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PRINCIPALLY GOLDIE*-LIFTING MODULES

ГОЛОВНІ ГОЛДІ*-ЛІФТИНГ МОДУЛІ

A module M is called a principal Goldie*-lifting if, for every proper cyclic submodule X of M, there is a direct summand D of M such that $X\beta^*D$. We focus our attention on principally Goldie*-lifting modules as a generalization of lifting modules. Various properties of these modules are presented.

Модуль називається головним Голді*-ліфтингом, якщо для кожного власного циклічного субмодуля X модуля M існує прямий доданок D з M такий, що $X\beta^*D$. Ми зосереджуємо нашу увагу на головних Голді*-ліфтинг модулях, що розглядаються як узагальнення ліфтинг модулів. Наведено різні властивості таких модулів.

1. Introduction. Throughout this paper, R denotes an associative ring with identity and all modules are unital right R-modules. $\mathrm{Rad}(M)$ will denote the Jacobson radical of M. Let M be an R-module and N, K be submodules of M. The submodule K of M will be denoted by $K \leq M$. K is called *small* (or superfluous) in M, denoted by $K \ll M$, if, for every submodule N of M, the equality $K + N = M$ implies $N = M$. K is called a *supplement* of N in M if K is minimal with respect to $N + K = M$, equivalently $K + N = M$ and $K \cap N \ll K$. A module M is called *supplemented* (*weakly supplemented*) if every submodule of M has a supplement (weak supplement) in M. A module M is $\oplus$-*supplemented* if every submodule of M has a supplement which is a direct summand of M. [1] defines principally supplemented modules and investigates their properties. A module M is said to be *principally supplemented* if for all cyclic submodule X of M there exists a submodule N of M such that $M = N + X$ and $N \cap X \ll N$. A module M is said to be $\oplus$-*principally supplemented* if, for each cyclic submodule X of M, there exists a direct summand D of M such that $M = D + X$ and $D \cap X \ll D$. A nonzero module M is said to be *hollow* if every proper submodule of M is small in M. A nonzero module M is said to be *principally hollow* if every proper cyclic submodule of M is small in M. Clearly, hollow modules are principally hollow. Given submodules $K \subseteq N \subseteq M$, the inclusion $K \hookrightarrow N$ is called *cosmall* in M, denoted by $K \overset{cs}{\hookrightarrow} N$, if $N/K \ll M/K$.

Lifting modules play an important role in module theory. Also their various generalizations are studied by many authors in [1, 2, 5–7, 9, 10]. A module M is called *lifting* if, for every submodule N of M, there is a decomposition $M = D \oplus D'$ such that $D \subseteq N$ and $D' \cap N \ll M$. A module M is called *principally lifting* if for all cyclic submodule X of M, there exists a decomposition $M = D \oplus D'$ such that $D \subseteq X$ and $D' \cap X \ll M$. A module M is said to be H-*supplemented* if, for every submodule N, there is a direct summand D of M such that $M = N + B$ holds if and only if $M = D + B$ for any submodule B of M. G. F. Birkenmeier et al. [2] defines β^* relation to study on the open problem 'Is every H-supplemented module supplemented?' in [7]. They say submodules X, Y of M are β^* equivalent, $X\beta^*Y$, if and only if $\dfrac{X+Y}{X}$ is small in $\dfrac{M}{X}$ and $\dfrac{X+Y}{Y}$ is small in $\dfrac{M}{Y}$. M is called *Goldie***-lifting* (or briefly, $\mathcal{G}^*$*-lifting*) if and only if for each $X \leq M$, there exists a direct summand D of M such that $X\beta^*D$. M is called *Goldie***-supplemented* (or briefly, $\mathcal{G}^*$*-supplemented*) if and only if for each $X \leq M$, there exists a supplement submodule S of M such that $X\beta^* S$ (see [2]).

Section 2 is based on principally Goldie*-lifting modules. These modules are considered as generalization of Goldie*-lifting modules. We give some necessary assumptions for a factor module or a direct summand of a principally Goldie*-lifting module to be principally Goldie*-lifting. Principally lifting, principally Goldie*-lifting and principally supplemented modules are compared. Finally, we show that principally lifting, principally Goldie*-lifting and $\oplus$-principally supplemented coincide on π-projective modules. In addition, one of the our aims is to determine the connection between principally Goldie*-lifting and Goldie*-lifting. As a consequence, we prove this relation under some restriction.

2. Principally Goldie*-lifting modules. In [2], G. F. Birkenmeier et al. defined β^* relation. We start this section by giving some properties of β^* relation without proofs. The proofs of the following notions can be found in [2]. Moreover, in [2], the authors introduced two notions called Goldie*-supplemented module and Goldie*-lifting module depend on the β^* relation. They showed that Goldie*-lifting modules and H-supplemented modules coincide in [2] (Theorem 3.6). In this section, we define principally Goldie*-lifting module (briefly principally $\mathcal{G}^*$-lifting module) as a generalization of $\mathcal{G}^*$-lifting module and investigate some properties of this module. In particular, we prove that principally $\mathcal{G}^*$-lifting and $\mathcal{G}^*$-lifting coincide under some conditions.

Definition 2.1 ([2], Definition 2.1). *Any submodules X, Y of M are β^* equivalent, $X\beta^*\, Y$, if and only if $\dfrac{X+Y}{X}$ is small in $\dfrac{M}{X}$ and $\dfrac{X+Y}{Y}$ is small in $\dfrac{M}{Y}$.*

Lemma 2.1 ([2], Lemma 2.2). *β^* is an equivalence relation.*

By [2, p. 43], the zero submodule is β^* equivalent to any small submodule.

Theorem 2.1 ([2], Theorem 2.3). *Let X, Y be submodules of M. The following are equivalent:*

(a) $X\beta^*Y$;

(b) $X \overset{cs}{\hookrightarrow} X+Y$ *and* $Y \overset{cs}{\hookrightarrow} X+Y$;

(c) *for each submodule A of M such that $X+Y+A = M$, then $X+A = M$ and $Y+A = M$;*

(d) *if $K \leq M$ with $X+K = M$, then $Y+K = M$, and if $H \leq M$ with $Y+H = M$, then $X+H = M$.*

Theorem 2.2 ([2], Theorem 2.6). *Let X, Y be submodules of M such that $X\beta^*Y$. Then*

1) *$X \ll M$ if and only if $Y \ll M$;*

2) *X has a (weak) supplement C in M if and only if C is a (weak) supplement for Y.*

Lemma 2.2. *Let $M = D \oplus D'$ and $A, B \leq D$. Then $A\beta^*B$ in M if and only if $A\beta^*B$ in D.*

Proof. $(\Rightarrow)$ Let $A\beta^*B$ in M and $A+B+N = D$ for some submodule N of D. Let us show that $A+N = D$ and $B+N = D$. Since $A\beta^*B$ in M,

$$M = D \oplus D' = A+B+N+D'$$

implies $A+N+D' = M$ and $B+N+D' = M$. By [11, p. 41], $A+N = D$ and $B+N = D$. From Theorem 2.1, we get $A\beta^*B$ in D.

$(\Leftarrow)$ Let $A\beta^*B$ in D. Then $\dfrac{A+B}{A} \ll \dfrac{D}{A}$ implies $\dfrac{A+B}{A} \ll \dfrac{M}{A}$. Similarly, $\dfrac{A+B}{B} \ll \dfrac{D}{B}$ implies $\dfrac{A+B}{B} \ll \dfrac{M}{B}$. This means that $A\beta^*B$ in M.

Lemma 2.3. *If a direct summand D of M is β^* equivalent to a cyclic submodule X of M, then D is also cyclic.*

Proof. Assume that $M = D \oplus D'$ for some submodules D, D' of M and X is a cyclic submodule of M which is β^* equivalent to D. By Theorem 2.1 (c), $M = X + D'$. Since $\dfrac{X + D'}{D'} = \dfrac{M}{D'} \cong D$ and X is cyclic, D is cyclic.

Definition 2.2. *A module M is called principally Goldie*-lifting (briefly principally $\mathcal{G}^*$-lifting) if for each cyclic submodule X of M, there exists a direct summand D of M such that $X\beta^*D$.*

Clearly, every $\mathcal{G}^*$-lifting module is principally $\mathcal{G}^*$-lifting. However, the converse does not hold as the next example shows.

***Example* 2.1.** Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathrm{Rad}(\mathbb{Q}) = \mathbb{Q}$, every cyclic submodule of $\mathbb{Q}$ is small in $\mathbb{Q}$. By [2] (Example 2.15), the $\mathbb{Z}$-module $\mathbb{Q}$ is principally $\mathcal{G}^*$-lifting. But the $\mathbb{Z}$-module $\mathbb{Q}$ is not supplemented. It follows from [2] (Theorem 3.6) that it is not $\mathcal{G}^*$-lifting.

A module M is said to be *radical* if $\mathrm{Rad}(M) = M$.

Lemma 2.4. *Every radical module is principally $\mathcal{G}^*$-lifting.*

Proof. Let $m \in M$. As M is radical, $mR \subseteq \mathrm{Rad}(M)$. By [11] (21.5), $mR \ll M$. So we get $mR\beta^*0$. Thus M is principally $\mathcal{G}^*$-lifting.

Theorem 2.3. *Let M be a module. Consider the following conditions*:

(a) *M is principally lifting*,

(b) *M is principally $\mathcal{G}^*$-lifting*,

(c) *M is principally supplemented.*

Then (a)$\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) Let $m \in M$. From (a), there is a decomposition $M = D \oplus D'$ such that $D \leq mR$ and $mR \cap D' \ll M$. Since $D \leq mR$, $\dfrac{mR + D}{mR} \ll \dfrac{M}{mR}$. By modularity, $mR = M \cap mR = (D \oplus D') \cap mR = D \oplus (mR \cap D')$. Then $\dfrac{mR}{D} \cong mR \cap D'$ and $\dfrac{M}{D} \cong D'$. If $mR \cap D' \ll M$, by [11] (19.3), $mR \cap D' \ll D'$. It implies that $\dfrac{mR + D}{D} \ll \dfrac{M}{D}$. Therefore it is seen that $mR\beta^*D$ from Definition 2.1. Hence M is principally $\mathcal{G}^*$-lifting.

(b) $\Rightarrow$ (c) Let $m \in M$. By the hypothesis, there exists a direct summand D of M such that $mR\beta^*D$. Since $M = D \oplus D'$ for some submodule D' of M and D' is a supplement of D, D' is a supplement of mR in M by [2] (Theorem 2.6 (ii)). Thus M is principally supplemented.

We expect that a principally $\mathcal{G}^*$-lifting module is principally lifting. But unfortunately, it is not true in general:

***Example* 2.2.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. From [10] (Example 3.7), we can say that M is a H-supplemented module. Then M is $\mathcal{G}^*$-lifting by [2] (Theorem 3.6). Since every $\mathcal{G}^*$-lifting module is principally $\mathcal{G}^*$-lifting, M is also principally $\mathcal{G}^*$-lifting. But from [1] (Examples 7.(3)), M is not principally lifting.

Theorem 2.4. *Let M be an indecomposable module. Then the following conditions are equivalent*:

(a) *M is principally lifting*,

(b) *M is principally hollow*,

(c) *M is principally $\mathcal{G}^*$-lifting.*

Proof. $(a) \Leftrightarrow (b)$ It is easy to see from [1] (Lemma 14).

(b) $\Rightarrow$ (c) Suppose that M is principally hollow and $m \in M$. Then $mR \ll M$. It means that $mR\beta^*0$.

(c) $\Rightarrow$ (b) Let mR be a proper cyclic submodule of M. By (c), there exists a decomposition $M = D \oplus D'$ such that $mR\beta^*D$. Since M is indecomposable, $D = M$ or $D = 0$. If $D = M$, from [2] (Corollary 2.8 (iii)), we obtain that $mR = M$, which is a contradiction. Thus D must be zero, that is, $mR\beta^*0$ and we have $mR \ll M$. Hence M is principally hollow.

We shall give the following example of modules which are principally supplemented but not principally $\mathcal{G}^*$-lifting.

***Example* 2.3.** Let F be a field, x and y commuting indeterminates over F. Let $R = F[x, y]$ be a polynomial ring and $I_1 = (x^2)$ and $I_2 = (y^2)$ be ideals of R and the ring $S = R/(x^2, y^2)$. Consider the S-module $M = \overline{x}S + \overline{y}S$. By [1] (Example 15), M is an indecomposable S-module and it is not principally hollow. Then from Theorem 2.4 M is not principally $\mathcal{G}^*$-lifting. Therefore it follows from [1] (Example 15) that M is principally supplemented.

A module M is said to be *principally semisimple* if every cyclic submodule of M is a direct summand of M.

Lemma 2.5. *Every principally semisimple module is principally $\mathcal{G}^*$-lifting.*

Proof. Let X be a cyclic submodule of M. By the assumption, X is a direct summand of M. Then $M = X \oplus X'$ for some submodule X' of M. Since β^* is an equivalence relation, we have $X\beta^*X$. Thus M is principally $\mathcal{G}^*$-lifting.

Recall that a submodule N of M is called *fully invariant* if for each endomorphism f of M, $f(N)$ is contained in N. Clearly 0 and M are fully invariant submodules of M. A module M is said to be a *duo module* provided every submodule of M is fully invariant. For example, if M is a simple right R-module, then M is a duo module but $M \oplus M$ is not duo (see [8]). A module M is called *distributive* if for all submodules A, B, C of M, $A + (B \cap C) = (A + B) \cap (A + C)$ or $A \cap (B + C) = (A \cap B) + (A \cap C)$ (see [3]).

Proposition 2.1. *Let $M = M_1 \oplus M_2$ be a duo module* (*or distributive module*). *Then M is principally $\mathcal{G}^*$-lifting if and only if M_1 and M_2 are principally $\mathcal{G}^*$-lifting.*

Proof. $(\Rightarrow)$ Take any $m \in M_1$. Since M is principally $\mathcal{G}^*$-lifting, then for $m \in M$, there exists a direct decomposition $M = D \oplus D'$ such that $mR\beta^*D$ in M for $D, D' \leq M$. As M is a duo module, it is obtained that $M_1 = (M_1 \cap D) \oplus (M_1 \cap D')$. We claim that $mR\beta^*(M_1 \cap D)$ in M_1. To prove this, it is enough to show that for some submodule A of M_1, $M_1 = mR + A$ and $M_1 = (M_1 \cap D) + A$. Let $M_1 = mR + (M_1 \cap D) + A$ for some submodule A of M_1. Then

$$M = M_1 \oplus M_2 = [mR + (M_1 \cap D) + A] \oplus M_2 = mR + D + A + M_2.$$

By Theorem 2.1, $M = D + A + M_2$ and $M = mR + A + M_2$. Because M is duo, we can write as $M_1 = M_1 \cap (D + A + M_2) = A + [M_1 \cap (D + M_2)] = A + (M_1 \cap D)$ and $M_1 = M_1 \cap (mR + A + M_2) = mR + A$. Again by Theorem 2.1, we get $mR\beta^*(M_1 \cap D)$ in M_1. Hence M_1 is principally $\mathcal{G}^*$-lifting. Similarly, it can be showed that M_2 is principally $\mathcal{G}^*$-lifting.

$(\Leftarrow)$ Let $m \in M$. If M is a duo module, for the cyclic submodule mR of M, $mR = (mR \cap M_1) \oplus (mR \cap M_2)$. If $M = M_1 \oplus M_2$, then $mR = m_1R + m_2R$ for some $m_1 \in M_1$ and $m_2 \in M_2$. So $mR \cap M_1 = m_1R$ and $mR \cap M_2 = m_2R$. Since M_1 and M_2 are principally $\mathcal{G}^*$-lifting, there are decompositions $M_1 = D_1 \oplus D_1'$ and $M_2 = D_2 \oplus D_2'$ such that $m_1R\beta^*D_1$ in M_1 and $m_2R\beta^*D_2$ in M_2. By Lemma 2.2, $m_1R\beta^*D_1$ and $m_2R\beta^*D_2$ in M. By [2] (Proposition 2.11), $(m_1R + m_2R)\beta^*(D_1 \oplus D_2)$. Since $mR = m_1R + m_2R$, we get $mR\beta^*(D_1 \oplus D_2)$.

Let M, N and P be R-modules. P is called *M-projective* if for each epimorphism $g\colon M \to N$ and each homomorphism $f: P \to N$, there exists a homomorphism $h\colon P \to M$ such that $gh = f$. If P is P-projective, then P is also called *self-injective* (or *quasi-injective*). An R-module M is said to be *π-projective* if for every two submodules U, V of M with $U + V = M$ there exists $f \in \mathrm{End}(M)$ with $\mathrm{Im}(f) \subset U$ and $\mathrm{Im}(1 - f) \subset V$. Clearly every self-projective module is also π-projective [11].

Proposition 2.2. *Let any cyclic submodule of M have a supplement which is a relatively projective direct summand of M. Then M is principally $\mathcal{G}^*$-lifting.*

Proof. Let $m \in M$. By the hypothesis, there exists a decomposition $M = D \oplus D'$ such that $M = mR + D'$ and $mR \cap D' \ll D'$. Because D is D'-projective, $M = A \oplus D'$ for some submodule A of mR by [7] (Lemma 4.47). So M is principally lifting. It follows from Theorem 2.3 that M is principally $\mathcal{G}^*$-lifting.

Proposition 2.3. *Let M be principally $\mathcal{G}^*$-lifting and N be a submodule of M. If $\dfrac{N+D}{N}$ is a direct summand of $\dfrac{M}{N}$ for any cyclic direct summand D of M, then $\dfrac{M}{N}$ is principally $\mathcal{G}^*$-lifting.*

Proof. Let $\dfrac{mR+N}{N}$ be a cyclic submodule of $\dfrac{M}{N}$ for $m \in M$. If M is principally $\mathcal{G}^*$-lifting, there exists a decomposition $M = D \oplus D'$ such that $mR\beta^*D$. Then D is also cyclic from Lemma 2.3. By the hypothesis, $\dfrac{D+N}{N}$ is a direct summand in $\dfrac{M}{N}$. We claim that $\dfrac{mR+N}{N}\beta^*\dfrac{D+N}{N}$. Consider the canonical epimorphism $\theta\colon M \to M/N$. By [2] (Proposition 2.9 (i)), $\theta(mR)\beta^*\theta(D)$, that is, $\dfrac{mR+N}{N}\beta^*\dfrac{D+N}{N}$. Thus $\dfrac{M}{N}$ is principally $\mathcal{G}^*$-lifting.

***Corollary* 2.1.** *Let M be principally $\mathcal{G}^*$-lifting. Then*

(a) *If M is a distributive (or duo) module, then any factor module of M is principally $\mathcal{G}^*$-lifting.*

(b) *Let N be a projection invariant, that is, $eN \subseteq N$ for all $e^2 = e \in \mathrm{End}(M)$. Then $\dfrac{M}{N}$ is principally $\mathcal{G}^*$-lifting. In particular, $\dfrac{M}{A}$ is principally $\mathcal{G}^*$-lifting for every fully invariant submodule A of M.*

Proof. (a) Let N be any submodule of M and D be a cyclic direct summand of M. Note that $M = D \oplus D'$ for some submodules D' of M. Therefore we have

$$\frac{M}{N} = \frac{D \oplus D'}{N} = \frac{D+N}{N} + \frac{D'+N}{N}.$$

We will show that $\dfrac{D+N}{N} \cap \dfrac{D'+N}{N} = 0$. Since M is distributive and $D \cap D' = 0$,

$$(D+N) \cap (D'+N) = \big((D+N) \cap D'\big) + \big((D+N) \cap N\big) = (D \cap D') + (N \cap D') + N = N.$$

We obtain $\dfrac{M}{N} = \dfrac{D+N}{N} \oplus \dfrac{D'+N}{N}$. By Proposition 2.3, $\dfrac{M}{N}$ is principally $\mathcal{G}^*$-lifting.

(b) Let D be a cyclic direct summand of M and N be a projection invariant of M. Then $M = D \oplus D'$ for some $D' \leq M$. For the projection map $\pi_D : M \to D$, $\pi_D^2 = \pi \in \mathrm{End}(M)$ and $\pi_D(N) \subseteq N$. So $\pi_D(N) = N \cap D$. Similarly, $\pi_{D'}(N) = N \cap D'$ for the projection map $\pi_{D'}: M \to D'$. Hence we have $N = (N \cap D) + (N \cap D')$. So

$$M = (D+N) + (D'+N) = \big[D + (N \cap D) + (N \cap D')\big] + (D'+N) =$$

$$= \big[D \oplus (N \cap D')\big] + (D' + N)$$

and, by modularity,

$$\big[D \oplus (N \cap D')\big] \cap (D' + N) = \big[D \cap (D' + N)\big] + (N \cap D') = (N \cap D) + (N \cap D') = N.$$

Thus it can be seen that $\dfrac{M}{N} = \dfrac{D \oplus (N \cap D')}{N} \oplus \dfrac{D' + N}{N}$. By Proposition 2.3, $\dfrac{M}{N}$ is principally $\mathcal{G}^*$-lifting.

Another consequence of Proposition 2.2 is given in the next result.

A module M is said to have the *summand sum property* (SSP) if the sum of any two direct summands of M is again a direct summand.

Proposition 2.4. *Let M be a principally $\mathcal{G}^*$-lifting module. If M has SSP, then any direct summand of M is principally $\mathcal{G}^*$-lifting.*

Proof. Let $M = N \oplus N'$ for some submodules N, N' of M. Our aim is to show that N is principally $\mathcal{G}^*$-lifting. Take any cyclic direct summand D of M. From the SSP property, we can write as $M = (D + N') \oplus T$ for some submodule T of M. Then

$$N \cong \frac{M}{N'} = \frac{D + N'}{N'} + \frac{T + N'}{N'}.$$

By modular law,

$$(D + N') \cap (T + N') = N' + \big[(D + N') \cap T\big] = N'.$$

So we obtain

$$\frac{M}{N'} = \frac{D + N'}{N'} \oplus \frac{T + N'}{N'}.$$

Using Proposition 2.3, it can be said that $N \cong \dfrac{M}{N'}$ is principally $\mathcal{G}^*$-lifting.

Next, we give a sufficient condition for $M/\operatorname{Rad}(M)$ is principally semisimple in case M is principally $\mathcal{G}^*$-lifting module.

Proposition 2.5. *Let M be principally $\mathcal{G}^*$-lifting and distributive module. Then $\dfrac{M}{\operatorname{Rad}(M)}$ is principally semisimple.*

Proof. Let $m \in M$. By the assumption, there exists a decomposition $M = D \oplus D'$ such that $mR\beta^* D$ for some submodule D, D' of M. By [2] (Theorem 2.6 (ii)), D' is a supplement of mR, that is, $M = mR + D'$ and $mR \cap D' \ll D'$. Then

$$\frac{M}{\operatorname{Rad}(M)} = \frac{mR + D'}{\operatorname{Rad}(M)} = \frac{mR + \operatorname{Rad}(M)}{\operatorname{Rad}(M)} + \frac{D' + \operatorname{Rad}(M)}{\operatorname{Rad}(M)}.$$

Because M is distributive,

$$\big(mR + \operatorname{Rad}(M)\big) \cap \big(D' + \operatorname{Rad}(M)\big) = (mR \cap D') + \operatorname{Rad}(M).$$

Since $mR \cap D' \ll D'$, so $mR \cap D' \subseteq \operatorname{Rad}(M)$. In this case, $\big(mR + \operatorname{Rad}(M)\big) \cap \big(D' + \operatorname{Rad}(M)\big) = \operatorname{Rad}(M)$. As a result, $\dfrac{mR + \operatorname{Rad}(M)}{\operatorname{Rad}(M)}$ is a direct summand in $\dfrac{M}{\operatorname{Rad}(M)}$, this means that $\dfrac{M}{\operatorname{Rad}(M)}$ is a principally semisimple module.

Proposition 2.6. *Let M be a principally $\mathcal{G}^*$-lifting module and* $\mathrm{Rad}(M) \ll M$. *Then* $\dfrac{M}{\mathrm{Rad}(M)}$ *is principally semisimple.*

Proof. Let $\dfrac{X}{\mathrm{Rad}(M)}$ be a cyclic submodule of $\dfrac{M}{\mathrm{Rad}(M)}$ for any submodule X of M containing $\mathrm{Rad}(M)$. Then $X = mR + \mathrm{Rad}(M)$ for some $m \in M$. By the assumption, there exists a decomposition $M = D \oplus D'$ such that $mR\beta^* D$ for submodules $D, D' \leq M$. It follows from [2] (Corollary 2.12) that $\big(mR + \mathrm{Rad}(M)\big)\beta^* D$. Moreover, D' is a supplement of $mR + \mathrm{Rad}(M)$ in M from by [2] (Theorem 2.6 (ii)). Then we have $M = mR + \mathrm{Rad}(M) + D'$ and $D' \cap \big(mR + \mathrm{Rad}(M)\big) = D' \cap X \ll D'$, that is, $D' \cap X \subseteq \mathrm{Rad}(M)$. On the other hand,

$$\frac{M}{\mathrm{Rad}(M)} = \frac{X}{\mathrm{Rad}(M)} + \frac{D' + \mathrm{Rad}(M)}{\mathrm{Rad}(M)}.$$

By modular law,

$$\frac{X}{\mathrm{Rad}(M)} \cap \frac{D' + \mathrm{Rad}(M)}{\mathrm{Rad}(M)} = \frac{(X \cap D') + \mathrm{Rad}(M)}{\mathrm{Rad}(M)}$$

and since $X \cap D' \subseteq \mathrm{Rad}(M)$, we obtain

$$\frac{M}{\mathrm{Rad}(M)} = \frac{X}{\mathrm{Rad}(M)} \oplus \frac{D' + \mathrm{Rad}(M)}{\mathrm{Rad}(M)}.$$

Theorem 2.5 ([4], 4.14). *Let M be π-projective and let $U, V \leq M$ be submodules with $M = U + V$.*

(1) *If U is a direct summand in M, then there exists $V' \subset V$ with $M = U \oplus V'$.*

(2) *If $U \cap V = 0$, then V is U-projective (and U is V-projective).*

(3) *If $U \cap V = 0$ and $V \cong U$, then M is self-projective.*

(4) *If U and V are direct summands of M, then $U \cap V$ is also direct summand of M.*

In general, it is not true that principally lifting and principally $\mathcal{G}^*$-lifting modules coincide. As we will see in the following theorem, we need π-projectivity condition.

Theorem 2.6. *Let M be a module. Consider the following conditions*:

(a) *M is principally lifting*,

(b) *M is principally $\mathcal{G}^*$-lifting*,

(c) *M is $\oplus$-principally supplemented.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *If M is π-projective, then* (c)$\Rightarrow$ (a) *holds.*

Proof. (a) $\Rightarrow$ (b) It follows from Theorem 2.3.

(b) $\Rightarrow$ (c) It follows from [2] (Theorem 2.6 (ii)).

(c) $\Rightarrow$ (a) Consider any $m \in M$. By the assumption, mR has a supplement D which is a direct summand in M, that is, $M = mR + D = D \oplus A$ and $mR \cap D \ll D$ for some submodule A of M. Since M is π-projective, there exists a complement D' of D such that $D' \subseteq mR$ by [4] (4.14 (1)). Then we have $M = D \oplus D'$. Thus M is principally lifting.

Proposition 2.7. *Let M be a π-projective module. Then M is principally $\mathcal{G}^*$-lifting if and only if every cyclic submodule X of M can be written as $X = D \oplus A$ such that D is a direct summand in M and $A \ll M$.*

Proof. $(\Rightarrow)$ Suppose M is principally $\mathcal{G}^*$-lifting and π-projective module. By Theorem 2.6, M is principally lifting. Then we observe that for any cyclic submodule X of M, there exists a direct decomposition $M = D \oplus D'$ such that $D \leq X$ and $X \cap D' \ll M$. By modularity, we conclude that $X = D \oplus (X \cap D')$.

$(\Leftarrow)$ Let X be any cyclic submodule of M. By the assumption and [5] (Lemma 2.10), M is principally lifting. Therefore from Theorem 2.6, M is principally $\mathcal{G}^*$-lifting.

Now we mention that principally $\mathcal{G}^*$-lifting and $\mathcal{G}^*$-lifting modules coincide under some conditions.

Proposition 2.8. *Let M be Noetherian and have SSP. Then M is principally $\mathcal{G}^*$-lifting if and only if M is $\mathcal{G}^*$-lifting.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ If M is Noetherian, for any submodule X of M there exist some $m_1, m_2, \ldots, m_n \in M$ such that $X = m_1R + m_2R + \ldots + m_nR$ by [11] (27.1). Since M is principally $\mathcal{G}^*$-lifting, there exist some direct summands $D_1, D_2, \ldots, D_n$ of M such that $m_1R\beta^*D_1$, $m_2R\beta^*D_2, \ldots, m_nR\beta^*D_n$. Then $D = D_1 + D_2 + \ldots + D_n$ is also a direct summand in M because of SSP. By [2] (Proposition 2.11), $X\beta^*D$. Hence M is $\mathcal{G}^*$-lifting.

Proposition 2.9. *Let any submodule N of M be a sum of a cyclic submodule X and a small submodule A in M. Then M is principally $\mathcal{G}^*$-lifting if and only if M is $\mathcal{G}^*$-lifting.*

Proof. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ Let N be any submodule of M and $N = X + A$ for a cyclic submodule X and a small submodule A of M. Since M is principally $\mathcal{G}^*$-lifting, there exists a direct summand D of M such that $X\beta^*D$. From [2] (Corollary 2.12), $(X + A)\beta^*D$, that is, $N\beta^*D$. Hence M is $\mathcal{G}^*$-lifting.

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