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JOINT UNIVERSALITY FOR L-FUNCTIONS FROM SELBERG CLASS AND PERIODIC HURWITZ ZETA-FUNCTIONS

СПІЛЬНА УНІВЕРСАЛЬНІСТЬ ДЛЯ L-ФУНКЦІЙ ІЗ КЛАСУ СЕЛЬБЕРГА ТА ПЕРІОДИЧНІ ДЗЕТА-ФУНКЦІЇ ХУРВІЦА

We obtain a joint universality theorem on the approximation of a collection of analytic functions by a collection of shifts consisting of L-functions from the Selberg class and periodic Hurwitz zeta-functions.

Встановлено теорему про спільну універсальність наближення сім'ї аналітичних функцій сім'єю зсувів, що складаються з L-функцій із класу Сельберга та періодичних дзета-функцій Хурвіца.

1. Introduction. Universality of zeta and L-functions is one of the most interesting phenomenons of analytic number theory. Roughly speaking, it means that every analytic function can be approximated with a given accuracy by shifts of the considered zeta or L-functions, uniformly on compact subsets of a certain region. The first result in the field belongs to S. M. Voronin who discovered [20] the universality property of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$. For a modern form of the Voronin theorem, we use the following notations. Let $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K, and by $H_0(K)$ the subclass of $H(K)$ of nonvanishing on K functions. Moreover, for a measurable set $A \subset \mathbb{R}$, we use the notation meas$A$ for the Lebesgue measure of the set A. Then the following statement is well-known, for the proof, see, for example, [7].

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \epsilon\right\} > 0.$$

The theorem shows that a given function $f(s) \in H_0(K)$ can be approximated by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, from a wide set having a positive lower density.

Later, it turned out that the majority of other classical zeta and L-functions also are universal in the above sense. A full survey on universality of zeta and L-functions is given in the excelent paper [12]. We focus over attention on the Selberg class [17] which is one of the most extensively studied objects of analytic number theory.

The Selberg class $\mathcal{S}$ contains Dirichlet series

$$\mathcal{L}(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s},$$

satisfying the following axioms:

1°) analytic continuation: there exists an integer $k \geq 0$ such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order;

2°) Ramanujan hypothesis: $a(m) \ll m^\epsilon$ with any $\epsilon > 0$, where the implied constant may depend on ϵ;

3°) functional equation: $\mathcal{L}(s)$ satisfies a functional equation

$$\Lambda_{\mathcal{L}}(s) = w\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$$

with positive real numbers Q, λ_j, and complex numbers μ_j, w, $\Re\mu_j \geq 0$ and $|w| = 1$;

4°) Euler product: $\mathcal{L}(s)$ has a product representation over primes p

$$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s),$$

where

$$\mathcal{L}_p(s) = \exp\left\{\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right\}$$

with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$.

Many authors investigated the structure of the class $\mathcal{S}$. For this, see, a survey paper [5] and subsequent works by J. Kaczorovski and A. Perelli.

For $\mathcal{L} \in \mathcal{S}$, define the degree $d_{\mathcal{L}}$ of $\mathcal{L}$ by

$$d_{\mathcal{L}} = 2\sum_{j=1}^{f} \lambda_j,$$

and let $D_{\mathcal{L}} = \left\{s \in \mathbb{C}\colon \max\left(\frac{1}{2}, 1 - \frac{1}{d_{\mathcal{L}}}\right) < \sigma < 1\right\}$. In [15] the analogue of Theorem 1.1 (with the strip of universality $D_{\mathcal{L}}$) was obtained for $\mathcal{L} \in \mathcal{S}$ satisfying additional hypothesis

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \leq x} \left|a(p)\right|^2 = \kappa, \tag{1.1}$$

where κ is some positive constant (depending on $\mathcal{L}$), $\pi(x) = \sum_{p \leq x} 1$. This result is an extension of previous result obtained by J. Steuding [18], who began to study the universality of functions from the Selberg class.

Theorem 1.1 is an example of universality theorems for zeta and L-functions with Euler products. The second group of universality theorems were proved for zeta-functions without Euler product. The simplest zeta-function without Euler product is the classical Hurwitz zeta-function. Let α, $0 < \alpha \leq 1$, be a fixed parameter. Then the Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and is meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1. A generalization of the function $\zeta(s, \alpha)$ is the periodic Hurwitz zeta-function. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

In virtue of periodicity of the sequence $\mathfrak{a}$, we have that, for $\sigma > 1$,

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{m=0}^{q-1} a_m \zeta\left(s, \frac{m+\alpha}{q}\right),$$

and the later equality gives meromorphic continuation to the whole complex plane for $\zeta(s, \alpha; \mathfrak{a})$ with a simple pole at the point $s = 1$. If

$$\sum_{m=0}^{q-1} a_m = 0,$$

then the function $\zeta(s, \alpha; \mathfrak{a})$ is entire one.

A more complicated is the joint universality of zeta-functions. In this case, a collection of analytic functions is simultaneously approximated by a collection of shifts of zeta or L-functions. The first joint universality theorem for Dirichlet L-functions was obtained by Voronin [21, 22]. Also, a joint universality when analytic functions are approximated by shifts of zeta-functions, having and having no Euler product, is possible. The first result in this direction belongs to Mishou [14]. In [6] a joint universality theorem was proved for $\zeta(s, \alpha; \mathfrak{a})$ and periodic zeta-function $\zeta(s; \mathfrak{b})$, which is defined, for $\sigma > 1$, by

$$\zeta(s; \mathfrak{b}) = \sum_{m=1}^{\infty} \frac{b_m}{m^s},$$

and $\mathfrak{b} = \{b_m : m \in \mathbb{N}\}$ is an another periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. The function $\zeta(s; \mathfrak{b})$, as $\zeta(s, \alpha; \mathfrak{a})$, has meromorphic continuation to the whole complex plane. In [8] Laurinčikas proved a generalization of the mentioned result. To state his theorem, we need some notation. Let $\mathfrak{b}_j = \{b_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k_j \in \mathbb{N}$ and $\zeta(s; \mathfrak{b}_j)$ be the corresponding periodic zeta-function, $j = 1, \ldots, r_1$, $r_1 > 1$. Denote by $k = [k_1, \ldots, k_{r_1}]$ the least common multiple of the periods $k_1, \ldots, k_{r_1}$, by $\eta_1, \ldots, \eta_{\varphi(k)}$ the reduced residue system modulo k, and define the matrix

$$B = \begin{pmatrix} b_{1\eta_1} & b_{2\eta_1} & \ldots & b_{r_1\eta_1} \\ b_{1\eta_2} & b_{2\eta_2} & \ldots & b_{r_1\eta_2} \\ \ldots & \ldots & \ldots & \ldots \\ b_{1\eta_{\varphi(k)}} & b_{2\eta_{\varphi(k)}} & \ldots & b_{r_1\eta_{\varphi(k)}} \end{pmatrix},$$

where $\varphi(k)$ is the Euler function. Moreover, we suppose that, for all primes p,

$$\sum_{\beta=1}^{\infty} \frac{|b_{jp^\beta}|}{p^{\beta/2}} \le c_j < 1, \quad j = 1, \ldots, r_1. \tag{1.2}$$

Further, let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}_0\}$ be an another periodic sequence of complex numbers with minimal period $q_j \in \mathbb{N}$ and $\zeta(s, \alpha_j; \mathfrak{a}_j)$ be the corresponding periodic Hurwitz zeta-function, $j = 1, \ldots, r_2$, $0 < \alpha_j \le 1$. Then, in [8] the following statement was proved.

Theorem 1.2. *Suppose that the sequences $\mathfrak{b}_1, \ldots, \mathfrak{b}_{r_1}$ are multiplicative, inequalities* (1.2) *are satisfied, the numbers $\alpha_1, \ldots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$, and* $\operatorname{rank}(B) = r_1$. *Let $K_1, \ldots, K_{r_1}, \hat{K}_1, \ldots, \hat{K}_{r_2} \in \mathcal{K}$, $f_1(s) \in H_0(K_1), \ldots, f_{r_1}(s) \in H_0(K_{r_1})$ and $\hat{f}_1(s) \in H(\hat{K}_1), \ldots$ $\ldots, \hat{f}_{r_1}(s) \in H(\hat{K}_{r_2})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} \left|\zeta(s+i\tau; \mathfrak{b}_j) - f(s)\right| < \varepsilon, \right.$$

$$\left. \sup_{1\le j\le r_2} \sup_{s\in \hat{K}_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - \hat{f}_j(s)| < \varepsilon \right\} > 0.$$

The aim of this paper is a joint universality theorem for the functions $\mathcal{L}(s) \in \mathcal{S}$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, where $0 < \alpha_j \le 1$ and $\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $q_{jl} \in \mathbb{N}$, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$. For $j = 1, \ldots, r$, let q_j be the least common multiple of $q_{j1}, \ldots, q_{jl_j}$, and

$$A_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \ldots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \ldots & a_{2jl_j} \\ \ldots & \ldots & \ldots & \ldots \\ a_{q_jj1} & a_{q_jj2} & \ldots & a_{q_jjl_j} \end{pmatrix}.$$

Moreover, as above, let $\mathcal{K}_{\mathcal{L}}$ be the class of compact subsets of the strip $D_{\mathcal{L}}$ with connected complements, and let $H_{0\mathcal{L}}(K)$, $K \in \mathcal{K}_{\mathcal{L}}$, be the class of continuous non-vanishing functions on K which are analytic in the interior of K.

Theorem 1.3. *Suppose that $\mathcal{L} \in \mathcal{S}$, hypothesis* (1.1) *is satisfied, the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(A_j) = l_j$, $j = 1, \ldots, r$. *Let $K \in \mathcal{K}_{\mathcal{L}}$, $f(s) \in H_{0\mathcal{L}}(K)$, and, for every $j = 1, \ldots, r$ and $l = 1, \ldots, l_j$, let $K_{jl} \in \mathcal{K}$ and $f_{jl}(s) \in H(K_{jl})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} \left|\mathcal{L}(s+i\tau) - f(s)\right| < \varepsilon, \right.$$

$$\left. \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} \left|\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)\right| < \varepsilon \right\} > 0.$$

Clearly, Theorem 1.3 implies the results of previous works [4, 10, 11, 16], where instead $\mathcal{L} \in \mathcal{S}$, the function $\zeta(s)$, zeta-functions of normalized Hecke cusp forms, zeta-functions of newforms, and zeta-functions of new forms with Dirichlet character were taken, respectively. For example, we can take, for $r = 2$,

$$\alpha_1 = 2^{-\sqrt[3]{2}}, \qquad \alpha_2 = 2^{-\sqrt[3]{4}},$$

because the numbers $\alpha_1 = 2^{\sqrt[3]{2}}$ and $\alpha_2 = 2^{\sqrt[3]{4}}$ are algebraically independent over $\mathbb{Q}$ [2].

2. Limit theorem. For the proof of Theorem 1.3, we will prove a limit theorem on weakly convergent probability measures in the space of analytic functions. Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic functions on G equipped with the topology of uniform convergence on compacta. Let

$$u = \sum_{j=1}^{r} l_j, \quad v = u + 1,$$

and

$$H^v = H^v(D_{\mathcal{L}}, D) = H(D_{\mathcal{L}}) \times H^u(D).$$

As usual, denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. For brevity, let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \ldots, \mathfrak{a}_{1l_1}, \ldots, \mathfrak{a}_{r1}, \ldots, \mathfrak{a}_{rl_r})$ and

$$Z(s_1, s, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) = \big(\mathcal{L}(s_1), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big).$$

In this section, we consider the weak convergence for

$$P_T(A) \stackrel{\text{df}}{=} \frac{1}{T} \operatorname{meas}\big\{\tau \in [0, T] \colon \ Z(s_1 + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\big\}, \quad A \in \mathcal{B}(H^v),$$

as $T \to \infty$. To state a limit theorem for P_T, we need a certain topological structure. Let $\gamma = \{s \in \mathbb{C} \colon |s| = 1\}$. Define

$$\hat{\Omega} = \prod_p \gamma_p \qquad \text{and} \qquad \Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the tori $\hat{\Omega}$ and Ω with the product topologies and pointwise multiplication are compact topological groups. Thus, on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ and $\big(\Omega, \mathcal{B}(\Omega)\big)$ the probability Haar measures $\hat{m}_H$ and m_H, respectively, exist, and we have the probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$, $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Moreover, for $j = 1, \ldots, r$, let $\Omega_j = \Omega$, and

$$\underline{\Omega} = \hat{\Omega} \times \Omega_1 \times \ldots \times \Omega_r.$$

Then again, $\underline{\Omega}$ is a compact topological Abelian group, and this gives one more probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$, where $\underline{m}_H$ is the probability Haar measure on $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to γ_p, $p \in \mathcal{P}$ ($\mathcal{P}$ is the set of all prime numbers), and by $\omega_j(m)$ the projection of $\omega_j \in \Omega_j$ to γ_m, $m \in \mathbb{N}_0$. After the above notation, on the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$ define the H^v-valued random element $Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})$, $\underline{\omega} = (\hat{\omega}, \omega_1, \ldots, \omega_r) \in \underline{\Omega}$, by the formula

$$Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) = \big(\mathcal{L}(s_1, \hat{\omega}), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \ldots$$
$$\ldots, \zeta(s, \omega_r, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \omega_r, \alpha_r; \mathfrak{a}_{rl_r})\big),$$

where, for $s_1 \in D_{\mathcal{L}}$,

$$\mathcal{L}(s_1, \hat{\omega}) = \sum_{m=1}^{\infty} \frac{a(m)\hat{\omega}(m)}{m^{s_1}}$$

with

$$\hat{\omega}(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \hat{\omega}^l(p), \quad m \in \mathbb{N},$$

and, for $s \in D$,

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \qquad j = 1, \ldots, r, \quad l = 1, \ldots, l_j.$$

We note that, for almost all $\hat{\omega} \in \hat{\Omega}$, $\mathcal{L}(s_1, \hat{\omega})$ has the Euler product [15], i.e.,

$$\mathcal{L}(s_1, \hat{\omega}) = \exp\left\{ \sum_p \sum_{k=1}^{\infty} \frac{b(p^k)\hat{\omega}^k(p)}{p^{ks_1}} \right\}.$$

Denote by P_Z the distribution of the random element $Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})$, i.e.,

$$P_Z(A) = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega}\colon\ Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\big), \qquad A \in \mathcal{B}(H^v).$$

Theorem 2.1. *Suppose that* $\mathcal{L} \in \mathcal{S}$, *hypothesis* (1.1) *is satisfied*, *and the numbers* $\alpha_1, \ldots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$. *Then* P_T *converges weakly to* P_Z *as* $T \to \infty$.

A way of the proof is sufficiently well-known, see, for example, similar theorems from [4, 10, 11], therefore, we will present only principal moments of the proof.

Lemma 2.1. *Suppose that the numbers* $\alpha_1, \ldots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$. *Then*

$$Q_T(A) \overset{\text{df}}{=} \frac{1}{T}\,\text{meas}\Big\{\tau \in [0, T]\colon \big((p^{-i\tau} : p \in \mathcal{P}),$$

$$\big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \ldots, \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)\big) \in A\Big\}, \quad A \in \mathcal{B}(\underline{\Omega}),$$

converges weakly to the Haar measure $\underline{m}_H$ *as* $T \to \infty$.

Proof of the lemma is given in [8] (Theorem 3). However, for conveniences of a reader, we rewrite the proof. It is well known that the dual group of $\underline{\Omega}$ is isomorphic to the group

$$G = \left(\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p \right) \bigoplus_{j=1}^{r} \left(\bigoplus_{m \in \mathbb{N}_0} \mathbb{Z}_{jm} \right),$$

where $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathcal{P}$ and $\mathbb{Z}_{jm} = \mathbb{Z}$ for all $m \in \mathbb{N}_0$ and $j = 1, \ldots, r$. An element $(\underline{k}, \underline{l}_1, \ldots, \underline{l}_r) = \big((k_p : p \in \mathcal{P}), (l_{1m} : m \in \mathbb{N}_0), \ldots, (l_{rm} : m \in \mathbb{N}_0)\big)$ of the group G, where only a finite number of integers k_p and $l_{1m}, \ldots, l_{rm}$ are distinct from zero, acts on $\underline{\Omega}$ by

$$(\hat{\omega}, \omega_1, \ldots, \omega_r) \to (\hat{\omega}^{\underline{k}}, \omega_1^{\underline{l}_1}, \ldots, \omega_r^{\underline{l}_r}) = \prod_{p \in \mathcal{P}} \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0} \omega_j^{l_{jm}}(m).$$

Therefore, the right-hand side of the latter equality defines characters of $\underline{\Omega}$, hence, the Fourier transform $g_T(\underline{k}, \underline{l}_1, \ldots, \underline{l}_r)$ of Q_T is of the form

$$g_T(\underline{k}, \underline{l}_1, \ldots, \underline{l}_r) = \int\limits_{\Omega} \left(\prod_{p\in\mathcal{P}} \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0} \omega_j^{l_{jm}}(m) \right) dQ_T.$$

Thus, by the definition of Q_T,

$$g_T(\underline{k}, \underline{l}_1, \ldots, \underline{l}_r) = \frac{1}{T} \int\limits_{0}^{T} \left(\prod_{p\in\mathcal{P}} p^{-i\tau k_p} \prod_{j=1}^{r} \prod_{m\in\mathbb{N}_0} (m+\alpha_j)^{-i\tau l_{jm}} \right) d\tau =$$

$$= \frac{1}{T} \int\limits_{0}^{T} \exp\left\{ -i\tau \left(\sum_{p\in\mathcal{P}} k_p \log p + \sum_{j=1}^{r} \sum_{m\in\mathbb{N}_0} l_{jm} \log(m+\alpha_j) \right) \right\} d\tau, \tag{2.1}$$

where, as above, only a finite number of integers k_p and $l_{1m}, \ldots, l_{rm}$ are distinct from zero. Clearly,

$$g_T(\underline{0}, \underline{0}, \ldots, \underline{0}) = 1. \tag{2.2}$$

Now suppose that $(\underline{k}, \underline{l}_1, \ldots, \underline{l}_r) \neq (\underline{0}, \underline{0}, \ldots, \underline{0})$. Since the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, the set $\big\{(\log(m+\alpha_1) : m \in \mathbb{N}_0), \ldots, (\log(m+\alpha_r) : m \in \mathbb{N}_0)\big\}$ is linearly independent over $\mathbb{Q}$. Indeed, suppose that there exist $l_{11}, \ldots, l_{1k_1}, \ldots, l_{r1}, \ldots, l_{rk_r} \in \mathbb{Z}\backslash\{0\}$ and $m_{11}, \ldots, m_{rk_1}, \ldots m_{r1}, \ldots, m_{rk_r}$ such that

$$l_{11} \log(m_{11}+\alpha_1) + \ldots + l_{1k_1} \log(m_{1k_1}+\alpha_1) + \ldots$$

$$\ldots + l_{r1} \log(m_{r1}+\alpha_r) + \ldots + l_{rk_r} \log(m_{rk_r}+\alpha_r) = 0.$$

Then

$$(m_{11}+\alpha_1)^{l_{11}} \ldots (m_{1k_1}+\alpha_1)^{l_{1k_1}} \ldots (m_{r1}+\alpha_r)^{l_{r1}} \ldots (m_{rk_r}+\alpha_r)^{l_{rk_r}} = 1.$$

From this, it follows that there exists a polynomial $p(x_1, \ldots, x_r)$ with integer coefficients such that $p(\alpha_1, \ldots, \alpha_r) = 0$, and this contradicts the algebraic independence of the numbers $\alpha_1, \ldots, \alpha_r$. It is well known that the set $\{\log p : p \in \mathcal{P}\}$ is linearly independent over $\mathbb{Q}$. Thus, assuming that the set

$$\big\{(\log p : p \in \mathcal{P}), (\log(m+\alpha_1) : m \in \mathbb{N}_0), \ldots, (\log(m+\alpha_r) : m \in \mathbb{N}_0)\big\} \tag{2.3}$$

is linearly dependent over $\mathbb{Q}$, we obtain, similarly as in the case of the set

$$\big\{(\log(m+\alpha_1) : m \in \mathbb{N}_0), \ldots, (\log(m+\alpha_r) : m \in \mathbb{N}_0)\big\},$$

the contradiction to algebraic independence of $\alpha_1, \ldots, \alpha_r$. Since the sums in (2.1) are finite, from the linear independence of (2.3), we find that, in the case $(\underline{k}, \underline{l}_1, \ldots, \underline{l}_r) \neq (\underline{0}, \underline{0}, \ldots, \underline{0})$,

$$\sum_{p\in\mathcal{P}} k_p \log p + \sum_{j=1}^{r} \sum_{m\in\mathbb{N}_0} l_{jm} \log(m+\alpha_j) \neq 0.$$

After integration, we find from (2.1) and (2.2)

$$g_T(\underline{k},\underline{l}_1,\ldots,\underline{l}_r)=$$

$$=\begin{cases}1, & \text{if} \quad (\underline{k},\underline{l}_1,\ldots,\underline{l}_r)=(\underline{0},\underline{0},\ldots,\underline{0}),\\ \dfrac{1-\exp\left\{-iT\left(\sum_{p\in\mathcal{P}}k_p\log p+\sum_{j=1}^{r}\sum_{m\in\mathbb{N}_0}l_{jm}\log(m+\alpha_j)\right)\right\}}{iT\left(\sum_{p\in\mathcal{P}}k_p\log p+\sum_{j=1}^{r}\sum_{m\in\mathbb{N}_0}l_{jm}\log(m+\alpha_j)\right)}, & \\ & \text{if} \quad (\underline{k},\underline{l}_1,\ldots,\underline{l}_r)\neq(\underline{0},\underline{0},\ldots,\underline{0}).\end{cases}$$

Hence,

$$\lim_{T\to\infty}g_T(\underline{k},\underline{l}_1,\ldots,\underline{l}_r)=\begin{cases}1, & \text{if} \quad (\underline{k},\underline{l}_1,\ldots,\underline{l}_r)=(\underline{0},\underline{0},\ldots,\underline{0}),\\ 0, & \text{if} \quad (\underline{k},\underline{l}_1,\ldots,\underline{l}_r)\neq(\underline{0},\underline{0},\ldots,\underline{0}).\end{cases} \tag{2.4}$$

It is not difficult to see, that the right-hand side of (2.4) is the Fourier transform of the Haar measure $\underline{m}_H$. Really, $\underline{m}_H$ is the product of Haar measures on each circle of $\underline{\Omega}$. The Haar measure on the unit circle coincides with the Lebesgue measure. Therefore, for example, denoting by μ_p the Haar measure on γ_p, we find that

$$\int\limits_{\gamma_p}\hat{\omega}^{k_p}(p)d\mu_p=\int\limits_0^1 e^{2\pi ik_px}dx=\begin{cases}1, & \text{if} \quad k_p=0,\\ 0, & \text{if} \quad k_p\neq0.\end{cases}$$

Now, the assertion of the theorem follows from general continuity theorems for probability measures on compact groups (if the Fourier transform converge, then the corresponding measure converges weakly to the measure of the limit of Fourier transform), see, for example [3] (Theorem 1.4.2).

Now, for fixed $\sigma_1>\dfrac{1}{2}$, let

$$u_n(m)=\exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\},\qquad m,n\in\mathbb{N},$$

and

$$u_n(m,\alpha_j)=\exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1}\right\},\qquad m\in\mathbb{N}_0,\quad n\in\mathbb{N},\quad j=1,\ldots,r,$$

and define

$$\mathcal{L}_n(s)=\sum_{m=1}^{\infty}\frac{a(m)u_n(m)}{m^s}$$

and

$$\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}u_n(m,\alpha_j)}{(m+\alpha_j)^s},\qquad j=1,\ldots,r,\quad l=1,\ldots,l_j.$$

Then we have that the series for $\mathcal{L}_n(s)$ converges absolutely for $\sigma>\max\left(\dfrac{1}{2},1-\dfrac{1}{d_{\mathcal{L}}}\right)$ [19], and the series for $\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})$ converges absolutely for $\sigma>\dfrac{1}{2}$. Also, we define

$$\mathcal{L}_n(s,\hat{\omega}) = \sum_{m=1}^{\infty} \frac{a(m)\hat{\omega}(m)u_n(m)}{m^s} \tag{2.5}$$

and

$$\zeta_n(s,\omega_j,\alpha_j;\mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)u_n(m,\alpha_j)}{(m+\alpha_j)^s}, \qquad j=1,\ldots,r, \quad l=1,\ldots,l_j. \tag{2.6}$$

In the next step of the proof, consider the weak convergence of the measures

$$P_{T,n}(A) = \frac{1}{T}\operatorname{meas}\big\{\tau\in[0,T]\colon Z_n(s_1+i\tau, s+i\tau, \underline{\alpha};\underline{\mathfrak{a}},\mathcal{L})\in A\big\}, \qquad A\in\mathcal{B}(H^v),$$

and

$$\hat{P}_{T,n}(A) = \frac{1}{T}\operatorname{meas}\big\{\tau\in[0,T]\colon\ Z_n(s_1+i\tau, s+i\tau, \underline{\omega}, \underline{\alpha};\underline{\mathfrak{a}},\mathcal{L})\in A\big\}, \qquad A\in\mathcal{B}(H^v),$$

as $T\to\infty$, where

$$Z_n(s_1,s,\underline{\alpha};\underline{\mathfrak{a}},\mathcal{L}) = \big(\mathcal{L}_n(s_1), \zeta_n(s,\alpha_1;\mathfrak{a}_{11}),\ldots,\zeta_n(s,\alpha_1;\mathfrak{a}_{1l_1}),\ldots$$
$$\ldots,\zeta_n(s,\alpha_r;\mathfrak{a}_{r1}),\ldots,\zeta_n(s,\alpha_r;\mathfrak{a}_{rl_r})\big)$$

and

$$Z_n(s_1,s,\underline{\omega},\underline{\alpha};\underline{\mathfrak{a}},\mathcal{L}) = \big(\mathcal{L}_n(s_1,\hat{\omega}), \zeta_n(s,\omega_1,\alpha_1;\mathfrak{a}_{11}),\ldots,\zeta_n(s,\omega_1,\alpha_1;\mathfrak{a}_{1l_1}),\ldots$$
$$\ldots,\zeta_n(s,\omega_r,\alpha_r;\mathfrak{a}_{r1}),\ldots,\zeta_n(s,\omega_r,\alpha_r;\mathfrak{a}_{rl_r})\big).$$

Lemma 2.2. *Suppose that the numbers $\alpha_1,\ldots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for every fixed $\underline{\omega}\in\underline{\Omega}$, the measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to the same probability measure P_n on $\big(H^v,\mathcal{B}(H^v)\big)$ as $T\to\infty$.*

Proof. We apply a standard method based on the preservation of weak convergence under continuous mapping, and Theorem 5.1 of [1]. Define the function $h_n\colon\underline{\Omega}\to H^v$ by the formula

$$h_n(\underline{\omega}) = Z_n(s_1,s,\underline{\omega},\underline{\alpha};\underline{\mathfrak{a}},\mathcal{L}).$$

The absolute convergence of the series (2.5) and (2.6) shows that the function h_n is continuous. Moreover,

$$h_n\left(\left(p^{-i\tau}: p\in\mathcal{P}\right), \big((m+\alpha_1)^{-i\tau}: m\in\mathbb{N}_0\big),\ldots,\big((m+\alpha_r)^{-i\tau}: m\in\mathbb{N}_0\big)\right) =$$
$$= Z_n(s_1+i\tau, s+i\tau, \underline{\alpha};\underline{\mathfrak{a}},\mathcal{L}).$$

Hence, $P_{T,n} = Q_T h_n^{-1}$. This, the continuity of h_n, Lemma 2.1 and Theorem 5.1 of [1] imply the weak convergence of the measure $P_{T,n}$ to $\underline{m}_H h_n^{-1}$ as $T\to\infty$.

Now, for a fixed $\underline{\omega}_0\in\underline{\Omega}$, let $h(\underline{\omega}) = \underline{\omega}\,\underline{\omega}_0$ for $\underline{\omega}\in\underline{\Omega}$. Then, obviously,

$$h_n\left(h\big((p^{-i\tau}: p\in\mathcal{P}), \big((m+\alpha_1)^{-i\tau}: m\in\mathbb{N}_0\big),\ldots,\big((m+\alpha_r)^{-i\tau}: m\in\mathbb{N}_0\big)\big)\right) =$$
$$= Z_n(s_1+i\tau, s+i\tau, \underline{\omega}_0, \underline{\alpha};\underline{\mathfrak{a}},\mathcal{L}).$$

Therefore, repeating the above arguments and using the invariance of the Haar measure $\underline{m}_H$, we find that the measure $\hat{P}_{T,n}$ also converges weakly to $\underline{m}_H h_n^{-1}$ as $T \to \infty$. Thus, the measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to the measure $P_n = \underline{m}_H h_n^{-1}$ as $T \to \infty$.

The change Z_n by Z requires certain approximation results. Denote by $\rho_{\mathcal{L}}$, ρ and ρ_v the metrices on $H(D_{\mathcal{L}})$, $H(D)$ and H^v, respectively, inducing the topology of uniform convergence on compacta. We note that, for $\underline{g} = (g, g_{11}, \ldots, g_{1l_1}, \ldots, g_{r1}, \ldots, g_{rl_r})$, $\underline{f} = (f, f_{11}, \ldots, f_{1l_1}, \ldots, f_{r1}, \ldots, f_{rl_r}) \in$ $\in H^v$,

$$\rho_v(\underline{g}, \underline{f}) = \max\left(\rho_{\mathcal{L}}(g, f), \max_{1 \le j \le r} \max_{1 \le l \le l_j} \rho(g_{jl}, f_{jl})\right).$$

Lemma 2.3. *We have*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_v\big(Z(s_1 + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}), Z_n(s_1 + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})\big) d\tau = 0.$$

Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_v\big(Z(s_1 + i\tau, s + i\tau, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}), Z_n(s_1 + i\tau, s + i\tau, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})\big) d\tau = 0.$$

Proof. By the definition of the metric ρ_v,

$$\frac{1}{T} \int_0^T \rho_v\big(Z(s_1 + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}), Z_n(s_1 + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})\big) d\tau \le$$

$$\le \frac{1}{T} \int_0^T \rho_{\mathcal{L}}\big(\mathcal{L}(s_1 + i\tau), \mathcal{L}_n(s_1 + i\tau)\big) d\tau +$$

$$+ \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{1}{T} \int_0^T \rho\big(\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}), \zeta_n(s + i\tau, \alpha_j; \mathfrak{a}_{jl})\big) d\tau. \tag{2.7}$$

Moreover, in view of Lemma 4.8 from [19],

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho_{\mathcal{L}}\big(\mathcal{L}(s_1 + i\tau), \mathcal{L}_n(s_1 + i\tau)\big) d\tau = 0, \tag{2.8}$$

and, by the proof of Lemma 2.4 from [11],

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}), \zeta_n(s + i\tau, \alpha_j; \mathfrak{a}_{jl})\big) d\tau = 0 \tag{2.9}$$

for $j = 1, \ldots, r$, $l = 1, \ldots, l_j$. Therefore, the first assertion of the lemma follows from (2.7)–(2.9). Similarly, in view of Lemma 4.10 from [19], for almost all $\hat{\omega}$ with respect to the measure $\hat{m}_H$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_{\mathcal{L}}\big(\mathcal{L}(s_1+i\tau,\hat{\omega}),\mathcal{L}_n(s_1+i\tau,\hat{\omega})\big)d\tau=0. \tag{2.10}$$

Let

$$\rho_u(g,f)=\max_{1\le j\le r}\max_{1\le l\le l_j}\rho(g_{jl},f_{jl})$$

and let $\widetilde{m}_H$ denote the Haar measure on $\big(\Omega_1\times\ldots\times\Omega_r,\mathcal{B}(\Omega_1\times\ldots\times\Omega_r)\big)$. Then formula (2.6) of [4] asserts that, for almost all $(\omega_1,\ldots,\omega_r)\in\Omega_1\times\ldots\times\Omega_r$ with respect to $\widetilde{m}_H$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_u\Big(\big(\zeta(s+i\tau,\omega_1,\alpha_1;\mathfrak{a}_{11}),\ldots,\zeta(s+i\tau,\omega_r,\alpha_r;\mathfrak{a}_{rl_r})\big),$$

$$\big(\zeta_n(s+i\tau,\omega_1,\alpha_1;\mathfrak{a}_{11}),\ldots,\zeta_n(s+i\tau,\omega_r,\alpha_r;\mathfrak{a}_{rl_r})\big)\Big)d\tau=0. \tag{2.11}$$

The measure $\underline{m}_H$ is the product of the measures $\hat{m}_H$ and $\widetilde{m}_H$. Therefore, the second assertion of the lemma follows from (2.10), (2.11) and the analogue of equality (2.7).

Lemma 2.4. *Suppose that $\mathcal{L}\in\mathcal{S}$, hypothesis* (1.1) *is satisfied and the numbers $\alpha_1,\ldots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measures P_T and*

$$\hat{P}_T(A)=\frac{1}{T}\text{meas}\big\{\tau\in[0,T]: Z(s_1+i\tau,s+i\tau,\underline{\omega},\underline{\alpha};\underline{\mathfrak{a}},\mathcal{L})\in A\big\},\qquad A\in\mathcal{B}(H^v),$$

both converge weakly, for almost all $\underline{\omega}\in\underline{\Omega}$, to the same probability measure P on $\big(H^v,\mathcal{B}(H^v)\big)$ as $T\to\infty$.

Proof. The properties of the class $\mathcal{S}$ implies that, for $\sigma>\frac{1}{2}$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\big|\mathcal{L}_n(\sigma+it)\big|^2dt=\sum_{m=1}^{\infty}\frac{|a(m)|u_n^2(m)}{m^{2\sigma}}\le\sum_{m=1}^{\infty}\frac{|a(m)|^2}{m^{2\sigma}}<\infty.$$

This and the Cauchy integral formula lead, for a compact set K of $D_{\mathcal{L}}$, to the estimate

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}\big|\mathcal{L}_n(s_1+i\tau)\big|d\tau\le C_K\left(\sum_{m=1}^{\infty}\frac{|a(m)|^2}{m^{2\sigma_K}}\right)^{1/2} \tag{2.12}$$

with some C_K and $\sigma_K>\max\left(\frac{1}{2},1-\frac{1}{d_{\mathcal{L}}}\right)$. By (2.5) of [11], for a compact subset K of D,

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}\big|\zeta_n(s+i\tau,\alpha_j;\mathfrak{a}_{jl})\big|d\tau\le B_K\left(\sum_{m=0}^{\infty}\frac{|a_{mjl}|^2}{(m+\alpha)^{2\hat{\sigma}_K}}\right)^{1/2} \tag{2.13}$$

with some $B_K>0$ and $\hat{\sigma}_K>\frac{1}{2}$ for all $j=1,\ldots,r$, $l=1,\ldots,l_j$.

Let θ be a random variable on a certain probability space $\big(\widetilde{\Omega},\mathcal{A},\mathbb{P}\big)$ and uniformly distributed on $[0,1]$. On this probability space define the H^v-valued random element $\underline{X}_{T,n}$ by the formula

$$\underline{X}_{T,n} = \underline{X}_{T,n}(s_1, s) =$$
$$= \big(X_{T,n}(s_1), X_{T,n,1,1}(s), \ldots, X_{T,n,1,l_1}(s), \ldots, X_{T,n,r,1}(s), \ldots, X_{T,n,r,l_r}(s)\big) =$$
$$= Z_n\left(s_1 + i\theta T, s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}\right).$$

Then, by Lemma 2.2, we have that

$$\underline{X}_{T,n} \xrightarrow{\mathcal{D}} \underline{X}_n, \tag{2.14}$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution, and

$$\underline{X}_n = \underline{X}_n(s_1, s) = \big(X_n(s_1), X_{n,1,1}(s), \ldots, X_{n,1,l_1}(s), \ldots, X_{n,r,1}(s), \ldots, X_{n,r,l_r}(s)\big)$$

is the H^v-valued random element with the distribution P_n (P_n is the limit measure in Lemma 2.2). Using (2.12)–(2.14), we prove, in a standard way, see, for example, [11, 19], that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight, i.e., that, for every $\epsilon > 0$, there exists a compact set $K = K(\epsilon) \subset H^v$ such that $P_n(K) > 1 - \epsilon$ for all $n \in \mathbb{N}$. Indeed, let the compact sets $\hat{K}_m$ and K_m come from the definition of the metric ρ_v (the definition of the metric ρ is given in [8]). Let

$$\hat{R}_m = C_{\hat{K}_m}\left(\sum_{m=1}^{\infty} \frac{|a(m)|^2)}{k^{2\sigma_{\hat{K}_m}}}\right)^{1/2}$$

and

$$R_{jlm} = B_{K_m}\left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_{K_m}}}\right)^{1/2}.$$

Let $\epsilon > 0$ be an arbitrary number, and

$$\hat{M}_m = \hat{R}_m 2^{m+1}\epsilon^{-1}, \qquad M_{jlm} = R_{jlm} 2^{u+m+1}\epsilon^{-1}, \quad m \in \mathbb{N}.$$

Then, in virtue of (2.12) and (2.13),

$$\limsup_{T\to\infty} \mathbb{P}\left(\left(\sup_{s_1 \in \hat{K}_m} \left|X_{T,n}(s_1)\right| > \hat{M}_m\right) \quad \text{or} \quad \left(\exists j, l\colon \sup_{s \in K_m} \left|X_{T,n,j,l}(s)\right| > M_{jlm}\right)\right) \le$$
$$\le \limsup_{T\to\infty} \mathbb{P}\left(\sup_{s_1 \in \hat{K}_m} \left|X_{T,n}(s_1)\right| > \hat{M}_m\right) +$$
$$+ \limsup_{T\to\infty} \mathbb{P}\left(\exists j, l\colon \sup_{s \in K_m} \left|X_{T,n,j,l}(s)\right| > M_{jlm}\right) \le$$
$$\le \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \sup_{s_1 \in \hat{K}_m} \left|\mathcal{L}(s_1 + i\tau)\right| > \hat{M}_m\right\} +$$
$$+ \limsup_{T\to\infty} \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \sup_{s \in K_m} \left|\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl})\right| > M_{jlm}\right\} \le$$

$$\leq \limsup_{T\to\infty} \frac{1}{T\hat{M}_m} \int_0^T \sup_{s_1\in\hat{K}_m} |\mathcal{L}(s_1+i\tau)| d\tau +$$

$$+\sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{T\to\infty} \frac{1}{TM_{jlm}} \int_0^T \sup_{s\in K_m} \big|\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl})\big| d\tau \leq \frac{\epsilon}{2^m}.$$

Hence, by (2.14),

$$\mathbb{P}\left(\left(\sup_{s_1\in\hat{K}_m} |X_n(s_1)| > \hat{M}_m\right) \quad \text{or} \quad \left(\exists j, l\colon \sup_{s\in K_m} \big|X_{n,j,l}(s)\big| > M_{jlm}\right)\right) \leq \frac{\epsilon}{2^m}, \qquad (2.15)$$

$j = 1, \ldots, r,\ l = 1, \ldots, l_j$. Define the set

$$K_\epsilon^v = \Big\{\underline{f} \in H^v\colon \sup_{s_1\in\hat{K}_m} \big|f(s_1)\big| \leq \hat{M}_m,\ \sup_{s\in K_m} \big|f_{jl}(s)\big| \leq M_{jlm},\ j = 1, \ldots, r,\ l = 1, \ldots, l_j,\ m \in \mathbb{N}\Big\}.$$

Then K_ϵ^v is a compact set in H^v. Moreover, by (2.15),

$$P_n\big(K_\epsilon^v\big) \geq 1 - \epsilon$$

for all $n \in \mathbb{N}$, i.e., $\{P_n : n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem [1] (Theorem 6.1). (If the family of probability measures is tight, then it is relatively compact), this family is relatively compact. Therefore, there exists a sequence $\underline{X}_{n_k}$ and a probability measure P on $\big(H^v, \mathcal{B}(H^v)\big)$ such that

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \qquad (2.16)$$

On $(\widetilde{\Omega}, \mathcal{A}, \mathbb{P})$, define one more H^v-valued random element $\underline{X}_T$ by the formula

$$\underline{X}_T = \underline{X}_T(s_1, s) = Z\big(s_1 + i\theta T, s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L})\big).$$

Then, in view of Lemma 2.3, we obtain that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\rho_v(\underline{X}_T, \underline{X}_{T,n}) \geq \epsilon\big) = 0.$$

This, (2.14) and (2.16) show that all hypothesis of Theorem 4.2 of [1] are satisfied. Therefore, by this theorem

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathcal{D}} P,$$

which is equivalent to the weak convergence of P_T to P as $T \to \infty$. From this, it follows that the measure P is independent on the sequence X_{n_k}. Thus,

$$\underline{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \qquad (2.17)$$

Similarly, using the H^v-valued random elements

$$Z_n\big(s_1 + i\theta T, s + i\theta T, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}\big)$$

and

$$Z\big(s_1 + i\theta T, s + i\theta T, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}\big)$$

as well as relation (2.17), we obtain that the measure $\hat{P}_T$ also converges weakly to P as $T \to \infty$.

***Proof of Theorem* 2.1.** In virtue of Lemma 2.4, it suffices to show that $P = P_Z$. Let A be a continuity set of the measure P. On the probability space $\big(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H\big)$, define the random variable ξ by the formula

$$\xi(\omega) = \begin{cases} 1, & \text{if} \quad Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4 implies the relation

$$\lim_{T\to\infty} \hat{P}_T(A) = P(A). \tag{2.18}$$

By the definition of ξ, the expectation $\mathbb{E}\xi$ is

$$\mathbb{E}\xi = \int\limits_{\underline{\Omega}} \xi d\underline{m}_H = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega} \colon Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\big) = P_Z(A). \tag{2.19}$$

For $\tau \in \mathbb{R}$, define the transformation Φ_τ of $\underline{\Omega}$ by

$$\Phi_\tau(\underline{\omega}) = \Big(\big(p^{-i\tau} : p \in \mathcal{P}\big), \big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \ldots, \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)\Big)\underline{\omega},$$

$\underline{\omega} \in \underline{\Omega}$. Lemma 7 of [8] asserts that the group of measurable measure preserving transformations $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is ergodic. Hence, the random process $\xi\big(\Phi_\tau(\underline{\omega})\big)$ is ergodic as well. Therefore, by the classical Birkhoff–Khintchine theorem,

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \xi\big(\Phi_\tau(\underline{\omega})\big) d\tau = \mathbb{E}\xi. \tag{2.20}$$

On the other hand, by the definitions of ξ and Φ_τ,

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \xi\big(\Phi_\tau(\underline{\omega})\big) d\tau = \frac{1}{T}\text{meas}\big\{\tau \in [0,T] \colon Z(s_1 + i\tau, s + i\tau, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\big\}.$$

This, (2.19) and (2.20) show that

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\big\{\tau \in [0,T] \colon Z(s_1 + i\tau, s + i\tau, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\big\} = P_Z(A).$$

Therefore, in view of (2.18) and the definition of $\hat{P}_T$, $P(A) = P_Z(A)$ for all continuity sets A of the measure P. Since all continuity sets constitute a determining class, we have that $P = P_Z$.

The theorem is proved.

3. Support. We recall that the support of P_Z is the minimal closed set $S_{P_Z} \subset H^v$ such that $P_Z(S_{P_Z}) = 1$. The set S_{P_Z} consists of all points $\underline{g} \in H^v$ such that, for every open neighbourhood G of $\underline{g}$, the inequality $P_Z(G) > 0$ is satisfied.

Let

$$S_{\mathcal{L}} = \big\{g \in H(D_{\mathcal{L}}) \colon g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\big\}.$$

Theorem 3.1. *Suppose that* $\mathcal{L} \in \mathcal{S}$, *hypothesis* (1.1) *is satisfied, the numbers* $\alpha_1, \ldots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$, *and that* $\operatorname{rank}(A_j) = l_j$, $j = 1, \ldots, r$. *Then the support of the measure* P_Z *is the set* $S_{\mathcal{L}} \times H^u(D)$.

Proof. We have that

$$H^v = H(D_{\mathcal{L}}) \times H^u(D). \tag{3.1}$$

Since the spaces $H(D_{\mathcal{L}})$ and $H^u(D)$ are separable, by (3.1),

$$\mathcal{B}(H^v) = \mathcal{B}\big(H(D_{\mathcal{L}})\big) \times \mathcal{B}\big(H^u(D)\big).$$

Therefore, it suffices to consider $P_Z(A)$ for $A = A_1 \times A_2$ with $A_1 \in H(D_{\mathcal{L}})$ and $A_2 \in H^u(D)$. The measure $\underline{m}_H$ is the product of the measures $\hat{m}_H$ and $\widetilde{m}_H$. Hence,

$$P_Z(A) = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega} \colon Z(s_1, s, \underline{\omega}, \underline{\alpha}; \underline{\mathfrak{a}}, \mathcal{L}) \in A\big) =$$

$$= \underline{m}_H\Big(\underline{\omega} \in \underline{\Omega} \colon \mathcal{L}(s_1, \hat{\omega}) \in A_1, \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in A_2\Big) =$$

$$= \hat{m}_H\big(\hat{\omega} \in \hat{\Omega} \colon\ \mathcal{L}(s_1, \hat{\omega}) \in A_1\big)\widetilde{m}_H\Big((\omega_1, \ldots, \omega_r) \in \Omega_1 \times \ldots \times \Omega_r \colon$$

$$\Big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\Big) \in A_2\Big). \tag{3.2}$$

By Proposition 3 of [15], the support of the random element $\mathcal{L}(s_1, \hat{\omega})$ is the set $S_{\mathcal{L}}$. We note that in [15], the $H(D_{\mathcal{L},N})$-valued random element, where

$$D_{\mathcal{L},N} = \left\{s \in \mathbb{C} \colon \max\left(\frac{1}{2}, 1 - \frac{1}{d_{\mathcal{L}}}\right) < \sigma < 1,\ |t| < N\right\},$$

is considered, however, the proof remains valid for the whole strip $D_{\mathcal{L}}$. Thus, $S_{\mathcal{L}}$ is a minimal closed subset of $H(D_{\mathcal{L}})$ such that

$$\hat{m}_H\left(\hat{\omega} \in \hat{\Omega} \colon \mathcal{L}(s, \hat{\omega}) \in S_{\mathcal{L}}\right) = 1. \tag{3.3}$$

Also, under hypotheses of the theorem, it was proved in [9], Theorem 3.1, that the support of the $H^u(D)$-valued random element $\big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big)$ is the set $H^u(D)$, i.e., $H^u(D)$ is a minimal closed subset of $H^u(D)$ such that

$$\widetilde{m}_H\Big((\omega_1, \ldots, \omega_r) \in \Omega_1 \times \ldots \times \Omega_r \colon \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in H^u(D)\Big) = 1.$$

Combining this with (3.3) and (3.2) gives the assertion of the theorem.

4. Proof of Theorem 1.3. Theorem 1.3 is a consequence of Theorems 2.1 and 3.1, and Mergelyan's theorem on the approximation of analytic functions by polynomials [13], see also [23].

***Proof of Theorem* 1.3.** By the Mergelyan theorem, there exist polynomials $p(s)$ and $p_{jl}(s)$ such that

$$\sup_{s \in K} \big|f(s) - p(s)\big| < \frac{\epsilon}{4} \tag{4.1}$$

and

$$\sup_{1 \le j \le r}\ \sup_{1 \le l \le l_j}\ \sup_{s \in K_{jl}} \big|f_{jl}(s) - p_{jl}(s)\big| < \frac{\epsilon}{2}. \tag{4.2}$$

Since $f(s) \neq 0$ on K, we have that $p(s) \neq 0$ on K as well if ϵ is small enough. Therefore, there exists a continuous branch of $\log p(s)$ on K which is analytic in the interior of K. This and the Mergelyan theorem show that there is a polynomial $q(s)$ such that

$$\sup_{s\in K} \left|p(s) - e^{q(s)}\right| < \frac{\epsilon}{4}.$$

Therefore, (4.1) implies that

$$\sup_{s\in K} \left|f(s) - e^{q(s)}\right| < \frac{\epsilon}{2}. \tag{4.3}$$

Define the set

$$G = \left\{(g, g_{11}, \ldots, g_{rl_r}) \in H^v : \sup_{s\in K} \left|g(s) - e^{q(s)}\right| < \frac{\epsilon}{2}, \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} \left|g_{jl}(s) - p_{jl}(s)\right| < \frac{\epsilon}{2}\right\}.$$

Then, in view of Theorem 3.1, G is an open neighbourhood of the element $\left(e^{q(s)}, p_{11}(s), \ldots, p_{rl_r}(s)\right)$ of the support of the measure P_Z. Consequently, $P_Z(G) > 0$. Therefore, by Theorem 2.1,

$$\liminf_{T\to\infty} P_T(G) \ge P_Z(G) > 0,$$

and, by the definition of G,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{\tau \in [0, T] : \sup_{s\in K} \left|\mathcal{L}(s + i\tau) - e^{q(s)}\right| < \frac{\epsilon}{2},\right.$$

$$\left.\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} \left|\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - p_{jl}(s)\right| < \frac{\epsilon}{2}\right\} > 0.$$

Combining this with inequalities (4.3) and (4.2) gives the assertion of Theorem 1.3.

References

1. *Billingsley P.* Convergence of probability measures. – New York: Wiley, 1968.
2. *Gel'fond A. O.* On algebraic independence of transcendental numbers of certain classes // Uspekhi Mat. Nauk. – 1949. – **4**, № 5. – P. 14 – 48.
3. *Heyer H.* Probability measures on locally compact groups. – Berlin: Springer-Verlag, 1977.
4. *Genys J., Macaitienė R., Račkauskienė S., Šiaučiūnas D.* A mixed joint universality theorem for zeta-functions // Math. Model. and Anal. – 2010. – **15**, № 4. – P. 431 – 446.
5. *Kaczorowski J., Perelli A.* The Selberg class: a survey // Number Theory in Progress. Proc. Intern. Conf. in honor of the 60th birthday of A. Schinzel (Zakopane, 1997). Vol. 2: Elementary and Analytic Number Theory. – Berlin: Walter De Gruyter, 1999. – P. 953 – 992.
6. *Kačinskaitė R., Laurinčikas A.* The joint distribution of periodic zeta-functions // Stud. Sci. Math. Hung. – 2011. – **48**. – P. 257 – 279.
7. *Laurinčikas A.* Limit theorems for the Riemann zeta-function. – Dordrecht etc.: Kluwer Acad. Publ., 1996.
8. *Laurinčikas A.* Joint universality of zeta-functions with periodic coefficients // Izv. RAN. Ser. Mat. – 2010. – **74**, № 3. – P. 79 – 102 (in Russian).
9. *Laurinčikas A., Skerstonaitė S.* Joint universality for periodic Hurwitz zeta-functions. II // New Directions in Value-Distribution Theory of Zeta and L-functions (Würzburg, 2008). – Aachen: Shaker-Verlag, 2009. – P. 161 – 169.
10. *Laurinčikas A., Šiaučiūnas D.* A mixed joint universality theorem for zeta-functions. III // Anal. Prob. Methods Number Theory. J. Kubilius Memorial Volume. – Vilnius: TEV, 2012. – P. 185 – 195.

11. *Macaitienė R.* On joint universality for the zeta-functions of newforms and periodic Hurwitz zeta-functions // RIMS Kôkyûroku Bessatsu: Functions in Number Theory and Their Probab. Aspects. – 2012. – **B34**. – P. 217–233.
12. *Matsumoto K.* A survey on the theory of universality for zeta and L-functions // Number Theory: Plowing and Starring Through High Wave Forms: Proc. 7th China – Jap. Sem. (Fukuoka 2013). Ser. Number Theory and its Appl. – 2015. – **11**. – P. 95 – 144.
13. *Mergelyan S. N.* Uniform approximations to functions of complex variable // Uspechi Mat. Nauk. – 1952. – **7**. – P. 31 – 122 (in Russian).
14. *Mishou H.* The joint value-distribution of the Riemann zeta function and Hurwitz zeta functions // Liet. Mat. Rink. – 2007. – **47**. – P. 62 – 80.
15. *Nagoshi H., Steuding J.* Universality for L-functions in the Selberg class // Lith. Math. J. – 2010. – **50 (163)**. – P. 293 – 311.
16. *Pocevičienė V., Šiaučiūnas D.* A mixed joint universality theorem for zeta-functions. II // Math. Model. and Anal. – 2014. – **19**. – P. 52 – 65.
17. *Selberg A.* Old and new conjectures and results about a class of Dirichlet series // Proc. Amalfi Conf. Anal. Number Theory (Maiori, 1989). – Salerno: Univ. Salerno, 1992. – P. 367 – 385.
18. *Steuding J.* On the universality for functions in the Selberg class // Proc. Sesion in Analytic Number Theory and Diophantine Equations (Bonn, 2002): Bonner Math. Schriften. – 2003. – **360**. – P. 22.
19. *Steuding J.* Value-distribution of L-functions // Lecture Notes Math. – 2007. – **1877**.
20. *Voronin S. M.* Theorem on the 'universality' of the Riemann zeta-function // Izv. Akad. Nauk SSSR. Ser. Mat. – 1975. – **39**. – P. 475 – 486 (in Russian).
21. *Voronin S. M.* On the functional independence of Dirichlet L-functions // Acta Arithm. – 1975. – **27**. – P. 493 – 503 (in Russian).
22. *Voronin S. M.* Analytic properties of generating functions of arithmetical objects: Diss. Doctor Phys.-Mat. Nauk. – Moscow, 1977 (in Russian).
23. *Walsh J. L.* Interpolation and approximation by rational functions in the complex domain // Amer. Math. Soc. Colloq. Publ. – 1960. – **20**.

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