mu^\theta f(z)}{R_\mu^\theta F(z)} \prec q(z).$$

Theorem 2.8 is proved.

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