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PROPERTIES OF THE LOGICAL CONSEQUENCE OPERATION AND ITS RELATIONSHIP WITH THE INDEPENDENCE OF PROPOSITIONAL LOGIC

ВЛАСТИВОСТІ ОПЕРАЦІЇ ЛОГІЧНОГО НАСЛІДКУ ТА ЇЇ ЗВ'ЯЗОК З НЕЗАЛЕЖНІСТЮ ПРОПОЗИЦІЙНОЇ ЛОГІКИ

We investigate the properties of the logical consequence operation and the characteristic features of independent sets of formulas. Further, we apply these results to propositional logic. Finally, we show under what conditions the results of addition of a formula to independent sets of formulas and the union of two independent sets of formulas are also independent, by using the operation of logical consequence, i.e., we establish a relationship between the logical consequence and the preservation of independence in propositional logic.

Вивчаються властивості операції логічного наслідку та характерні особливості незалежних множин формул. Отримані результати застосовуються до пропозиційної логіки. Крім того, встановлено, за яких умов результат додавання формули до незалежних множин формул та об'єднання двох незалежних множин формул також є незалежним за операцією логічного наслідку, тобто встановлено співвідношення між логічним наслідком та збереженням незалежності у пропозиційній логіці.

1. Introduction. The logical consequence undisputedly is the central concept of logic. The main purpose of logic is to tell us what follows logically from what. Logical consequence is a relation between a given set of formulas and the formulas that logically follow. But this concept has a long story before the adoption of its validity. In a series of papers, published early 1930's [14], Tarski describes his logical perspective as follows: our goal is to study the properties of deductive systems. A deductive system, or a formal theory, is the set of all formulas which follows formally from a set of formulas; more precisely, the formal theory T includes a set of axioms A and a set of inference rules R. Then the set of logical consequences of a formula φ in T was defined as the smallest set of formulas of T that contains φ and the axioms in A, and is closed under the rules in R.

The need for semantic definitions of the same concepts arose when Tarski realized that there was a serious gap between the proof theoretical definitions and the intuitive concepts: many intuitive consequences of the formal theories were undetectable by standard system of proof. His conclusion was that proof theory can provide only a partical account of the metalogic [5].

Tarski described the *intuitive* content of the concept logical consequence as follows. Let Σ be a set of formulas and φ be a formula that follows from Σ.

(C) If φ is a logical consequence of Σ, then φ is a *necessary consequence* of Σ in the sense that it cannot be the case that all formulas in Σ are true and φ is false.

Further, he introduced the notion of model and proposed the *formal* definition of logical consequence in terms of models.

(LC) The sentence φ follows logically from Σ if and only if every model of Σ is also a model of φ.

Now the definition of logical truth immediately follows:

(LT) The sentence φ of a formal theory T is logically true if and only if every model of T is a model of φ.

That (LC) satisfies (C) is a very simple observation [10]:

Suppose that φ is a logical consequence of Σ, i.e., φ is true in all models in which all the formulas in Σ are true. Now assume that φ is not a necessary consequence of Σ. Then it is possible that all the elements of Σ are true and φ is false. But in that case there is a model in which all the members of Σ come out true and φ happens to be false. Contradiction.

From now on $\Sigma \vdash \varphi$ will mean that φ is a logical consequence of Σ. More on logical consequence can be found, for instance, in [1, 4, 7 – 9, 16]; works in [6] and [11] can be considered as main sources on the subject.

In this work, we deal with properties of the logical consequence operation and its relationship with independence of propositional logic. We prove that under which conditions the addition of a formula to independent sets of formulas, and the union of two independent sets of formulas are also independent by using the logical consequence operation. Briefly, we establish a relationship between logical consequence and the preservation of independence in propostional logic.

2. Properties of logical consequence $\boldsymbol{Cn}$. In this section, we give definitions about consequence operation Cn. Further, we indicate basic properties of consequence operation. As we shall see in the remaining sections, they are the basis of this work.

Definition 2.1. *Let L be any formal language. A consequence operation on L is a function $Cn\colon 2^L \to 2^L$ such that the following holds*:

(i) $\Sigma \subseteq Cn(\Sigma)$, *all* $\Sigma \subseteq L$ (*reflexivity*),

(ii) *for all* $\Sigma_1,\ \Sigma_2 \subseteq L$, *if* $\Sigma_1 \subseteq \Sigma_2$ *then* $Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$ (*monotonacity*),

(iii) *for all* $\Sigma \subseteq L$, $Cn(Cn(\Sigma)) = Cn(\Sigma)$ (*idempotency*).

Definition 2.2. *Cn is a structural consequence operation if and only if $f(Cn(\Sigma)) \subseteq Cn(f(\Sigma))$ for every endomorphism f of L.*

Definition 2.3. *Cn is a finite (or algebraic) consequence operation if and only if $Cn(\Sigma) = = \bigcup\{Cn(\nabla)\colon \nabla \subseteq \Sigma, \nabla finite\}$, where Σ is the subset of any formal language L.*

Definition 2.4. *If Cn is finite and structural*, *then Cn is standart.*

Definition 2.5 [3]**.** (i) *Cn is stronger than Cn' $(Cn' \leq Cn)$ if and only if for all Σ, $Cn'(\Sigma) \subseteq \subseteq Cn(\Sigma)$.*

(ii) *Cn is properly stronger than Cn' $(Cn' < Cn)$ if and only if Cn is stronger than Cn' and there is a Σ such that $Cn'(\Sigma) \subset Cn(\Sigma)$.*

Definition 2.6 [3]**.** (i) *Cn is consistent if and only if $Cn\varnothing) \neq L$.*

(ii) *Cn is compact if and only if for each $\Sigma \subseteq L\colon$ if $Cn(\Sigma) = L$, then there exists a finite $\Sigma' \subseteq \Sigma$ such that $Cn(\Sigma') = L$.*

(iii) *Σ is a Cn-theory if and only if $Cn(\Sigma) = \Sigma$.*

(iv) *Σ is Cn-consistent if and only if $Cn(\Sigma) \neq L$.*

(v) *Σ is Cn-complete if and only if for all $A\colon$ if $\Sigma \cup \{A\}$ is consistent, then $A \in Cn(\Sigma)$.*

(vi) *Σ is Cn-maximally consistent if and only if Σ is consistent, and there does not exist a consistent Σ' such that $\Sigma \subset \Sigma'$.*

(vii) *Σ is Cn-axiom system for Σ' if and only if $Cn(\Sigma) = Cn(\Sigma')$.*

(viii) *A is Cn-independent in Σ if and only if $A \in \Sigma$ and $A \notin Cn(\Sigma - \{A\})$.*

(ix) *A is Cn-tautology if and only if $A \in Cn(\varnothing)$.*

Definition 2.7 [3]. *A set Ω is a closure system if and only if it is closed intersection. Namely, if $\Gamma \subseteq \Omega$, then $\bigcap \Gamma \in \Omega$.*

Let $\langle L, Cn\rangle$ denote a logic on a language L and a consequence operation Cn. Also, $(Cn \cap Cn')(\Sigma) = Cn(\Sigma) \cap Cn'(\Sigma)$ and $(Cn \cup Cn')(\Sigma) = Cn(\Sigma) \cup Cn'(\Sigma)$ for all $\Sigma \subseteq L$. Now we can give some results on Cn.

Lemma 2.1. *Let Σ_1 and Σ_2 be sets in $\langle L, Cn\rangle$. Then the following identities hold:*

(i) $Cn(Cn(\Sigma_1) \cup Cn(\Sigma_2)) = Cn(\Sigma_1 \cup \Sigma_2)$,

(ii) $Cn(Cn(\Sigma_1) \cap Cn(\Sigma_2)) = Cn(\Sigma_1) \cap Cn(\Sigma_2)$.

Lemma 2.2. *Let $\langle L, Cn\rangle$ and $\langle L, Cn'\rangle$ be consistent. Then $\langle L, Cn \cap Cn'\rangle$ is also consistent.*

Remark **2.1.** In general the union of $\langle L, Cn\rangle$ and $\langle L, Cn'\rangle$ is not consistent. For example, let $Cn(\varnothing) = \Psi(\varnothing \neq \Psi \neq L)$ and $Cn'(\varnothing) = L - \Psi$. We get $Cn(\varnothing) \cup Cn'(\varnothing) = L$. Therefore, $\langle L, Cn \cup Cn'\rangle$ is not consistent.

Lemma 2.3. *Let $\langle L, Cn\rangle$ and $\langle L, Cn'\rangle$ be compact. Then $\langle L, Cn \cap Cn'\rangle$ and $\langle L, Cn \cup Cn'\rangle$ are also compact.*

Lemma 2.4. *If Σ_1 and Σ_2 are Cn-theory, then $Cn(\Sigma_1) \cap Cn(\Sigma_2) = Cn(\Sigma_1 \cap \Sigma_2)$.*

Definition 2.8. *A set $T \subseteq L$ is closed under Cn if and only if $T = Cn(T)$.*

Lemma 2.5. *Every Cn-theory is closed.*

In this study, the class of all closed sets in a logic $\langle L, Cn\rangle$ is denoted by $\Im = \{T \in 2^L : T = Cn(T)\}$. We can easily show that $(\Im, \subseteq)$ is a partial order structure. At the same time, for every $\Gamma \subseteq \Im$, $\mathrm{Sup}(\Gamma) = \mathrm{Cn}(\bigcup \Gamma) \in \Im$ and $\mathrm{Inf}(\Gamma) = \mathrm{Cn}(\bigcap \Gamma) \in \Im$. Every subset of $\Im$ has a Sup and Inf. Hence, $(\Im, \subseteq)$ is a complete lattice.

Lemma 2.6. *Let A be the any subset of tautologies set (it means that $Cn(A) = Cn(\varnothing)$). L is the maximal element and $Cn(A)$ is the minimal element in $\Im$.*

Lemma 2.7. *Let $\Sigma \neq L$. Σ is closed Cn-complete if and only if Σ is Cn-maximally consistent.*

Proof. Let Σ be closed Cn-complete. For all A, if $\Sigma \cup \{A\}$ is Cn-consistent, then $\{A\} \subseteq \Sigma$. Therefore, Σ is Cn-maximally consistent. We assume that Σ is Cn-maximally consistent. For all A, if $\Sigma \cup \{A\}$ is Cn-consistent, then $\{A\} \subseteq \Sigma$ and also $A \in Cn(\Sigma)$, because Σ is Cn-maximally consistent. Hence, Σ is Cn-complete. If $A \in Cn(\Sigma)$, then $\Sigma \cup \{A\}$ is Cn-consistent. Since Σ is Cn-maximally consistent, $\{A\} \subseteq \Sigma$. So, $A \in \Sigma$. Therefore, $Cn(\Sigma) = \Sigma$.

Lemma 2.8. *If Σ_1 and Σ_2 are Cn-consistent, then $\Sigma_1 \cap \Sigma_2$ is also Cn-consistent.*

Remark **2.2.** The union of any two Cn-consistent sets is not in general Cn-consistent.

Lemma 2.9 [13]. *Let $Cn_1 \leq Cn_2 \leq Cn_3 \leq \ldots$ be an infinite chain of finite consequence operations and $Cn = \mathrm{Sup}\{Cn_i : i = 1, 2, 3, \ldots\}$. Therefore*

$$Cn(\Sigma) = \bigcup_{i \in \mathbb{N}} Cn_i(\Sigma)$$

for every $\Sigma \subseteq L$.

Lemma 2.10. *Let $Cn_1 \leq Cn_2 \leq Cn_3 \leq \ldots$ be an infinite chain of finite structural consequence operations and $Cn = \mathrm{Sup}\{Cn_i : i = 1, 2, 3, \ldots\}$. Then Cn is also structural consequence operation.*

Proof. Let f be any endomorphism and $\Sigma \subseteq L$:

$$f(Cn(\Sigma)) = f\left(\bigcup_{i \in \mathbb{N}} Cn_i(\Sigma)\right) \ \text{(by Lemma 2.9)} = \bigcup_{i \in \mathbb{N}} f(Cn_i(\Sigma)) \ \text{(since } f \text{ is an endomorphism)} \subseteq$$

$$\subseteq \bigcup_{i\in\mathbb{N}} Cn_i(f(\Sigma)) \text{ (by Definition 2.2)} = Cn(f(\Sigma)) \text{ (by Lemma 2.9).}$$

Therefore, Cn is a structural consequence operation.

3. Properties of propositional logical consequence Cn. The notion of propositional consequence operation includes all systems of classical propositional logic [1, 12]. And, we use L_{PROP} for describing language of propostional logic which is generated by the connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$.

Throughout the study φ, φ_1, $\varphi_2, \ldots$ and Σ, Σ_1, $\Sigma_2, \ldots$ denote propositional logical formulas and sets of propositional logical formulas, respectively. $Cn(\Sigma)$ stands for the set of logical consequences of Σ; that is, $\varphi \in Cn(\Sigma)$ if and only if $\Sigma \vdash \varphi$.

Definition 3.1 [3]**.** *A consequence operation Cn is a propositional consequence operation if and only if for $\varphi_1,\ \varphi_2 \in L_{PROP}$ and $\Sigma \subseteq L_{PROP}$:*

($\neg$) $\varphi_1 \in Cn(\Sigma)$ *if and only if* $Cn(\Sigma \cup \{\neg\varphi_1\}) = L_{PROP}$,

($\wedge$) $Cn(\Sigma \cup \{\varphi_1 \wedge \varphi_2\}) = Cn(\Sigma \cup \{\varphi_1, \varphi_2\})$,

($\vee$) $Cn(\Sigma \cup \{\varphi_1 \vee \varphi_2\}) = Cn(\Sigma \cup \{\varphi_1\}) \cap Cn(\Sigma \cup \{\varphi_2\})$,

($\rightarrow$) $\varphi_1 \rightarrow \varphi_2 \in Cn(\Sigma)$ *if and only if* $\varphi_2 \in Cn(\Sigma \cup \{\varphi_1\})$.

Definition 3.2. *A set of formulas Σ is said to be independent if for all φ belonging to Σ, φ is not a logical consequence of $\Sigma \setminus \{\varphi\}$; in symbols, if $\Sigma \setminus \{\varphi\} \nvdash \varphi$. Equivalently, Σ is independent if there is a model for $(\Sigma \setminus \{\varphi\}) \cup \{\neg\varphi\}$, where $\neg$ is the negation operation.*

Definition 3.3. *Two set of formulas are said to be equivalent if any formula of the one set is a consequence of the other set and conversely. Equivalently, two sets are equivalent when they have the same models.*

We recall the following results.

Lemma 3.1 [11]**.** *Given $\varphi_1, \varphi_2, \varphi_3$ and Σ_1, Σ_2, we have*

(i) $\varphi_1 \vdash \varphi_1$;

(ii) *if* $\Sigma_1 \vdash \varphi_1$, *then* $\Sigma_1 \cup \Sigma_2 \vdash \varphi_1$;

(iii) *if* $\Sigma_1 \cup \{\varphi_2\} \vdash \varphi_1$, *and* $\Sigma_2 \vdash \varphi_2$, *then* $\Sigma_1 \cup \Sigma_2 \vdash \varphi_1$;

(iv) $\Sigma_1 \vdash \varphi_1$ *if and only if* $\Sigma_2 \vdash \varphi_1$ *for some finite subset* Σ_2 *of* Σ_1;

(v) $\Sigma_1 \cup \{\varphi_1\} \vdash \varphi_2$ *if and only if* $\Sigma_1 \vdash \varphi_1 \rightarrow \varphi_2$;

(vi) $\Sigma_1 \cup \{\varphi_1, \varphi_2\} \vdash \varphi_3$ *if and only if* $\Sigma_1 \cup \{\varphi_1 \wedge \varphi_2\} \vdash \varphi_3$.

Note that we can write $\Sigma_1, \varphi_1, \varphi_2 \vdash \varphi_3$ for $\Sigma_1 \cup \{\varphi_1, \varphi_2\} \vdash \varphi_3$.

In this section, we prove some results on Cn in propositional logic, which are of set theoretical nature.

Lemma 3.2. *Given Σ_1 and Σ_2, we have*

$$Cn(\Sigma_1) \cup Cn(\Sigma_2) \subseteq Cn(\Sigma_1 \cup \Sigma_2).$$

Proof. Let $\varphi \in Cn(\Sigma_1) \cup Cn(\Sigma_2)$. Then $\varphi \in Cn(\Sigma_1)$ or $\varphi \in Cn(\Sigma_2)$. If $\varphi \in Cn(\Sigma_1)$, then $\Sigma_1 \vdash \varphi$, hence $\Sigma_1 \cup \Sigma_2 \vdash \varphi$ by Lemma 2.1 (ii). It follows that $\varphi \in Cn(\Sigma_1 \cup \Sigma_2)$. If $\varphi \in Cn(\Sigma_2)$, then exactly the same way we obtain $\varphi \in Cn(\Sigma_1 \cup \Sigma_2)$, so

$$Cn(\Sigma_1) \cup Cn(\Sigma_2) \subseteq Cn(\Sigma_1 \cup \Sigma_2).$$

***Corollary* 3.1.** *For a family of formulas Σ_k, we have*

$$\bigcup_{k=1}^{n} Cn(\Sigma_k) \subseteq Cn\left(\bigcup_{k=1}^{n} \Sigma_k\right).$$

Proof. Just use induction on n.
The following result is obvious.
Result 3.1. *Given Σ_1 and Σ_2, we have*

$$\Sigma_1 \cup \Sigma_2 \subseteq Cn(\Sigma_1) \cup Cn(\Sigma_2).$$

***Corollary* 3.2.** *For a family of formulas Σ_k, we have*

$$\bigcup_{k=1}^{n} \Sigma_k \subseteq \bigcup_{k=1}^{n} Cn(\Sigma_k).$$

Thus, we get the following inclusions:

$$\bigcup_{k=1}^{n} \Sigma_k \subseteq \bigcup_{k=1}^{n} Cn(\Sigma_k) \subseteq Cn\left(\bigcup_{k=1}^{n} \Sigma_k\right).$$

Let Σ_1 and Σ_2 be any set of formulas in L_{PROP}. By the definition of Cn, we have that

$$\text{if } \Sigma_1 \subseteq \Sigma_2, \quad \text{then } Cn(\Sigma_1) \subseteq Cn(\Sigma_2).$$

Lemma 3.3. *Given Σ_1 and Σ_2, we have*

$$Cn(\Sigma_1 \cap \Sigma_2) \subseteq Cn(\Sigma_1) \cap Cn(\Sigma_2).$$

From the above results, we obtain the following inclusions:

$$\bigcap_{k=1}^{n} \Sigma_k \subseteq Cn\left(\bigcap_{k=1}^{n} \Sigma_k\right) \subseteq \bigcap_{k=1}^{n} Cn(\Sigma_k).$$

Lemma 3.4. *Let Σ_1 and Σ_2 be sets of formulas and A be any subset of a set which includes only tautologies. If $Cn(\Sigma_1) \cap Cn(\Sigma_2) = Cn(\varnothing)$, then $\Sigma_1 \cap \Sigma_2 = (\varnothing)$ or $\Sigma_1 \cap \Sigma_2 = A$.*

Proof. Assume that φ is any formula which is not a tautology and $\varphi \in \Sigma_1 \cap \Sigma_2$. Then $\varphi \in Cn(\Sigma_1) \cap Cn(\Sigma_2)$. We obtain the conclusion $Cn(\Sigma_1) \cap Cn(\Sigma_2) \neq Cn(\varnothing)$. It is contradiction.

***Remark* 3.1.** The converse of this theorem does not hold. To see this, consider the following formulas from formal number theory:

$\varphi_1 : \quad \forall xy(x' = y' \to x = y)$,

$\varphi_2 : \quad \forall xy(x + y' = (x + y)')$,

$\varphi_3 : \quad \forall xy(xy' = xy + x')$.

where $'$ is the successor function on $\mathbb{N}$. Then let $\Sigma_1 = \{\varphi_1 \wedge \varphi_2\}$ and $\Sigma_2 = \{\varphi_1 \wedge \varphi_3\}$. Now $\Sigma_1 \cap \Sigma_2 = \varnothing$, but $Cn(\Sigma_1) \cap Cn(\Sigma_2) \neq Cn(\varnothing)$, since $\Sigma_1 \vdash \varphi_1$ and $\Sigma_2 \vdash \varphi_1$.

Theorem 3.1. *Let Σ be Cn-complete and $\varphi \notin Cn(\Sigma)$. Then for all $\psi \in L_{PROP}$, $\varphi \to \psi \in$ $\in Cn(\Sigma)$.*

Proof. Assume that Σ is Cn-complete and $\varphi \notin Cn(\Sigma)$. Then $\Sigma \cup \{\varphi\}$ is not consistent. So, we hold that $\Sigma \cup \{\varphi\} \vdash \psi$. By the Lemma 3.1 (v), $\Sigma \vdash \varphi \rightarrow \psi$.

Theorem 3.2. *Any theory is complete in L_{PROP} if and only if it is maximal consistent.*

Proof. $(\Rightarrow)$ Let Σ be theory and complete. If $\varphi \notin Cn(\Sigma)$, then $\Sigma \cup \{\varphi\}$ is not consistent. As $\Sigma = Cn(\Sigma)$, Σ is maximal consistent.

$(\Leftarrow)$ It is clear.

4. Independent sets of propositional logical formulas and Cn. In this section, we state and prove some results on independent sets of propositional logical formulas related to logical consequence.

First of all, let Σ be an independent set of propositional logical formulas and φ be a formula. If we add a formula φ to Σ, and φ is pairwise independent with each member of Σ, it is possible that $\Sigma \cup \{\varphi\}$ is not independent. And the union of two independent sets of propositional logical formulas could not be independent. In the first instance, we ask that "under which conditions is $\Sigma \cup \{\varphi\}$ independent?". Further, "when is the union of two independent sets of propositional logical formulas independent?".

These observations allows us to deduce some results on independent sets of propositional logical formulas with respect to logical consequence operation.

Theorem 4.1. *Let Σ be an independent set of propositional logical formulas. If $Cn((\Sigma \cup \{\varphi_{n+1}\}) \setminus \{\varphi_k\}) = Cn(\Sigma \setminus \{\varphi_k\}) \cup Cn(\{\varphi_{n+1}\})$, where the union of each $\varphi_k \in \Sigma$ and $\{\varphi_{n+1}\}$ is independent set, then*

$$\Sigma \cup \{\varphi_{n+1}\} \text{ is independent if and only if } \Sigma \nvdash \varphi_{n+1}.$$

Proof. Set $\overline{\Sigma} = \Sigma \cup \{\varphi_{n+1}\}$ and suppose that $\overline{\Sigma}$ is independent. Then $\overline{\Sigma} \setminus \{\varphi_{n+1}\} \nvdash \varphi_{n+1}$. As $\overline{\Sigma} \setminus \{\varphi_{n+1}\} = \Sigma$, we have $\Sigma \nvdash \varphi_{n+1}$. Thus condition is necessary.
For the converse, assume $\Sigma \nvdash \varphi_{n+1}$. Then

$$\overline{\Sigma} \setminus \{\varphi_{n+1}\} \nvdash \varphi_{n+1}. \tag{4.1}$$

Suppose that $\Sigma \cup \{\varphi_{n+1}\} \setminus \{\varphi_k\} \nvdash \varphi_k$ for each $\varphi_k \in \Sigma$. Since Σ is independent, we have

$$\Sigma \setminus \{\varphi_k\} \nvdash \varphi_k, \tag{4.2}$$

and as $\{\varphi_k, \varphi_{n+1}\}$ is independent for every $\varphi_k \in \Sigma$ it follows that

$$\{\varphi_{n+1}\} \nvdash \varphi_k. \tag{4.3}$$

From (4.2) and (4.3), we get $\Sigma \setminus \{\varphi_k\} \nvdash \varphi_k$ and $\{\varphi_{n+1}\} \nvdash \varphi_k$. Thus by the hypothesis

$$\varphi_k \notin Cn(\Sigma \setminus \{\varphi_k\}) \text{ and } \varphi_k \notin Cn(\{\varphi_{n+1}\}) \Rightarrow \ \varphi_k \notin Cn(\Sigma \setminus \{\varphi_k\}) \cup Cn(\{\varphi_{n+1}\})) \Rightarrow$$

$$\Rightarrow \ \varphi_k \notin Cn(\Sigma \setminus \{\varphi_k\} \cup \{\varphi_{n+1}\}) \ \Rightarrow \ \varphi_k \notin Cn((\Sigma \cup \{\varphi_{n+1}\}) \setminus \{\varphi_k\}).$$

Thereby we obtain

$$(\Sigma \cup \{\varphi_{n+1}\}) \setminus \{\varphi_k\} \nvdash \varphi_k. \tag{4.4}$$

From (4.1) and (4.4), we conclude that $\Sigma \cup \{\varphi_{n+1}\}$ is independent.

Theorem 4.1 is proved.

Theorem 4.2. *Let Σ_1 and Σ_2 be disjoint independent sets of propositional logical formulas and let*

$$Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) = \begin{cases} Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2), & \textit{if} \quad \varphi_k \in \Sigma_1, \\ Cn(\Sigma_1) \cup Cn(\Sigma_2 \setminus \{\varphi_k\}), & \textit{if} \quad \varphi_k \in \Sigma_2. \end{cases}$$

Then for $\Sigma = \Sigma_1 \cup \Sigma_2$ to be independent it is necessary and sufficient that $\Sigma_1 \nvdash \varphi_k$ for every $\varphi_k \in \Sigma_2$ and $\Sigma_2 \nvdash \varphi_l$ for each $\varphi_l \in \Sigma_1$.

Proof. $(\Rightarrow)$ Let Σ be independent. Then, $\Sigma \setminus \{\varphi_m\} \nvdash \varphi_m$ for all $\varphi_m \in \Sigma$. As $\Sigma_1 \cap \Sigma_2 = \varnothing$, either $\varphi_m \in \Sigma_1$ or $\varphi_m \in \Sigma_2$ is hold.

Suppose that $\varphi_m \in \Sigma_2$. As Σ is independent, we obtain $\Sigma_1 \cup \Sigma_2 \setminus \{\varphi_m\} \nvdash \varphi_m$. From hence, we get $(\Sigma_1 \setminus \{\varphi_m\}) \cup (\Sigma_2 \setminus \{\varphi_m\}) \nvdash \varphi_m$. Therefore, we find $\Sigma_1 \setminus \{\varphi_m\} \nvdash \varphi_m$ and $\Sigma_2 \setminus \{\varphi_m\} \nvdash \varphi_m$. As $\Sigma_1 \cap \Sigma_2 = \varnothing$ and $\varphi_m \in \Sigma_2$, we get $\Sigma_1 \setminus \{\varphi_m\} = \Sigma_1$. Thus, $\Sigma_1 \nvdash \varphi_m$ is hold for all $\varphi_m \in \Sigma_2$. Similarly it is proved the condition of $\varphi_m \in \Sigma_1$.

$(\Leftarrow)$ Suppose that $\Sigma_1 \nvdash \varphi_k$ for every $\varphi_k \in \Sigma_2$ and $\Sigma_2 \nvdash \varphi_l$ for each $\varphi_l \in \Sigma_1$ and $\Sigma = \Sigma_1 \cup \Sigma_2$. Then either $\varphi_m \in \Sigma_1$ or $\varphi_m \in \Sigma_2$ is hold.

Suppose that $\varphi_m \in \Sigma_2$. We obtain $\Sigma_1 \nvdash \varphi_m$ because of the hypothesis. Therefore, $\Sigma_2 \setminus \{\varphi_m\} \nvdash \varphi_m$ and $\Sigma_1 \nvdash \varphi_m$. It means that $\varphi_m \notin Cn(\Sigma_2 \setminus \{\varphi_m\})$ and $\varphi_m \notin Cn(\Sigma_1)$. So, $\varphi_m \notin$ $\notin (Cn(\Sigma_2 \setminus \{\varphi_m\}) \cup Cn(\Sigma_1))$. We obtain $\varphi_m \notin (Cn(\Sigma_2 \setminus \{\varphi_m\}) \cup \Sigma_1)$ from the hypothesis. Thus $\varphi_m \notin (Cn(\Sigma_2 \cup \Sigma_1) \setminus \{\varphi_m\})$ namely,

$$((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_m\}) \nvdash \varphi_m.$$

For $\varphi_m \in \Sigma_1$, we get $((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_m\}) \nvdash \varphi_m$ by using same method. Therefore, Σ is independent.

Theorem 4.2 is proved.

Theorem 4.3. *Let $\{\Sigma_i : i \in \mathbb{N}\}$ be a family of independent sets. Let $Cn((\bigcup_{i=1}^n \Sigma_i) \setminus \{\varphi_k\}) =$ $= \bigcup_{i=1}^n Cn(\Sigma_i \setminus \{\varphi_k\})$ and $\Sigma_i \cap \Sigma_j = \varnothing$ for $i \neq j$, $i, j \in \mathbb{N}$. Then for $\bigcup_{i=1}^n \Sigma_i$ to be independent it is necessary and sufficient that for each $\varphi_k \in (\bigcup_{i=1}^n \Sigma_i) \setminus \Sigma_j$, $\Sigma_j \nvdash \varphi_k$.*

Proof. It is proved by using induction on sets of propositional logical formulas.

Theorem 4.4. *Let Σ_1 and Σ_2 be independent sets of propositional logical formulas such that $Cn(\Sigma_1) \cap Cn(\Sigma_2) = Cn(\varnothing)$. Then in order to have*

$$Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) = \begin{cases} Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2), & \textit{if} \quad \varphi_k \in \Sigma_1, \\ Cn(\Sigma_1) \cup Cn(\Sigma_2 \setminus \{\varphi_k\}), & \textit{if} \quad \varphi_k \in \Sigma_2, \end{cases}$$

it is necessary and sufficient that $\Sigma_1 \cup \Sigma_2$ is independent.

Proof. $(\Rightarrow)$ Let $Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) = Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2)$ for all $\varphi_k \in \Sigma_1$. If $\varphi_k \in Cn(\Sigma_1)$ and $\Sigma_1 \setminus \{\varphi_k\} \nvdash \varphi_k$, then φ_k is not a tautology and $\varphi_k \notin Cn(\Sigma_2)$ because of $Cn(\Sigma_1) \cap Cn(\Sigma_2) = Cn(\varnothing)$. Thus $\varphi_k \notin Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2)$. By the hypothesis, $\varphi_k \notin$ $\notin Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\})$. Therefore, $(\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\} \nvdash \varphi_k$.

It is similarly showed that for all $\varphi_m \in \Sigma_2$. As a conclusion, $(\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\} \nvdash \varphi_k$ for all $\varphi \in \Sigma_1 \cup \Sigma_2$. It means that $\Sigma_1 \cup \Sigma_2$ is independent.

$(\Leftarrow)$ Let $\Sigma_1 \cup \Sigma_2$ be independent. If $\Sigma_1 \cup \Sigma_2$ is independent then Σ_1 and Σ_2 are also independent. If $\varphi_k \notin Cn(\Sigma_1 \setminus \{\varphi_k\})$ for $\varphi_k \in \Sigma_1$, then φ_k is not a tautology. And also $\varphi_k \notin Cn(\Sigma_2)$ because $Cn(\Sigma_1) \cap Cn(\Sigma_2) = Cn(\varnothing)$. As $\Sigma_1 \cup \Sigma_2$ is independent, therefore $\varphi_k \notin Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\})$ for $\varphi_k \in \Sigma_1$. It means that $Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) \subseteq Cn((\Sigma_1) \setminus \{\varphi_k\}) \cup Cn(\Sigma_2)$. We use similar argument for $\varphi_k \in \Sigma_2$. After that we obtain $Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) \subseteq Cn(\Sigma_1) \cup Cn(\Sigma_2 \setminus \{\varphi_k\})$.

Namely,

$$Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) \subseteq \begin{cases} Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2), & \text{if } \varphi_k \in \Sigma_1, \\ Cn(\Sigma_1) \cup Cn(\Sigma_2 \setminus \{\varphi_k\}), & \text{if } \varphi_k \in \Sigma_2. \end{cases} \tag{4.5}$$

As $Cn(\Sigma_1) \cap Cn(\Sigma_2) = Cn(\varnothing)$, $\Sigma_1 \cap \Sigma_2$ does not include any formula without tautology. Thereby $((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) = ((\Sigma_1 \setminus \{\varphi_k\}) \cup \Sigma_2)$ for $\varphi_k \in \Sigma_1$ and $((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) = (\Sigma_1 \cup (\Sigma_2 \setminus \{\varphi_k\}))$ for $\varphi_k \in \Sigma_2$ since φ_k is not a tautology. By using Lemma 3.2 we obtain

$$Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) \supseteq \begin{cases} Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2), & \text{if } \varphi_k \in \Sigma_1, \\ Cn(\Sigma_1) \cup Cn(\Sigma_2 \setminus \{\varphi_k\}), & \text{if } \varphi_k \in \Sigma_2. \end{cases} \tag{4.6}$$

Therefore, by using (4.5) and (4.6), we get

$$Cn((\Sigma_1 \cup \Sigma_2) \setminus \{\varphi_k\}) = \begin{cases} Cn(\Sigma_1 \setminus \{\varphi_k\}) \cup Cn(\Sigma_2), & \text{if } \varphi_k \in \Sigma_1, \\ Cn(\Sigma_1) \cup Cn(\Sigma_2 \setminus \{\varphi_k\}), & \text{if } \varphi_k \in \Sigma_2. \end{cases}$$

Theorem 4.4 is proved.

5. Conclusion. In this paper, we give some results on consequence operation and independent sets of propositional logical formulas by means of logical consequence operation. It turns out that there is a bridge between propositional logical consequence operation and independent sets of logical formulas. Our main results in here answer the question under which conditions the sets of propositional logical formulas preserve independence.

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