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NUMERICAL SOLUTIONS OF FRACTIONAL SYSTEM, TWO-POINT BVPS USING ITERATIVE REPRODUCING KERNEL ALGORITHM

ЧИСЕЛЬНИЙ РОЗВ'ЯЗОК ДРОБОВИХ СИСТЕМ ДВОТОЧКОВИХ ГРАНИЧНИХ ЗАДАЧ ЗА ДОПОМОГОЮ ІТЕРАТИВНОГО ВІДНОВЛЮЮЧОГО ЯДЕРНОГО АЛГОРИТМУ

We propose an efficient computational method, namely, the iterative reproducing kernel method for the approximate solution of fractional-order systems of two-point time boundary-value problems in the Caputo sense. Two extended inner-product spaces are constructed in which the boundary conditions of the systems are satisfied. The reproducing kernel functions are constructed to get an accurate algorithm for the investigation of fractional systems. The developed procedure is based on generating the orthonormal basis with an aim to formulate the solution throughout the evolution of the algorithm. The analytic solution is represented in the form of a series in the reproducing kernel Hilbert space with readily computed components. In this connection, some numerical examples are presented to show the good performance and applicability of the developed algorithm. The numerical results indicate that the proposed algorithm is a powerful tool for the solution of fractional models arising in different fields of sciences and engineering.

Запропоновано ефективний обчислювальний метод, а саме ітеративний відновлюючий ядерний алгоритм для наближеного розв'язування систем дробового порядку для двоточкових часових граничних задач у сенсі Капуто. Побудовано два розширені гільбертові простори, в яких виконуються граничні умови для систем. Також побудовано відновлювальні ядерні функції, щоб отримати точний алгоритм для вивчення дробових систем. Розроблена процедура базується на генерації ортонормального базису з метою формулювання розв'язку для всієї еволюції алгоритму. Аналітичний розв'язок представлено у вигляді ряду у відновлювальному ядерному просторі Гільберта з компонентами, що легко обчислюються. У зв'язку з цим ми наводимо деякі чисельні приклади, щоб продемонструвати гарну роботу та застосовність розробленого алгоритму. Чисельні результати показують, що даний алгоритм є потужним інструментом для розв'язування дробових моделей, які з'являютъся в різних областях науки і техніки.

1. Introduction. Recently, fractional differential equations received increasing attention as a superb tool for modeling many problems in different fields of sciences and engineering. This concept is not unique and there exist several definitions of fractional-order derivative including Grunwald – Letnikov's definition, Riemann – Liouville's definition, Caputo's definition, and Riesz's definition. This generalized calculus is an extension of the classical calculus theory of noninteger order [1 – 5]. On the other hand, the fractional derivatives supply a popularity implement for the definition of memory and hereditary characteristics that involve the whole history of the function in a weighted form. In this sense, the FDEs have a nonlocal property, which means that the next state of the system depends not only upon the current state but also upon the history of all previous states. This is the fundamental advantage of using FDEs compared with classical integer-order counterpart. Therefore, there has been increasing interest in the subject of a fractional calculus which can give a more realistic interpretation of natural phenomena. Moreover, several systems in interdisciplinary fields can be described by FDEs including turbulence, signal processing, and quantum evolution. In spite of this, most of nonlinear fractional systems do not have closed form solutions, so analytical and numerical methods must be used.

The purpose of this study is to investigate and implement a computational iterative technique, the reproducing kernel method (RKM), in finding approximate solutions for a certain class of fractional system, two-point BVPs in Caputo sense. More specifically, we consider system of differential equations of fractional-order in the following form:

$$D^{\alpha_i} u_i(t) = f_i\big(t, u_i(t), u_i'(t)\big), \quad 0 < t < T, \tag{1}$$

associated with two-point boundary conditions

$$u_i(0) = a_i, \qquad u(T) = b_i, \quad i = 1, 2, \ldots, N, \tag{2}$$

where $a_i, b_i \in \mathbb{R}$, $1 < \alpha_i \leq 2$, D^{α_i} denotes the Caputo fractional derivative of order α_i, $i = 1, 2, \ldots, N$, $f_i(t, u_i, u_i') \in \mathcal{W}_1[0, T]$, $i = 1, 2, \ldots, N$, are sufficiently analytical given functions such that BVPs (1) and (2) satisfies the existence and uniqueness of the solutions, and $u_i \in \mathcal{W}_3[0, T]$ are unknown functions to be determined.

The RKM was developed as an efficient numerical method for treating different kind of singular differential equations [6, 7], integral equations [8], integrodifferential equations [9 – 14], and fuzzy differential equations [15 – 17]. It is an alternative process for getting analytic Taylor series solution. It has been successfully put into practiced to handle the approximate solution of periodic boundary-value problems [18, 19], the approximate solution of MHD squeezing fluid flow [20], the solution of difference equations [21], Duffing equations with integral boundary conditions [22], and parabolic problems with nonclassical conditions [23]. While the numerical solvability for different categories can be found in [24 – 26].

The present analysis extends the application of the RKM for obtaining approximate solutions of FDE system in Caputo sense. The structure of the present article is as follows. In Section 2, we utilized some necessary definitions and results from the fractional calculus theory. In Section 3, theoretical and analytical basis with representation of solutions are introduced in Hilbert space. In Section 4, numerical examples are simulated to show the reasonableness of the theory and to demonstrate the high performance of the proposed method. Finally, some conclusions are summarized in the last section.

2. Background and preliminaries. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary-order that generalizes the notions of integer-order differentiation and integration. Herein, we adopt the Caputo fractional derivative sense which is a modification of Riemann – Liouville sense because the initial conditions that defined during the formulation of the system are similar to those conventional conditions of integer-order. In this section, the main descriptions and features of the fractional calculus theory are illustrated. For more details about the mathematical properties of FDEs, we refer to [2 – 5].

Definition 1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $\rho > \mu$ such that $f(x) = x^\rho f_1(x)$, where $f_1(x)$ is continuous in $[0, \infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(x) \in C_\mu$, $n \in \mathbb{N}$.*

Definition 2. *The Riemann – Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as*

$$J_s^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_s^x (x - \xi)^{\alpha - 1} f(\xi)\, d\xi, \qquad \alpha > 0, \quad x > s \geq 0,$$

$$J_s^0 f(x) = f(x),$$

where Γ *is the well-known Gamma-function.*

The operator J_s^α has the following properties: for $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta > 0$, $x > s \geq 0$, $c \in \mathbb{R}$, and $\gamma > -1$, one can get $J_s^\alpha J_s^\beta f(x) = J_s^{\alpha+\beta} f(x) = J_s^\beta J_s^\alpha f(x)$, $J_s^\alpha c = \dfrac{c}{\Gamma(\alpha+1)}(x-s)^\alpha$, and $J_s^\alpha (x-s)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}(x-s)^{(\alpha+\gamma)}$.

The Riemann – Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Thus, we shall introduce a modified fractional differential operator D_s^α proposed by Caputo in his work on the theory of viscoelasticity.

Definition 3. *The Riemann – Liouville fractional derivative of order* $\alpha > 0$ *of* $f \in C_{-1}^n$, $n \in \mathbb{N}$, *is defined as*

$$\widehat{D_s^\alpha} f(x) = \begin{cases} \dfrac{d^n}{dx^n} J_s^{n-\alpha} f(x), & n-1 < \alpha < n, \quad x > s \geq 0, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n. \end{cases}$$

Definition 4. *The Caputo fractional derivative of order* $\alpha > 0$ *of* $f \in C_{-1}^n$, $n \in \mathbb{N}$, *is defined as*

$$D_s^\alpha f(x) = \begin{cases} J_s^{n-\alpha} f^{(n)}(x), & n-1 < \alpha < n, \quad x > s \geq 0, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n. \end{cases}$$

***Remark* 1.** For $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $x > s \geq 0$, and $f \in C_{-1}^n$, one can get

$$J_s^\alpha D_s^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(s^+) \frac{(x-s)^k}{k\,!},$$

$$D_s^\alpha J_s^\alpha f(x) = f(x).$$

The operator D_s^α has the following properties: for $f \in C_{-1}^n$, $\alpha > 0$, $x > s \geq 0$, $c \in \mathbb{R}$, and $\gamma > -1$, one can get $D_s^\alpha c = 0$, and $D_s^\alpha (x-s)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(x-s)^{(\gamma-\alpha)}$.

3. Theoretical and analytical basis of the method. In this section, we construct a representation solution for fractional system associated to given boundary conditions, in which the solution provided in terms of a rapidly convergent series in the reproducing kernel space with components that can be elegantly computed.

Definition 5. *Let* $\mathcal{H}$ *be a Hilbert space of function* $\mathcal{F}: \Omega \to \mathcal{H}$ *on a set* Ω. *A function* $K: \Omega \times \Omega \to \mathbb{R}$ *is a reproducing kernel of* $\mathcal{H}$ *if the following conditions are satisfied: firstly,* $K(\cdot, \tau) \in \mathcal{H}$ *for each* $\tau \in \Omega$; *secondly,* $\big\langle \mathcal{F}(\cdot), K(\cdot, \tau) \big\rangle = \mathcal{F}(\tau)$ *for each* $\mathcal{F} \in \mathcal{H}$ *and each* $\tau \in \Omega$.

Definition 6. *The reproducing kernel Hilbert space* $\mathcal{W}_1[0,T]$ *is defined as* $\mathcal{W}_1[0,T] = \{u(t)$ *is one-variable absolutely continuous real-valued function on* $[0,T]$ *and* $u'(t) \in L^2[0,T]\}$. *The inner product and the norm of* $\mathcal{W}_1[0,T]$ *are given, respectively, by*

$$\big\langle u_1(t), u_2(t) \big\rangle_{\mathcal{W}_1} = u_1(0)u_2(0) + \int_0^T u_1'(\xi) u_2'(\xi) d\xi,$$

and $\|u(t)\|_{\mathcal{W}_1}^2 = \langle u(t), u(t) \rangle_{\mathcal{W}_1}$, *where* $u_1, u_2 \in \mathcal{W}_1[0,T]$.

***Remark* 2.** The Hilbert space $\mathcal{W}_m[a,b]$, is called a reproducing kernel if for each fixed t in $[a,b]$, there exist a function $K_t(s) \in \mathcal{W}_m[a,b]$ such that $\langle u(s), K_t(s)\rangle_{\mathcal{W}_m} = u(t)$ for any $u(s) \in \mathcal{W}_m[a,b]$ and $s \in [a,b]$.

Theorem 1. *The Hilbert space $\mathcal{W}_1[0,T]$ is a complete reproducing kernel with the reproducing kernel function*

$$\mathcal{R}_t(s) = \begin{cases} 1+s, & s \le t, \\ 1+t, & s > t. \end{cases}$$

Anyhow, to solve system (1) and (2) by using the RKM, it is necessary to construct the reproducing kernel space $\mathcal{W}_3[0,T]$ in which every function satisfies the homogenous boundary conditions $u(0) = 0$ and $u(T) = 0$.

Definition 7. *The reproducing kernel Hilbert space $\mathcal{W}_3[0,T]$ is defined as $\mathcal{W}_3[0,T] = \{u(t) :$ u, u', u'' are one-variable absolutely continuous real-valued functions on $[0,T]$ and $u'''(t) \in L^2[0,T]$, $u(0) = u(T) = 0\}$. The inner product and the norm of $\mathcal{W}_3[0,T]$ are given, respectively, by*

$$\big\langle u_1(t), u_2(t)\big\rangle_{\mathcal{W}_3} = \sum_{i=0}^{1} u_1^{(i)}(0)u_2^{(i)}(0) + u_1(T)u_2(T) + \int_0^T u_1'''(\xi)u_2'''(\xi)\,d\xi, \tag{3}$$

and $\|u(t)\|^2_{\mathcal{W}_3} = \langle u(t), u(t)\rangle_{\mathcal{W}_3}$, where $u_1, u_2 \in \mathcal{W}_3[0,T]$.

Lemma 1. *The Hilbert space $\mathcal{W}_3[0,T]$ is a complete reproducing kernel with the reproducing kernel function*

$$Q_t(s) = \begin{cases} \sum_{i=0}^{6} p_i(t)s^{i-1}, & s \le t, \\ \sum_{i=0}^{6} q_i(t)s^{i-1}, & t > s, \end{cases}$$

where the unknown coefficients $p_i(t)$ and $q_i(t)$, $i = 1, \ldots, 6$, can be uniquely obtained by utilizing the following assumptions:

$$\begin{aligned} &Q_t(0) = 0, \qquad Q_t(T) = 0, \qquad Q_t^{(3)}(0) = 0, \\ &Q_t^{(i)}(T) = 0, \quad i = 3,4, \qquad Q_t'(0) + Q_t^{(4)}(0) = 0, \\ &Q_t^{(i)}(s^+) = Q_t^{(i)}(s^-), \quad i = 0,1,\ldots,4, \\ &Q_t^{(5)}(s^+) - Q_t^{(5)}(s^-) = -1. \end{aligned} \tag{4}$$

Consequently, by using the Mathematica for handling the above-mentioned generalized differential equations (4), the reproducing kernel function is given by

$$Q_t(s) = \begin{cases} \dfrac{s}{120T^2}\left[\begin{array}{l} -6T^3t^2s + t^2s(-120 + t^3 + s^3) - 5Tt(-24s + t^3s + \\ \qquad +t(-24 + s^3)) + T^2(10t^3s - s^4 + 5t(-24 + s^3))) \end{array}\right], & s \le t, \\ \dfrac{-t}{120T^2}\left[\begin{array}{l} 6T^3ts^2 - ts^2(-120 + t^3 + s^3) + T^2(t^4 + 120s - \\ \qquad -5t^3s - 10ts^3) + 5Ts(-24s + t^3s + t(-24 + s^3)) \end{array}\right], & s > t. \end{cases} \tag{5}$$

Herein, it is worth to mention that $\{u_n(t)\}_{n=1}^{\infty}$ is a compact subset of the space $C[0,T]$, which means that $\{u_n(t)\}_{n=1}^{\infty}$ are equicontinuous functions. To see this, use the property of $Q_t(s)$ such

that

$$|u_n(t_2) - u_n(t_1)| = \left|\langle u(s), Q_{t_2}(s) - Q_{t_1}(s)\rangle_{\mathcal{W}_3}\right| \leq$$

$$\leq \|u(s)\|_{\mathcal{W}_3} \|Q_{t_2}(s) - Q_{t_1}(s)\|_{\mathcal{W}_3} \leq M \|Q_{t_2}(s) - Q_{t_1}(s)\|_{\mathcal{W}_3}.$$

By "Mean-value theorem of differentials" and the symmetry of $Q_t(s)$, it follows that

$$\left|Q_{t_2}(s) - Q_{t_1}(s)\right| = |Q_s(t_2) - Q_s(t_1)| = \left|\frac{d}{dt}Q_s(t)\right|_{t=\tau} |t_2 - t_1| \leq N|t_2 - t_1|.$$

Thus, if $\gamma \leq |t_2 - t_1| \leq \dfrac{\epsilon}{MN}$, then $|u_n(t_2) - u_n(t_1)| < \epsilon$.

In order to illustrate the RKHS methodology of the proposed model, we consider the differential operator $L_{\mathcal{W}} : \mathcal{W}_3[0,T] \to \mathcal{W}_1[0,T]$ such that $L_{\mathcal{W}}u(t) = D^{\alpha}u(t)$. Then, BVPs (4) and (5) can be equivalently converted into the form

$$\begin{aligned} &L_{\mathcal{W}}u_i(t) = f_i\left(t, u_i(t), u_i'(t)\right), \\ &u_i(0) = 0, \qquad u_i(T) = 0, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{6}$$

where $u_i(t) \in \mathcal{W}_3[0,T]$ and $f_i(\tau, u_i, v_i) \in \mathcal{W}_1[0,T]$ as $u_i = u_i(\tau)$, $v_i = u_i'(\tau) \in \mathcal{W}_3[0,T]$, $\tau \in [0,T]$.

Let $\varphi_i(t) = Q_{t_i}(t)$ and $\psi_i(t) = L^*_{\mathcal{W}}\varphi_i(t)$, where $\{t_i\}_{i=1}^{\infty}$ is countable dense subset of $[0,T]$, and $L^*_{\mathcal{W}}$ is the adjoint operator of $L_{\mathcal{W}}$. Thus, in terms of the properties of reproducing-kernel, it holds $\langle u(t), \psi_i(t)\rangle_{\mathcal{W}_3} = \langle u(t), L^*_{\mathcal{W}}\varphi_i(t)\rangle_{\mathcal{W}_3} = \langle L_{\mathcal{W}}u(t), \varphi_i(t)\rangle_{\mathcal{W}_1} = L_{\mathcal{W}}u(t_i), i = 1, 2, \ldots$.

Lemma 2. *The operator $L_{\mathcal{W}} : \mathcal{W}_3[0,T] \to \mathcal{W}_1[0,T]$ is a bounded linear operator.*

Proof. It is so easy to see that $L_{\mathcal{W}}$ is a linear operator. Thus, it is enough to show that $L_{\mathcal{W}}$ is bounded operator. From the space $\mathcal{W}_1[0,T]$, we have

$$\|L_{\mathcal{W}}u(t)\|^2_{\mathcal{W}_1} = \|D^{\alpha}u(t)\|^2_{\mathcal{W}_1} = \langle D^{\alpha}u(t), D^{\alpha}u(t)\rangle_{\mathcal{W}_1} = [D^{\alpha}u(0)]^2 + \int_0^T \left[\frac{d}{d\xi}D^{\alpha}u(\xi)\right]^2 d\xi.$$

By reproducing property of $Q_t(s)$ and since $D^{\alpha}Q_t(s)$ is uniformly bounded about t and s, we obtain $u(t) = \left\langle u(s), Q_t(s)\right\rangle_{\mathcal{W}_3}$, $L_{\mathcal{W}}u(t) = \left\langle u(s), D^{\alpha}Q_t(s)\right\rangle_{\mathcal{W}_3}$ and $\dfrac{d}{dt}L_{\mathcal{W}}u(t) = \left\langle u(s), \dfrac{d}{dt}D^{\alpha}Q_t(s)\right\rangle_{\mathcal{W}_3}$. By Schwarz inequality, we get

$$|L_{\mathcal{W}}u(t)| = \left|\left\langle u(s), D^{\alpha}Q_t(s)\right\rangle_{\mathcal{W}_3}\right| \leq \|u\|_{\mathcal{W}_3}\|D^{\alpha}Q_t(s)\|_{\mathcal{W}_3} = \mu_1\|u\|_{\mathcal{W}_3}$$

and

$$\left|\frac{d}{dt}L_{\mathcal{W}}u(t)\right| = \left|\left\langle u(s), \frac{d}{dt}D^{\alpha}Q_t(s)\right\rangle_{\mathcal{W}_3}\right| \leq \|u\|_{\mathcal{W}_3}\left\|\frac{d}{dt}D^{\alpha}Q_t(s)\right\|_{\mathcal{W}_3} = \mu_2\|u\|_{\mathcal{W}_3},$$

where μ_1 and μ_2 are positive constants.

Thus $\left[D^{\alpha}u(0)\right]^2 \leq \mu_1^2\|u\|^2_{\mathcal{W}_3}$ and $\displaystyle\int_0^T \left[\frac{d}{d\xi}D^{\alpha}u(\xi)\right]^2 d\xi \leq T\mu_2^2\|u\|^2_{\mathcal{W}_3}$. Hence, $\|L_{\mathcal{W}}u(t)\|_{\mathcal{W}_1} \leq$ $\leq \mu\|u(t)\|_{\mathcal{W}_3}$, where $\mu = \sqrt{\mu_1^2 + T\mu_2^2}$.

Lemma 2 is proved.

Theorem 2. *Let* $\{t_i\}_{i=1}^{\infty}$ *be a dense subset of the interval* $[0,T]$, *then the sequence* $\left\{\psi_i^{\upsilon}(t)\right\}_{i=1}^{\infty}$, $\upsilon = 1,2,\ldots,N$, *is a complete function system of the space* $\mathcal{W}_3[0,T]$ *such that* $\psi_i^{\upsilon}(t) = D_s^{\alpha} Q_t(s)|_{s=t_i}$.

Proof. From reproducing property of $Q_t(s)$, we get

$$\psi_i^{\upsilon}(t) = L_{\mathcal{W}}^{*}\varphi_i^{\upsilon}(t) = \left\langle L_{\mathcal{W}}^{*}\varphi_i^{\upsilon}(s), Q_t(s)\right\rangle_{\mathcal{W}_3} =$$

$$= \left\langle \varphi_i^{\upsilon}(s), L_{\mathcal{W}} Q_t(s)\right\rangle_{\mathcal{W}_1} = L_{\mathcal{W}} Q_t(t_i) = D_s^{\alpha} Q_t(s)\big|_{s=t_i}.$$

Since $\{t_i\}_{i=1}^{\infty}$ is dense in the interval $[0,T]$. For each $u_{\upsilon}(t)$ in $\mathcal{W}_3[0,T]$, if $\left\langle u_{\upsilon}(t), \psi_i^{\upsilon}(t)\right\rangle_{\mathcal{W}_3} = \left\langle L_{\mathcal{W}} u_{\upsilon}(t), \varphi_i^{\upsilon}(t)\right\rangle_{\mathcal{W}_1} = L_{\mathcal{W}} u_{\upsilon}(t_i) = 0$, $i = 1,2,\ldots$, then from the density of $\{t_i\}_{i=1}^{\infty}$ and continuity of $u_{\upsilon}(t)$, $\upsilon = 1,2,\ldots,N$, we have $u_{\upsilon}(t) = 0$.

Theorem 2 is proved.

The reproducing kernel solution will be obtained by calculating a truncated series based on the orthonormal functions $\left\{\bar{\psi}_i^{\upsilon}(t)\right\}_{i=1}^{\infty}$ of the space $\mathcal{W}_3[0,T]$, which is constructed from $\{\psi_i^{\upsilon}(t)\}_{i=1}^{\infty}$ by using the Gram – Schmidt process such that

$$\bar{\psi}_i^{\upsilon}(t) = \sum_{k=1}^{i} \mu_{ik}^{\upsilon} \psi_k^{\upsilon}(t), \tag{7}$$

where μ_{ik}^{υ} are orthogonalizatio coefficients, $\mu_{ii}^{\upsilon} > 0$, $i = 1,2,\ldots,n$.

Theorem 3. *If* $\{t_i\}_{i=1}^{\infty}$ *is dense on the interval* $[0,1]$ *and* $u_{\upsilon}(t) \in \mathcal{W}_3[0,T]$ *is a unique solution of Eq.* (6), *then the exact solution could be represented by*

$$u_{\upsilon}(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \mu_{ik}^{\upsilon} f_{\upsilon}\left(t_k, u_{\upsilon}(t_k), u_{\upsilon}'(t_k)\right) \bar{\psi}_i^{\upsilon}(t). \tag{8}$$

Proof. For each $u_{\upsilon}(t) \in \mathcal{W}_3[0,T]$, the series $\sum_{i=1}^{\infty} \left\langle u_{\upsilon}(t), \bar{\psi}_i^{\upsilon}(t)\right\rangle_{\mathcal{W}_3} \bar{\psi}_i^{\upsilon}(t)$ is convergent. From the Fourier series expansion, $u_{\upsilon}(t)$ can be written as follows:

$$u_{\upsilon}(t) = \sum_{i=1}^{\infty} \left\langle u_{\upsilon}(t), \bar{\psi}_i^{\upsilon}(t)\right\rangle_{\mathcal{W}_3} \bar{\psi}_i^{\upsilon}(t) = \sum_{i=1}^{\infty} \left\langle v(t), \sum_{k=1}^{i} \mu_{ik}^{\upsilon} \psi_k^{\upsilon}(t)\right\rangle_{\mathcal{W}_3} \bar{\psi}_i(t) =$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \mu_{ik}^{\upsilon} \left\langle u_{\upsilon}(t), L_{\mathcal{W}}^{*}\varphi_k^{\upsilon}(t)\right\rangle_{\mathcal{W}_3} \bar{\psi}_i^{\upsilon}(t) =$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \mu_{ik}^{\upsilon} \left\langle L_{\mathcal{W}} u_{\upsilon}(t), \varphi_k^{\upsilon}(t)\right\rangle_{\mathcal{W}_1} \bar{\psi}_i^{\upsilon}(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \mu_{ik}^{\upsilon} L_{\mathcal{W}} u_{\upsilon}(t_k) \bar{\psi}_i^{\upsilon}(t) =$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \mu_{ik}^{\upsilon} f_{\upsilon}\left(t_k, u_{\upsilon}(t_k), u_{\upsilon}'(t_k)\right) \bar{\psi}_i^{\upsilon}(t).$$

Therefore, the form of Eq. (8) is the exact solution of Eq. (6).

Theorem 3 is proved.

Since $\mathcal{W}_3[0,T]$ is a Hilbert space, the series $\sum_{i=1}^{\infty}\sum_{k=1}^{i}\mu_{ik}^{\upsilon}\langle L_{\mathcal{W}}u_{\upsilon}(t),\varphi_k^{\upsilon}(t)\rangle_{\mathcal{W}_1}\bar{\psi}_i^{\upsilon}(t)<\infty$. Hence, the truncated series

$$u_{\upsilon,n}(t)=\sum_{i=1}^{n}\sum_{k=1}^{i}\mu_{ik}^{\upsilon}f_{\upsilon}\left(t_k,u_{\upsilon}(t_k),u'_{\upsilon}(t_k)\right)\bar{\psi}_i^{\upsilon}(t) \tag{9}$$

is convergent in the sense of $\|\cdot\|_{\mathcal{W}_3[0,T]}$ and the numerical solution of Eq. (6) can be directly calculated by Eq. (9).

***Corollary* 1.** *The approximate solution $u_{\upsilon,n}(t)$ and its derivative $u_{\upsilon,n}^{(m)}(t)$, $m=1,2$, are converging uniformly to the exact solution $u_{\upsilon}(t)$ and its derivative $u_{\upsilon}^{(m)}(t)$ as $n\to\infty$, respectively.*

Proof. For any $t\in[0,T]$, it easy to see that

$$\left|u_{\upsilon,n}^{(i)}(t)-u_{\upsilon}^{(i)}(t)\right|=\left|\left\langle u_{\upsilon,n}(t)-u_{\upsilon}(t),Q_t^{(i)}(s)\right\rangle_{\mathcal{W}_3}\right|\le$$

$$\le\left\|Q_t^{(i)}(s)\right\|_{\mathcal{W}_3}\|u_{\upsilon,n}(t)-u_{\upsilon}(t)\|_{\mathcal{W}_3}\le$$

$$\le M_i^{\upsilon}\|u_{\upsilon,n}(t)-u_{\upsilon}(t)\|_{\mathcal{W}_3},\qquad M_i^{\upsilon}\in\mathbb{R},\quad i=0,1,2.$$

Hence, if $\|u_{\upsilon,n}(t)-u_{\upsilon}(t)\|_{\mathcal{W}_3}\to 0$ as $n\to\infty$, then the approximate solution $u_{\upsilon,n}^{(i)}(t), i=0,1,2$, are converge uniformly to the exact solution $u_{\upsilon}(t)$ and its derivative, respectively.

***Remark* 3.** In order to apply the IRM for solving system (6) numerically, we have the following two cases based on the structure of the function f_{υ}.

Case 1. If system (6) is linear, then the exact and approximate solutions can be obtained directly from equations (8) and (9), respectively.

Case 2. If system (6) is nonlinear, then the exact and approximate solutions can be obtained by using the following process: according to exact solution in equation (8), the representation of the solution of system (6) can be denoted by

$$u_{\upsilon}(t)=\sum_{i=1}^{\infty}B_i^{\upsilon}\bar{\psi}_i^{\upsilon}(t),$$

where $B_i^{\upsilon}=\sum_{k=1}^{i}\mu_{ik}^{\upsilon}f_{\upsilon}\left(t_k,u_{\upsilon,k-1}(t_k),u'_{\upsilon,k-1}(t_k)\right)$. So, we will approximate the unknown B_i^{υ} using the known Λ_i^{υ} as follows: set the initial data such that $u_{\upsilon,0}(t_1)=u'_{\upsilon,0}(t_1)=0$, and define the n-term approximation to $u_{\upsilon}(t)$ by

$$u_{\upsilon,n}(t)=\sum_{i=1}^{n}\Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}(t), \tag{10}$$

where the coefficients Λ_i^{υ} of $\bar{\psi}_i^{\upsilon}(t)$, $i=1,2,\ldots,n$, are given by

$$\begin{aligned}
&\Lambda_1^{\upsilon} = \mu_{11}^{\upsilon} f_{\upsilon}\left(t_1, u_{\upsilon,0}(t_1), u'_{\upsilon,0}(t_1)\right) \Longrightarrow u_{\upsilon,1}(t) = \Lambda_1^{\upsilon}\bar{\psi}_1^{\upsilon}(t),\\
&\Lambda_2^{\upsilon} = \sum_{k=1}^{2} \mu_{2k}^{\upsilon} f_{\upsilon}\left(t_k, u_{\upsilon,k-1}(t_k), u'_{\upsilon,k-1}(t_k)\right) \Longrightarrow u_{\upsilon,2}(t) = \sum_{i=1}^{2} \Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}(t),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Lambda_n^{\upsilon} = \sum_{k=1}^{n} \mu_{nk}^{\upsilon} f_{\upsilon}\left(t_k, u_{\upsilon,k-1}(t_k), u'_{\upsilon,k-1}(t_k)\right) \Longrightarrow u_{\upsilon,n}(t) = \sum_{i=1}^{n} \Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}(t).
\end{aligned} \tag{11}$$

In the iterative process of Eq. (10), we can guarantee that the numerical solution u_n satisfies the constraints conditions of Eq. (6).

For the error behavior, if $\varepsilon_n = |u_n(t) - u(t)|$, where $u_n(t)$ is given in Eq. (10). Then, one can write $\|\varepsilon_n\|^2_{\mathcal{W}_3} = \left\|\sum_{i=n+1}^{\infty} \Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}\right\|^2_{\mathcal{W}_3} = \sum_{i=n+1}^{\infty} (\Lambda_i^{\upsilon})^2$ and $\|\varepsilon_{n-1}\|^2_{\mathcal{W}_3} = \left\|\sum_{i=n}^{\infty} \Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}\right\|^2_{\mathcal{W}_3} = \sum_{i=n}^{\infty} (\Lambda_i^{\upsilon})^2$. Clearly, $\{\varepsilon_n\}_{n=1}^{\infty}$ is decreasing in the sense of $\|\cdot\|_{\mathcal{W}_3}$. Since $\sum_{i=1}^{\infty} \Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}(t)$ is convergent series, then $\|\varepsilon_n\|_{\mathcal{W}_3} \to 0$ as $n \to \infty$.

4. Applications and numerical examples. In this section, numerical examples are studied to demonstrate the performance, accuracy and applicability of the present method for both linear and nonlinear problems. Results obtained are compared with the exact solution of each example and are found to be in good agreement with each other. In the process of computation, all the symbolic and numerical computations performed by using Mathematica software package.

Algorithm 1. To approximate the solution $u_n(t)$ of $u(t)$ for Eqs. (1) and (2), do the following steps.

Step 1. Fixed t in $[0, T]$ and set $s \in [0, T]$;

if $s \le t$, let $Q_s(t) = \sum_{i=1}^{6} p_i(t) s^{i-1}$;

else let $Q_s(t) = \sum_{i=1}^{6} Q_i(t) s^{i-1}$.

Step 2. Choose n collocation points and do the following subroutine:

set $t_i = \dfrac{i-1}{N-1}$, $i = 1, 2, \ldots, N$;

set $\psi_i^{\upsilon}(t) = \mathcal{D}_s^{\alpha} Q_s(t)|_{s=t_i}$.

Step 3. Obtain the orthogonalization coefficients $\mu_{i\rho}^{\upsilon}$ as follows:

let $c_{ik}^{\nu} = \left\langle \psi_i^{\upsilon}(t), \overline{\psi}_k^{\upsilon}(t) \right\rangle_{\mathcal{W}_3}$, and do the following subroutine:

for $i = 1$, set $\mu_{11}^{\upsilon} = \|\psi_1^{\upsilon}\|^{-1}_{\mathcal{W}_3}$;

for $i = 2, \ldots, N$, set $\mu_{ii}^{\upsilon} = \left(\|\psi_i^{\upsilon}\|^2_{\mathcal{W}_3} - \sum_{k=1}^{i-1} (c_{ik}^{\upsilon})^2\right)^{-1/2}$;

else (for $\rho < i$), set $\mu_{i\rho}^{\upsilon} = -\left(\sum_{k=\rho}^{i-1} c_{ik}^{\upsilon}\mu_{k\rho}^{\upsilon}\right)\left(\|\psi_i^{\upsilon}\|_{\mathcal{W}_3}^2 - \sum_{k=1}^{i-1}(c_{ik}^{\upsilon})^2\right)^{-1/2}$.

Step 4. For $i = 1, 2, \ldots, N$, set

$\overline{\psi}_i^{\upsilon}(t) = \sum_{k=1}^{i} \mu_{ik}^{\upsilon}\psi_k^{\upsilon}(t)$.

Step 5. Set $t_1 = 0$, and choose an initial approximation $u_0(t_1) = u(t_1)$, $u_0'(t_1) = u'(t_1)$;

for $i = 1$, set $\Lambda_1^{\upsilon} = \mu_{11}^{\upsilon} f_{\upsilon}\left(t_1, u_{\upsilon,0}(t_1), u'_{\upsilon,0}(t_1)\right)$ and $u_{\upsilon,1}(t) = \Lambda_1^{\upsilon}\overline{\psi}_1^{\upsilon}$;

for $i = 2, 3, \ldots, n$, set $\Lambda_i^{\upsilon} = \sum_{k=1}^{i} \mu_{nk}^{\upsilon} f_{\upsilon}\left(t_k, u_{\upsilon,k-1}(t_k), u'_{\upsilon,k-1}(t_k)\right)$;

set $u_{\upsilon,n}(t) = \sum_{i=1}^{n} \Lambda_i^{\upsilon}\bar{\psi}_i^{\upsilon}(t)$.

Outcome: the numerical solution $u_{\upsilon,n}(t)$.

Stop.

By applying Algorithm 1 throughout the numerical computations, we present some tabulate data and graphical results that discussed quantitatively at some selected grid points on $[0, 1]$.

***Example* 1.** Consider the following linear fractional system:

$$\begin{aligned} D^{\alpha_1}u_1(t) + u_2'(t) &= \frac{1}{2}e^{\frac{t}{2}} + (2+t)e^t, \\ D^{\alpha_2}u_2(t) - u_1(t) + u_1'(t) &= \frac{1}{4}e^{\frac{t}{2}} + e^t, \end{aligned} \tag{12}$$

with two-point boundary conditions

$$\begin{aligned} u_1(0) &= 0, \qquad u_1(1) = e, \\ u_2(0) &= 1, \qquad u_2(1) = \sqrt{e}, \end{aligned} \tag{13}$$

where $1 < \alpha_i \leq 2$, $i = 1, 2$, $0 \leq t \leq 1$ and $u_1(t), u_2(t) \in \mathcal{W}_3[0, 1]$.

The exact solutions at $\alpha_1 = \alpha_2 = 2$ are $u_1(t) = te^t$ and $u_2(t) = e^{\frac{t}{2}}$. Using the proposed method, taking $t_i = \dfrac{i-1}{N-1}$, $i = 1, 2, \ldots, N$. The numerical results at some selected grid points are given in Tables 1 and 2.

***Example* 2.** Consider the following nonlinear fractional system:

$$\begin{aligned} D^{\alpha_1}u_1(t) &= u_1(t)u_2'(t) + 2e^t - e^{2t}, \\ D^{\alpha_2}u_2(t) &= \ln(u_2(t)) - 2u_1'(t) + 3e^t - 1, \\ D^{\alpha_3}u_3(t) &= u_2^2(t)u_3(t) - u_1(t) + e^{-t} - 1, \end{aligned} \tag{14}$$

with two-point boundary conditions

Table 1. Numerical results for solution $u_1(t)$ in Example 1

t	α_1			
	2	1.9	1.75	1.5
0.1	0.110518	0.110523	0.110998	0.184325
0.2	0.244283	0.244292	0.249768	0.317781
0.3	0.404961	0.404975	0.417821	0.476476
0.4	0.596735	0.596753	0.600455	0.681002
0.5	0.824367	0.824392	0.932478	0.944067
0.6	1.093281	1.093310	1.110259	1.348214
0.7	1.409642	1.409670	1.293007	1.337854
0.8	1.780443	1.780490	1.978111	1.984125
0.9	2.213665	2.213712	2.423435	2.629512
1.0	2.718286	2.718361	2.880985	2.973447

Table 2. Numerical results for solution $u_2(t)$ in Example 1

t	α_2			
	2	1.9	1.75	1.5
0.1	1.05127059	1.05128524	1.051230521	1.051415781
0.2	1.10516988	1.10517581	1.105068425	1.105237792
0.3	1.16183265	1.16185772	1.161668114	1.161423754
0.4	1.22140061	1.22143654	1.221170831	1.221227945
0.5	1.28402267	1.28410678	1.283725058	1.286784210
0.6	1.34985544	1.34987211	1.349486888	1.352745691
0.7	1.41906353	1.41912458	1.418620408	1.419824567
0.8	1.49181998	1.49190054	1.491298115	1.497587212
0.9	1.56830674	1.56843729	1.567701336	1.574500647
1.0	1.64871504	1.64887596	1.648020689	1.654788521

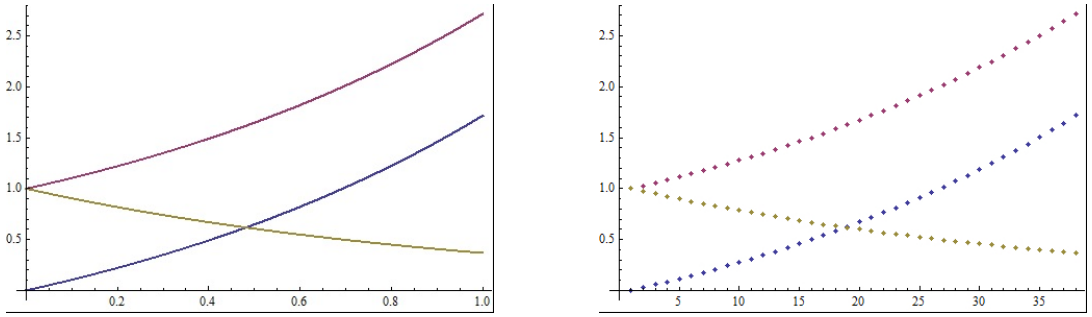
$$\begin{aligned} u_1(0) &= 0, \qquad u_1(1) = e-1, \\ u_2(0) &= 1, \qquad u_2(1) = e, \\ u_3(0) &= 1, \qquad u_3(1) = \frac{1}{e}, \end{aligned} \tag{15}$$

where $1 < \alpha_i \le 2$, $0 \le t \le 1$ and $u_i(t) \in \mathcal{W}_3[0,1]$, $i = 1,2,3$.

The exact solutions at $\alpha_1 = \alpha_2 = \alpha_3 = 2$ are $u_1(t) = e^t - 1$, $u_2(t) = e^t$ and $u_3(t) = e^{-t}$. Using the proposed method, taking $t_i = \dfrac{i-1}{N-1}$, $i = 1, 2, \ldots, N$. The numerical results at some selected grid points are given in Table 3 and Fig. 1.

Table 3. Numerical solutions and errors at $\alpha_1 = \alpha_2 = \alpha_3 = 2$, for Example 2

t_i	$u_1(t)$		$u_2(t)$		$u_3(t)$	
	Absolute Error	Relative Error	Absolute Error	Relative Error	Absolute Error	Relative Error
0.16	9.9129×10^{-7}	9.1507×10^{-7}	1.0969×10^{-6}	5.2652×10^{-7}	2.2703×10^{-6}	1.9349×10^{-6}
0.32	2.0486×10^{-6}	1.7457×10^{-6}	2.2681×10^{-6}	1.0435×10^{-6}	4.7692×10^{-6}	3.4672×10^{-6}
0.48	3.1704×10^{-6}	2.4939×10^{-6}	3.5107×10^{-6}	1.5457×10^{-6}	7.4503×10^{-6}	4.6269×10^{-6}
0.64	4.3776×10^{-6}	3.1788×10^{-6}	4.8481×10^{-6}	2.0394×10^{-6}	1.0321×10^{-5}	5.4861×10^{-6}
0.80	5.6937×10^{-6}	3.8166×10^{-6}	6.3057×10^{-6}	2.5305×10^{-6}	1.3392×10^{-5}	6.1058×10^{-6}
0.96	7.1443×10^{-6}	4.4207×10^{-6}	7.9124×10^{-6}	3.0245×10^{-6}	1.6677×10^{-5}	6.5366×10^{-6}



(*a*) (*b*)

Fig. 1. Plots of system in Example 2 at $\alpha_i = 2,\ i = 1, 2, 3,\ n = 101$: exact solutions (*a*) and approximate solutions (*b*).

The results of numerical analysis are approximate as much as is required within a logical error ratio that will be stored in a fixed number of digits. It is clear from the tables that the numerical solutions are in close agreement with the exact solutions for all examples, while the accuracy is in advanced by using only few term of the RKM iterations. This is an indication of stability of the presented method. In Fig. 1, the approximation values within a graphically plotted indicate that the solution approach smoothly to the t-axis by satisfying their boundary conditions. Indeed, decreasing the step-size increases the accuracy of the results while increasing the time required to simulate the problem.

5. Concluding remarks. The main concern of this work has been to propose an efficient numeric technique for the solutions of a class of time-fractional system in Caputo sense subjected to appropriate boundary conditions. The goal has been achieved by introducing the IRKM to solve this class of FDEs. A regularization procedure based on the reproducing kernel theory is utilized to improve the regularity and localization of the method. The behavior of approximate solution for different values of fractional-order α is shown quantitatively as well as graphically. We can conclude that the IRKM is powerful and promising technique in finding approximate solution for both linear and nonlinear problems. In the proposed algorithm, the solution $u(t)$ and the approximate solution $u_n(t)$ are represented in the form of series in $\mathcal{W}_3[0, T]$. Moreover, the approximate solution and its derivative converge uniformly to the exact solution and its derivative, respectively.

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