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# ON DARBOUX VECTOR IN LORENTZIAN 5-SPACE

# ПРО ВЕКТОР ДАРБУ В 5-ПРОСТОРІ ЛОРЕНЦА

We introduce the Darboux vector in the Lorentzian 5-space. We give some characterizations of this vector in the space. In addition, we consider some special cases in the space.

Введено вектор Дарбу в 5-просторі Лоренца. Наведено деякі харатеристики даного вектора в цьому просторі. Крім того, розглянуто деякі частинні випадки в цьому ж просторі.

**1. Introduction.** Let $X = (x_1, x_2, x_3, x_4, x_5)$ and $Y = (y_1, y_2, y_3, y_4, y_5)$ be two non-zero vectors in Lorentzian 5-space $\mathbb{L}^5$. For $X,\ Y \in \mathbb{L}^5$

$$g(X,Y) = \langle X, Y \rangle = -x_1y_1 + \sum_{i=2}^{5} x_iy_i,$$

is called *Lorentzian inner product*. The couple $\{\mathbb{L}^5, \langle,\rangle\}$ is called *Lorentzian* 5*-space*. Then the vector $X$ of $\mathbb{L}^5$ is called: **(i)** time-like if $\langle X, X\rangle < 0$, **(ii)** space-like if $\langle X, X\rangle > 0$ or $X = 0$, **(iii)** null (or light-like) vector if $\langle X, X\rangle = 0$ and $X \neq 0$. The set of a null vectors in $T_p(\mathbb{L}^5)$ is called the nullcone at $p \in \mathbb{L}^5$.

Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{L}^5$ can be locally be space-like, time-like or null, if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $X$ is given by $\|X\| = \sqrt{|\langle X, X\rangle|}$. Therefore, $X$ is a unit vector if $\langle X, X\rangle = \pm 1$. Next, vectors $X$, $Y$ in $\mathbb{L}^5$ are said to be orthogonal if $\langle X, Y\rangle = 0$. The velocity of the curve $\alpha$ is given by $\|\alpha'(s)\|$ . Thus, a space-like or a time-like $\alpha$ is said to be parametrized by arclength function $s$, if $\langle \alpha'(s), \alpha'(s)\rangle = \pm 1$ [1].

**2. Basic definitions of $\mathbb{L}^5$.**

**Definition 1.** *Let $\alpha\colon I \longrightarrow \mathbb{L}^5$ be a curve in $\mathbb{L}^5$ and $k_1, k_2, k_3, k_4$ be the Frenet curvatures of $\alpha$. Then for the unit tangent vector $V_1 = \alpha'(s)$ over $M$ the $i$th e-curvature function $m_i$, $1 \le i \le 5$, is defined by*

$$m_i = \begin{cases} 0, & i = 1, \\ \dfrac{\varepsilon_1\varepsilon_2}{k_1}, & i = 2, \\ \left[\dfrac{d}{dt}(m_{i-1}) + \varepsilon_{i-2}m_{i-2}k_{i-2}\right]\dfrac{\varepsilon_i}{k_{i-1}}, & 2 < i \le 5, \end{cases} \tag{1}$$

*where $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1$.*

**Definition 2.** *Let $\alpha\colon I \longrightarrow \mathbb{L}^5$ be a unit speed non-null curve. The curve $\alpha$ is called Frenet curve of osculating order $d$, $d \le 5$, if its 5th order derivatives $\alpha'(s), \alpha''(s), \ldots, \alpha^{v}(s)$ are linearly independent and $\alpha'(s), \alpha''(s), \ldots, \alpha^{vi}(s)$ are no longer linearly independent for all $s \in I$. For each Frenet curve of order 5 one can associate an orthonormal 5-frame $\{V_1, V_2, V_3, V_4, V_5\}$ along $\alpha$ (such that $\alpha'(s) = V_1$) called the Frenet frame $k_1$, $k_2$, $k_3$, $k_4\colon I \longrightarrow \mathbb{R}$ called the Frenet curvatures, such that the Frenet formulas is defined in the usual way*;

$$\begin{aligned}
\nabla_{V_1} V_1 &= \varepsilon_2 k_1 V_2, \\
\nabla_{V_1} V_2 &= -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3, \\
\nabla_{V_1} V_3 &= -\varepsilon_2 k_2 V_2 + \varepsilon_4 k_3 V_4, \\
\nabla_{V_1} V_4 &= -\varepsilon_3 k_3 V_3 + \varepsilon_5 k_4 V_5, \\
\nabla_{V_1} V_5 &= -\varepsilon_4 k_4 V_4,
\end{aligned} \tag{2}$$

*where* $V_1,\ V_2,\ V_3,\ V_4,$ *and* $V_5$ *are orthogonal vectors satisfying equations*

$$\langle V_1, V_1 \rangle = -1,$$

$$\langle V_i, V_i \rangle = 1, \quad 2 \leq i \leq 5,$$

*and* $\nabla$ *is the Levi-Civita connection of* $\mathbb{L}^5$.

**Definition 3.** *Let* $\alpha$ *be a non-null curve of osculating order* 5. *The harmonic functions*

$$H_j : I \longrightarrow \mathbb{R}, \quad 0 \leq j \leq 3,$$

*defined by*

$$H_0 = 0,$$

$$H_1 = \frac{k_1}{k_2},$$

$$\left[\frac{d}{dt}(H_{j-1}) + \varepsilon_{j-2} H_{j-2} k_j\right] \frac{\varepsilon_j}{k_{j+1}}, \quad 2 \leq j \leq 3,$$

*are called the harmonic curvatures of* $\alpha$, *where* $k_1,\ k_2,\ k_j,\ k_{j+1}$ *are Frenet curvatures of* $\alpha$ *which are not necessarily constant and* $\varepsilon_j = \langle V_j, V_j \rangle = \pm 1$ [2].

**Definition 4.** *Let* $\alpha$ *be a non-null curve of osculating order* 5. *Then* $\alpha$ *is called a general helix of* $rank$ 3 *if*

$$\sum_{i=1}^{3} (H_i)^2 = c,$$

*holds, where* $c \neq 0$ *is a real constant.*

***Corollary*** **1.** *If* $\alpha$ *is a general helix of* $rank$ 3, *then*

$$H_1^2 + H_2^2 + H_3^2 = c.$$

***Proof.*** By the use of above definition, we obtain the corollary.

## 3. The Darboux vector in $\mathbb{L}^5$.

**Theorem 1.** *Let* $\alpha$ *be a non-null curve of osculating order* 5 *in* $\mathbb{L}^5$, *then*

$$\nabla_{V_j} V_i = \frac{\varepsilon_i}{m_{i+1}} V_{i+1}, \quad i, j = 1,$$

$$\nabla_{V_j} V_{i+1} = -\frac{\varepsilon_{i+1}}{m_{i+1}} V_i + \frac{m'_{i+1}}{m_{i+2}} V_{i+2}, \quad i, j = 1,$$

$$\nabla_{V_j} V_{i+1} = \left( \frac{-\varepsilon_i \varepsilon_{i+1} m_i m_i' - \varepsilon_{i-1} \varepsilon_{i+1} m_{i-1} m_{i-1}'}{m_i m_{i+1}} \right) V_i +$$

$$+ \left( \frac{m_{i+1} m_{i+1}' + \varepsilon_i \varepsilon_{i+1} m_i m_i' + \varepsilon_{i-1} \varepsilon_{i+1} m_{i-1} m_{i-1}'}{m_{i+1} m_{i+2}} \right) V_{i+2}, \qquad j = 1, \quad i = 2, 3,$$

$$\nabla_{V_j} V_{i+1} = \left( \frac{-\varepsilon_i \varepsilon_{i+1} m_i m_i' - \varepsilon_{i-1} \varepsilon_{i+1} m_{i-1} m_{i-1}' - \varepsilon_{i-2} \varepsilon_{i+1} m_{i-2} m_{i-2}'}{m_i m_{i+1}} \right) V_i, \qquad j = 1, \quad i = 4,$$

*where $m_i$ are the $i$th $e$-curvature functions and $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$.*

***Corollary 2.*** *Let $\alpha$ be a non-null curve of osculating order 5 in $\mathbb{L}^5$, then*

$$\nabla_{V_1} V_1 = \frac{\varepsilon_1}{m_2} V_2,$$

$$\nabla_{V_1} V_2 = -\frac{\varepsilon_2}{m_2} V_1 + \frac{m_2'}{m_3} V_3,$$

$$\nabla_{V_1} V_3 = -\varepsilon_2 \varepsilon_3 \frac{m_2'}{m_3} V_2 + \left( \frac{m_3 m_3' + \varepsilon_2 \varepsilon_3 m_2 m_2'}{m_3 m_4} \right) V_4,$$

$$\nabla_{V_1} V_4 = \left( \frac{-\varepsilon_3 \varepsilon_4 m_3 m_3' - \varepsilon_2 \varepsilon_4 m_2 m_2'}{m_3 m_4} \right) V_3 + \left( \frac{m_4 m_4' + \varepsilon_3 \varepsilon_4 m_3 m_3' + \varepsilon_2 \varepsilon_4 m_2 m_2'}{m_4 m_5} \right) V_5,$$

$$\nabla_{V_1} V_5 = \left( \frac{-\varepsilon_4 \varepsilon_5 m_4 m_4' - \varepsilon_3 \varepsilon_5 m_3 m_3' - \varepsilon_2 \varepsilon_5 m_2 m_2'}{m_4 m_5} \right) V_4.$$

***Proof.*** By the use of a previous theorem, we obtain the corollary.

**Theorem 2.** *Let $\alpha$ be a non-null curve of osculating order 5 in $\mathbb{L}^5$, then*

$$\nabla_{V_1} V_1 = \varepsilon_2 k_2 H_1 V_2,$$

$$\nabla_{V_1} V_2 = -\varepsilon_1 k_2 H_1 V_1 + \varepsilon_3 \frac{k_1}{H_1} V_3,$$

$$\nabla_{V_1} V_3 = -\varepsilon_2 \frac{k_1}{H_1} V_2 + \varepsilon_2 \varepsilon_4 \frac{H_1'}{H_2} V_4,$$

$$\nabla_{V_1} V_4 = -\varepsilon_2 \varepsilon_3 \frac{H_1'}{H_2} V_3 + \varepsilon_3 \varepsilon_5 \left( \frac{H_2 H_2' + \varepsilon_1 \varepsilon_2 H_1 H_1'}{H_2 H_3} \right) V_5,$$

$$\nabla_{V_1} V_5 = -\varepsilon_3 \varepsilon_4 \left( \frac{H_2 H_2' + \varepsilon_1 \varepsilon_2 H_1 H_1'}{H_2 H_3} \right) V_4,$$

*where $H_1,\ H_2,\ H_3$ are harmonic curvature of $\alpha$ and $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$ for $1 \le i \le 5$.*

***Proof.*** By using definition of harmonic curvatures, we obtain the theorem.

**Theorem 3.** *Let $\alpha\colon \longrightarrow \mathbb{L}^5$ be a non-null curve of osculating order 5 given over the Frenet frame $\{V_1, V_2, V_3, V_4, V_5\}$. If $m_i,\ 2 \le i \le 5$, are the $i$th e-curvature functions, $H_i,\ 1 \le i \le 3$, are the harmonic curvatures and $m_i' = \dfrac{dm_i}{ds},\ 2 \le i \le 5$, then the following relations hold*:

$$\det(m_2', m_3', m_4', m_5') = 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{3} (H_i)^2 = \text{constant},$$

$$\det(m_2', m_3', m_4', m_5') = 0 \quad \Longleftrightarrow \quad \sum_{i=2}^{5} m_i^2 = \text{constant}.$$

***Proof.*** The proof can seen by using the definitions of $i$th e-curvature functions and harmonic curvatures.

**Definition 5.** *Let $\alpha$ be a non-null curve of osculating order 5 in $\mathbb{L}^5$, with Frenet curvatures $k_1, k_2, k_3, k_4$. Let us denote*

$$a_0 = k_2 k_4,$$

$$a_j = \varepsilon_j \frac{k_{2j-1}}{k_{2j}} a_{j-1}, \quad 1 \leq j \leq 2, \quad k_{2j} \neq 0.$$

*The Darboux vector in $\mathbb{L}^5$ is defined by*

$$D(s) = \sum_{j=0}^{2} a_j V_{2j+1} = a_0 V_1 + a_1 V_3 + a_2 V_5,$$

*where $V_{2j+1}$ are the Frenet vectors of $\alpha$.*

***Corollary* 3.** *Let $\alpha$ be a non-null curve of osculating order 5 in $\mathbb{L}^5$, then*

$$\frac{a_j}{a_{j-1}} = \varepsilon_j \frac{k_{2j-1}}{k_{2j}},$$

*where $1 \leq j \leq 2,\ k_{2j} \neq 0$.*

**Lemma 1.** *The derivative of the Darboux vector $D(s)$ is* [4].

$$D'(s) = a_0' V_1 + a_1' V_3 + a_2' V_5.$$

**Definition 6.** *The point $\alpha(s_0)$ is called Darboux vertex of $\alpha$ if the first derivative of the Darboux vector $D(s)$ is vanishing at that point* [3].

**Theorem 4.** *Let $\alpha$ be a non-null curve of osculating order 5 in $\mathbb{L}^5$, with Frenet curvatures $k_1$, $k_2$, $k_4$ and harmonic curvatures $H_1$, $H_2$. Let us denote*

$$a_0 = \frac{k_1 k_4}{H_1},$$

$$a_1 = \varepsilon_1 H_1 k_2 k_4,$$

$$a_2 = \varepsilon_1 \frac{H_1'}{H_2} k_1,$$

*where $H_1 \neq 0,\ H_2 \neq 0,\ \varepsilon_1 = \langle V_1, V_1 \rangle = \pm 1$.*

***Proof.*** By using definition of harmonic curvature, we get the result.

The following theorem is true.

**Theorem 5.** *Let $\alpha \colon I \longrightarrow \mathbb{L}^5$ be a non-null curve of osculating order 5. Then*

$$H_1 = \varepsilon_1 \frac{a_1}{a_0}, \quad a_0 \neq 0,$$

$$H_2 = \varepsilon_1 \frac{a_0 H_1 H_1'}{a_2 k_4}, \qquad a_2 \neq 0, \quad k_4 \neq 0,$$

*where* $H_1,\ H_2$ *are harmonic curvatures of* $\alpha,\ k_4$ *is Frenet curvature of* $\alpha,\ \varepsilon_1 = \langle V_1, V_1\rangle = \pm 1$ *and* $a_0,\ a_1,\ a_2 \in R.$

***Proof.*** By the use of a previous theorem, we obtain the theorem.

***Corollary* 4.** (i) *If the vector* $V_1$ *is space-like, then*

$$H_1 = \frac{a_1}{a_0}, \quad a_0 \neq 0,$$

$$H_2 = \frac{a_0 H_1 H_1'}{a_2 k_4}, \qquad a_2 \neq 0, \quad k_4 \neq 0.$$

(ii) *If the vector* $V_1$ *is time-like, then*

$$H_1 = -\frac{a_1}{a_0}, \quad a_0 \neq 0,$$

$$H_2 = -\frac{a_0 H_1 H_1'}{a_2 k_4}, \qquad a_2 \neq 0, \quad k_4 \neq 0.$$

**Theorem 6.** *Let* $\alpha$ *be a non-null curve of osculating order* 5 *in* $\mathbb{L}^5$*,with Frenet curvatures* $k_1,\ k_2,\ k_3,\ k_4$*. Then the curve* $\alpha$ *has a Darboux vertex at point* $\alpha(s)$ *if and only if*

$$\varepsilon_i \left(\frac{k_i}{k_{i+1}}\right)' = 0, \qquad k_{i+1} \neq 0, \quad 1 \leq i \leq 3,$$

*where* $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1.$

***Corollary* 5.** (i) *If the vector* $V_1$ *is space-like, then*

$$(H_1)' = 0.$$

(ii) *If the vector* $V_1$ *is time-like, then*

$$-(H_1)' = 0.$$

***Corollary* 6.** *If* $\alpha\colon I \longrightarrow \mathbb{L}^5$ *has a Darboux vertex at the point* $\alpha(s_0)$*, then* $\alpha$ *is a general helix of order* 3 [3].

**4. ccr-Curve in $\mathbb{L}^5$.**

**Definition 7.** *A curve* $\alpha\colon I \longrightarrow \mathbb{L}^5$ *is said to have constant curvature ratios* (*that is to say, it is a ccr-curve*) *if all the quotients* $\varepsilon_i\left(\dfrac{k_{i+1}}{k_i}\right)$ *are constant,* $k_i \neq 0$*. Here,* $k_i, k_{i+1},\ 1 \leq i \leq 3,$ *are Frenet curvatures of* $\alpha$*, and* $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1,\ 1 \leq i \leq 3.$

**Theorem 7.** (i) *For* $i = 1$*, the ccr-curve is* $\dfrac{a_1}{a_0 (H_1)^2}.$

(ii) *For* $i = 2$*, the ccr-curve is* $\varepsilon_1 \dfrac{a_2 H_1}{(k_1)^2}.$

(iii) *For* $i = 3$*, the ccr-curve is* $\varepsilon_2 \varepsilon_3 \dfrac{a_0 H_2}{k_2 H_1'}.$

*Here* $H_1, H_2$ *are harmonic curvatures of* $\alpha$*;* $k_1, k_2$ *are Frenet curvatures of* $\alpha$*;* $\varepsilon_i = \langle V_i, V_i\rangle = \pm 1,\ 1 \leq i \leq 3,$ *and* $a_0,\ a_1,\ a_2 \in R.$

***Proof.*** The proof can be easily seen by using the definitions of the harmonic curvature and ccr-curve.

***Corollary* 7.** (i) *If the vector* $V_1$ *is space-like, then the ccr-curve is* $\dfrac{a_2 H_1}{(k_1)^2}$, *where* $\varepsilon_1 = 1$.

(ii) *If the vector* $V_1$ *is time-like, then the ccr-curve is* $-\dfrac{a_2 H_1}{(k_1)^2}$, *where* $\varepsilon_1 = -1$.

(iii) *If the vectors* $V_2, V_3$ *are space-like, then the ccr-curve is* $\dfrac{a_0 H_2}{k_2 H_1'}$, *where* $\varepsilon_2 = \varepsilon_3 = 1$.

(iv) *If the vector* $V_2$ *is time-like, then the ccr-curve is* $-\dfrac{a_0 H_2}{k_2 H_1'}$, *where* $\varepsilon_2 = -1$ *and* $\varepsilon_3 = 1$.

(v) *If the vector* $V_3$ *is time-like, then the ccr-curve is* $-\dfrac{a_0 H_2}{k_2 H_1'}$, *where* $\varepsilon_3 = -1$ *and* $\varepsilon_2 = 1$.

***Corollary* 8.** $\alpha$ *is a ccr-curve in* $\mathbb{L}^5 \Leftrightarrow \varepsilon_1 (H_1)^{-1} = \text{constant}$.

***Corollary* 9.** *For* $\alpha\colon I \longrightarrow \mathbb{L}^5$ *curve, we have*

$$\varepsilon_i \left(\frac{k_i}{k_{i+1}}\right)' = \varepsilon_i \left(\frac{k_{i+1}}{k_i}\right), \qquad 1 \leq i \leq 3,$$

*that is the Darboux vertex is equal to the constant curvature ratio.*

Now, we will calculate Darboux vector and Darboux vertex of the unit speed time-like curve in $\mathbb{L}^5$ studied in [5].

**5. An example.**

***Example* 1.** Let us consider the following curve:

$$\alpha(s) = \left(\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sin s, s, \cos s\right),$$

$$V_1(s) = \alpha'(s) = \left(\sqrt{3}\cosh s, \sqrt{3}\sinh s, \cos s, 1, -\sin s\right),$$

where $\langle \alpha'(s), \alpha'(s)\rangle = -1$. One can easily see that $\alpha(s)$ is an unit speed time-like curve. We express the following differentiations:

$$\alpha''(s) = \left(\sqrt{3}\sinh s, \sqrt{3}\cosh s, -\sin s, 0, -\cos s\right) \Rightarrow$$

$$\Rightarrow \alpha'''(s) = \left(\sqrt{3}\cosh s, \sqrt{3}\sinh s, -\cos s, 0, \sin s\right) \Rightarrow$$

$$\Rightarrow \alpha^{iv}(s) = \left(\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sin s, 0, \cos s\right) \Rightarrow$$

$$\Rightarrow \alpha^{v}(s) = \left(\sqrt{3}\cosh s, \sqrt{3}\sinh s, \cos s, 0, -\sin s\right).$$

So, we have the first curvature as

$$\left\|\alpha''(s)\right\| = k_1(s) = 2.$$

Moreover we can write second, third, fourth and fifth Frenet vectors of the curve, respectively,

$$V_2(s) = \varepsilon_2 \left(\frac{\sqrt{3}}{2}\sinh s, \frac{\sqrt{3}}{2}\cosh s, -\frac{1}{2}\sin s, 0, -\frac{1}{2}\cos s\right),$$

$$V_3(s) = \frac{1}{\sqrt{14}}\left(-3\sqrt{3}\cosh s, -3\sqrt{3}\sinh s, -5\cos s, -4, 5\sin s\right),$$

$$V_4(s) = \mu\left(-\frac{1}{2}\sinh s, -\frac{1}{2}\cosh s, -\frac{3}{2}\sin s, 0, -\frac{3}{2}\cos s\right)$$

and

$$V_5(s) = \mu\left(-\frac{1}{\sqrt{14}}\cosh s, -\frac{1}{\sqrt{14}}\sinh s, \sqrt{\frac{3}{14}}\cos s, -\sqrt{\frac{2}{7}}, -\sqrt{\frac{3}{14}}\sin s\right),$$

where $\mu$ is taken $\mp 1$ to make $+1$ determimant of $\{V_1(s), V_2(s), V_3(s), V_4(s), V_5(s)\}$ matrix. In addition to, we can write second, third, fourth curvatures and harmonic curvature of $\alpha(s)$, respectively,

$$k_2(s) = \sqrt{14}, \qquad k_3(s) = \sqrt{\frac{3}{14}}, \qquad k_4(s) = \sqrt{\frac{2}{7}}, \qquad H_1 = \frac{2}{\sqrt{14}},$$

$$a_0 = 2, \qquad a_1 = \varepsilon_1\left(\frac{2\sqrt{2}}{\sqrt{7}}\right), \qquad a_2 = 0.$$

Now, we will calculate ccr-curves of $\alpha(s)$ in $\mathbb{L}^5$. If the vector $V_1$ is time-like, then $\mu = 1$, $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_3 = 1$

$$\varepsilon_1\frac{k_2}{k_1} = -\frac{\sqrt{14}}{2} = \text{constant}, \qquad \varepsilon_2\frac{\mathrm{k}_3}{\mathrm{k}_2} = \frac{\sqrt{3}}{14} = \text{constant} \qquad \text{and} \qquad \varepsilon_3\frac{\mathrm{k}_4}{\mathrm{k}_3} = \frac{2}{\sqrt{3}} = \text{constant}.$$

Thus, $\alpha(s)$ is a ccr-curve in $\mathbb{L}^5$. Also, we obtain, respectively, Darboux vector and Darboux vertex of $\alpha(s)$ in $\mathbb{L}^5$,

(i) If the vector $V_1$ is time-like, then $\mu = 1$, $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_3 = 1$,

$$D(s) = \left(\frac{20\sqrt{3}}{7}\cosh s, \frac{20\sqrt{3}}{7}\sinh s, \frac{24}{7}\cos s, \frac{22}{7}, -\frac{24}{7}\sin s\right)$$

and

$$D'(s) = \left(\frac{20\sqrt{3}}{7}\sinh s, \frac{20\sqrt{3}}{7}\cosh s, -\frac{24}{7}\sin s, 0, -\frac{24}{7}\cos s\right).$$

(ii) If the vector $V_1$ is time-like, then Darboux vector and Darboux vertex are as in (i) when $\mu = -1$, $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_3 = 1$, since $a_2 = 0$.

## References

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