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N. Durna (Cumhuriyet Univ., Sivas, Turkey)

SUBDIVISION OF SPECTRA FOR SOME LOWER TRIANGULAR DOUBLE-BAND MATRICES AS OPERATORS ON c_0

ПІДРОЗДІЛ СПЕКТРІВ ДЛЯ ДЕЯКИХ НИЖНЬО-ТРИКУТНИХ ДВОРЯДКОВИХ МАТРИЦЬ ЯК ОПЕРАТОРІВ НА c_0

The generalized difference operator $\Delta_{a,b}$ was defined by El-Shabrawy: $\Delta_{a,b}x = \Delta_{a,b}(x_n) = (a_n x_n + b_{n-1}x_{n-1})_{n=0}^{\infty}$ with $x_{-1} = b_{-1} = 0$, where (a_k) and (b_k) are convergent sequences of nonzero real numbers satisfying certain conditions. We completely determine the approximate point spectrum, the defect spectrum, and the compression spectrum of the operator $\Delta_{a,b}$ in the sequence space c_0.

Узагальнений різницевий оператор $\Delta_{a,b}$ було визначено Ель-Шабраві: $\Delta_{a,b}x = \Delta_{a,b}(x_n) = (a_n x_n + b_{n-1}x_{n-1})_{n=0}^{\infty}$ при $x_{-1} = b_{-1} = 0$, де (a_k), (b_k) — збіжні послідовності ненульових дійсних чисел, що задовольняють деякі умови. Повністю визначено наближений точковий спектр, дефектний спектр та стискувальний спектр оператора $\Delta_{a,b}$ у просторі послідовностей c_0.

1. Introduction. Spectral theory is an important part of functional analysis. It has numerous applications in many parts of mathematics and physics including matrix theory, function theory, complex analysis, differential and integral equations, control theory and quantum physics. For example, in quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star.

In numerical analysis, matrices from finite element or finite difference problems are often banded. Such matrices can be viewed as descriptions of the coupling between the problem variables; the bandedness corresponds to the fact that variables are not coupled over arbitrarily large distances. Such matrices can be further divided — for instance, banded matrices exist where every element in the band is nonzero. These often arise when discretizing one-dimensional problems.

Problems in higher dimensions also lead to banded matrices, in which case the band itself also tends to be sparse. For instance, a partial differential equation on a square domain (using central differences) will yield a matrix with a bandwidth equal to the square root of the matrix dimension, but inside the band only 5 diagonals are nonzero. Unfortunately, applying Gaussian elimination (or equivalently an LU decomposition) to such a matrix results in the band being filled in by many nonzero elements. And so, the resolvent set of the band operators is important for solving such problems.

Many problems in computational physics can be reduced to linear algebra problems. In this laboratory you will use several fundamental techniques of computational linear algebra to solve physics problems common in many different areas of science. In order to solve the problem with some variations you will need to solve a system of linear equations by Gauss – Jordan Elimination, "LU decomposition plus back substitution," matrix inversion and matrix diagonalization.

In recent years, spectral theory has witnessed an explosive development. There are many types of spectra, both for one or several commuting operators, with important applications, for example the approximate point spectrum, Taylor spectrum, local spectrum, essential spectrum etc.

1.1. The spectrum. Let X and Y be the Banach spaces, and $L\colon X \to Y$ also be a bounded linear operator. By $R(L)$, we denote the range of L, i.e.,

$$R(L) = \{y \in Y : y = Lx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach spaces and $L \in B(X)$, then the adjoint L^* of L is a bounded linear operator on the dual X^* of X defined by $\big(L^* f\big)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$.

Let $L\colon D(L) \to X$ be a linear operator, defined on $D(L) \subset X$, where $D(L)$ denotes the domain of L and X is a complex normed linear space. For $L \in B(X)$ we associate a complex number λ with the operator $(\lambda I - L)$ denoted by L_λ defined on the same domain $D(L)$, where I is the identity operator. The inverse $(\lambda I - L)^{-1}$, denoted by L_λ^{-1} is known as the resolvent operator of L_λ.

A regular value of L is a complex number λ of L such that L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X.

The resolvent set of L is the set of all such regular values a of L, denoted by $\rho(L, X)$. Its complement is given by $\mathbb{C}\backslash\rho(L; X)$ in the complex plane $\mathbb{C}$ is called the spectrum of L, denoted by $\sigma(L, X)$. Thus the spectrum $\sigma(L, X)$ consist of those values of $\lambda \in \mathbb{C}$, for which L_λ is not invertible.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: the point (discrete) spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. Further $\lambda \in \sigma_p(L, X)$ is called the eigen value of L. We say that $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L if the resolvent operator L_λ^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the resolvent operator L_λ^{-1} exists, but its domain of definition (i.e., the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X; in this case L_λ^{-1} may be bounded or unbounded. Together with the point spectrum, these two subspectra form a disjoint subdivision

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \tag{1}$$

of the spectrum of L.

Also the spectrum $\sigma(L, X)$ is partitioned into three sets which are not necessarily disjoint as follows:

we call a sequence $(x_k)_k$ in X a Weyl sequence for L if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$.

We call the set

$$\sigma_{ap}(L, X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \ \lambda I - L\}$$

the approximate point spectrum of L. Moreover, the subspectrum

$$\sigma_\delta(L, X) := \{\lambda \in \mathbb{C} : \lambda I - L \text{ is not surjective}\}$$

is called defect spectrum of L. There exists another subspectrum,

$$\sigma_{co}(L, X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\}$$

which is often called compression spectrum in the literature. Clearly, $\sigma_p(L, X) \subseteq \sigma_{ap}(L, X)$ and $\sigma_{co}(L, X) \subseteq \sigma_\delta(L, X)$. Moreover, comparing these subspectra with those in (1) we note that

$$\sigma_r(L,X) = \sigma_{co}(L,X)\backslash\sigma_p(L,X)$$

and

$$\sigma_c(L,X) = \sigma(L,X)\backslash[\sigma_p(L,X) \cup \sigma_{co}(L,X)].$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 1 ([4], Proposition 1.3)**.** *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations*:

(a) $\sigma(L^*, X^*) = \sigma(L, X)$,
(b) $\sigma_c(L^*, X^*) \subseteq \sigma_{ap}(L, X)$,
(c) $\sigma_{ap}(L^*, X^*) = \sigma_\delta(L, X)$,
(d) $\sigma_\delta(L^*, X^*) = \sigma_{ap}(L, X)$,
(e) $\sigma_p(L^*, X^*) = \sigma_{co}(L, X)$,
(f) $\sigma_{co}(L^*, X^*) \supseteq \sigma_p(L, X)$,
(g) $\sigma(L, X) = \sigma_{ap}(L, X) \cup \sigma_p(L^*, X^*) = \sigma_p(L, X) \cup \sigma_{ap}(L^*, X^*)$.

1.2. Goldberg's classification of spectrum. If $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$,
(II) $\overline{R(T)} = X$, but $R(T) \neq X$,
(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,
(2) T^{-1} exists but discontinuous,
(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: (I_1), (I_2), (I_3), (II_1), (II_2), (II_3), (III_1), (III_2), (III_3). If an operator is in state (III_2) for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [12]).

If λ is a complex number such that $T = \lambda I - L \in (\mathrm{I}_1)$ or $T = \lambda I - L \in (\mathrm{II}_1)$, then $\lambda \in \rho(L, X)$. All scalar values of λ not in $\rho(L, X)$ comprise the spectrum of L. The further classification of $\sigma(L, X)$ gives rise to the fine spectrum of L. That is, $\sigma(L, X)$ can be divided into the subsets $(\mathrm{I}_2)\,\sigma(L, X) = \varnothing$, $(\mathrm{I}_3)\,\sigma(L, X)$, $(\mathrm{II}_2)\,\sigma(L, X)$, $(\mathrm{II}_3)\,\sigma(L, X)$, $(\mathrm{III}_1)\,\sigma(L, X)$, $(\mathrm{III}_2)\,\sigma(L, X)$, $(\mathrm{III}_3)\,\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, (III_2) (say), then we write $\lambda \in \in (\mathrm{III}_2)\,\sigma(L, X)$.

By the definitions given above, we can write Table 1.

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. We shall write ℓ_∞, c, c_0 and bv for the space of all bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_1, ℓ_p, bv_p we denote the spaces of all absolutely summable sequences, p-absolutely summable sequences and p-bounded variation sequences, respectively.

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We summarize the knowledge in the existing literature concerned with the spectrum and the fine spectrum. The fine spectrum of the Cesaro operator on the sequence space ℓ_p for $1 < p < \infty$ has been studied by Gonzalez [13].

Table 1

Possibility		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L, X)$	$-$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L, X)$	$\lambda \in \sigma_c(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$

Also, Wenger [19] examined the fine spectrum of the integer power of the Cesaro operator over c, and Rhoades [16] generalized this result to the weighted mean methods. Reade [15] worked the spectrum of the Cesaro operator over the sequence space c_0. The spectrum of the Rhaly operators on the sequence spaces c_0 and c is studied by Yildirim [17] and the fine spectrum of the Rhaly operators on the sequence space c_0 is studied by Yildirim [18]. In the last year, several authors have investigated spectral divisions of generalized differance matrices. For example, Akhmedov and El-Shabrawy [1, 2] have studied the spectrum and fine spectrum of the generalized lower triangle double-band matrix Δ_v over the sequence spaces c_0, c and ℓ_p, where $1 < p < \infty$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv is investigated by Kayaduman and Furkan [14] and c_0 and c, is investigated by Altay and Başar [3] etc.

The above-mentioned articles, concerned with the decomposition of spectrum which defined by Goldberg. However, in [8] Durna and Yildirim have investigated subdivision of the spectra for factorable matrices on c_0 and in [6] Basar, Durna and Yildirim have investigated subdivisions of the spectra for genarilized difference operator over certain sequence spaces.

2. The fine spectrum of the operator $\Delta_{a,b}$ on c_0. The generalized difference operator $\Delta_{a,b}$ has been defined by El-Shabrawy [9]. Let (a_k) and (b_k) are two convergent sequences of nonzero real numbers satisfying

$$\lim_{k\to\infty} a_k = a > 0 \qquad \text{and} \qquad \lim_{k\to\infty} b_k = b \neq 0. \tag{2}$$

We consider the operator $\Delta_{a,b} : c_0 \to c_0$, which is defined as follows:

$$\Delta_{a,b}x = \Delta_{a,b}(x_k) = (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty} \qquad \text{with} \quad x_{-1} = b_{-1} = 0.$$

It is easy to verify that the operator $\Delta_{a,b}$ can be represented by a lower triangular double-band matrix of the form

$$\Delta_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & \dots \\ b_0 & a_1 & 0 & \dots \\ 0 & b_1 & a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that, if (a_k) and (b_k) are constant sequences, say $a_k = r \neq 0$ and $b_k = s \neq 0$ for all $k \in \mathbb{N}$, then the operator $\Delta_{a,b}$ is reduced to the operator $B(r,s)$ and the results for the subdivisions of the spectra for generalized difference operator $\Delta_{a,b}$ over c_0, c, ℓ_p and bv_p have been studied in [6].

2.1. Subdivision of the spectrum of $\Delta_{a,b}$ on c_0. If $T : c_0 \to c_0$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_1 \to \ell_1$ is defined by the transpoze of the matrix A. It is well known that the dual space c_0^* of c is isomorphic to ℓ_1.

The spectra and the fine spectra of the operator $\Delta_{a,b}$ over the sequence space c_0 has been studied by El-Shabrawy [10]. In this subsection we summarize the main results.

Theorem 1 ([10], Theorem 2.1). *Let $D = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$ and $E = \{a_k : k \in \mathbb{N}, |a_k - a| > |b|\}$. Then $\sigma(\Delta_{a,b}, c_0) = D \cup E$.*

Theorem 2 ([10], Theorem 2.2). *$\sigma_p(\Delta_{a,b}, c_0) = E \cup K$, where*

$$K = \left\{ a_j : j \in \mathbb{N},\ \ |a_k - a| = |b|,\ \ \prod_{i=m}^{\infty} \frac{b_{i-1}}{a_j - a_i} \ \text{ diverges to zero for some } \ m \in \mathbb{N} \right\}.$$

Theorem 3 ([10], Theorem 2.3). *$\sigma_p\big(\Delta_{a,b}^*, c_0^*\big) = \big\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\big\} \cup E \cup H$, where*

$$H = \left\{ \lambda \in \mathbb{C} : |\lambda - a| = |b|,\ \sum_{k=0}^{\infty} \left| \prod_{i=0}^{k} \frac{\lambda - a_i}{b_i} \right| < \infty \right\}.$$

Theorem 4 ([10], Theorem 2.5). *$\sigma_r(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup (H \backslash K)$.*

Theorem 5 ([4], Theorem 2.6). *$\sigma_c(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \backslash H$.*

Lemma 1 ([12], Theorem II 3.11). *The adjoint operator T^* is onto if and only if T has a bounded inverse.*

Lemma 2 ([12], Theorem II 3.7). *A linear operator T has a dense range if and only if the adjoint operator T^* is one-to-one.*

Lemma 3. *If $\lim_{k\to\infty} a_k = a \neq 1$ for all $k \in \mathbb{N}$ where $a_k \neq 0$ for all $k \in \mathbb{N}$, then the product $\prod_k a_k$ is divergent.*

Lemma 4. *For $p, r \in \mathbb{N}$,*

$$\sum_{n=p}^{\infty} \left(\sum_{k=r}^{n-r} a_k b_{nk} \right) = \sum_{k=r}^{\infty} a_k \left(\sum_{n=p}^{\infty} b_{nk} \right),$$

where (a_k) and (b_{nk}) are nonnegative real numbers and $p \geq 2r$.

Proof. We have

$$\sum_{n=p}^{\infty} \left(\sum_{k=r}^{n-r} a_k b_{nk} \right) = \sum_{k=r}^{p-r} a_k b_{pk} + \sum_{k=r}^{p+1-r} a_k b_{pk} + \sum_{k=r}^{p+2-r} a_k b_{pk} + \sum_{k=r}^{p+3-r} a_k b_{pk} + \ldots =$$

$$= (a_r b_{pr} + a_{r+1} b_{p,r+1} + a_{r+2} b_{p,r+2} + \ldots + a_{p-r} b_{p,p-r}) +$$

$$+ (a_r b_{p+1,r} + a_{r+1} b_{p+1,r+1} + a_{r+2} b_{p+1,r+2} + \ldots + a_{p+1-r} b_{p+1,p+1-r}) +$$

$$+ (a_r b_{p+2,r} + a_{r+1} b_{p+2,r+1} + a_{r+2} b_{p+2,r+2} + \ldots + a_{p+2-r} b_{p+2,p+2-r}) + \ldots =$$

$$= a_r (b_{pr} + b_{p+1,r} + b_{p+2,r} + \ldots) + a_{r+1} (b_{p,r+1} + b_{p+1,r+1} + b_{p+2,r+1} + \ldots) +$$

$$+ a_{r+2} (b_{p,r+2} + b_{p+1,r+2} + b_{p+2,r+2} + \ldots) + \ldots =$$

$$= a_r \sum_{n=p}^{\infty} b_{nr} + a_{r+1} \sum_{n=p}^{\infty} b_{n,r+1} + a_{r+2} \sum_{n=p}^{\infty} b_{n,r+2} + \ldots =$$

$$= \sum_{k=r}^{\infty} a_k \left(\sum_{n=p}^{\infty} b_{nk} \right).$$

Theorem 6. $(\mathrm{III}_1)\, \sigma(\Delta_{a,b}, \ell_p) = \{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\} \cup (H \backslash K)$.

Proof. Let we investigate whether the operator $(\lambda I - \Delta_{a,b})^* = \lambda I - \Delta_{a,b}^*$ is surjective or not. Does there exist $x \in \ell_1$ for $y \in \ell_1$ such that $(\lambda I - \Delta_{a,b}^*)x = y$? If for $y \in \ell_1$, then $(\lambda I - \Delta_{a,b}^*)x = y$ and we get

$$(\lambda - a_0)\, x_0 - b_0 x_1 = y_0,$$

$$(\lambda - a_1)\, x_1 - b_1 x_2 = y_1,$$

$$(\lambda - a_2)\, x_2 - b_2 x_3 = y_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(\lambda - a_n)\, x_n - b_n x_{n+1} = y_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore, we obtain

$$x_1 = \frac{\lambda - a_0}{b_0} x_0 - \frac{1}{b_0} y_1,$$

$$x_2 = \frac{\lambda - a_1}{b_1} x_1 - \frac{1}{b_1} y_1 = \frac{\lambda - a_0}{b_0} \frac{\lambda - a_1}{b_1} x_0 - \frac{1}{b_0} \frac{\lambda - a_1}{b_1} y_0 - \frac{1}{b_1} y_1,$$

$$x_3 = \frac{\lambda - a_2}{b_2} x_2 - \frac{1}{b_2} y_2 = \frac{\lambda - a_0}{b_0} \frac{\lambda - a_1}{b_1} \frac{\lambda - a_2}{b_2} x_0 -$$

$$- \frac{1}{b_0} \frac{\lambda - a_1}{b_1} \frac{\lambda - a_2}{b_2} y_0 - \frac{1}{b_1} \frac{\lambda - a_2}{b_2} y_1 - \frac{1}{b_2} y_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, we get

$$x_n = x_0 \prod_{k=0}^{n-1} \frac{\lambda - a_k}{b_k} + \sum_{k=1}^{n-1} \frac{y_{k-1}}{b_{k-1}} \prod_{i=k}^{n-1} \frac{\lambda - a_i}{b_i} + \frac{y_{n-1}}{b_{n-1}}, \quad n \geq 2.$$

Now, we must show that $x \in \ell_1$. That is, is the series $\sum_{n=0}^{\infty} |x_n|$ covergent? We obtain

$$\sum_{n=0}^{\infty} |x_n| = |x_0| + |x_1| + \sum_{n=2}^{\infty} |x_n| \leq$$

$$\leq |x_0| + |x_1| + |x_0| \sum_{n=2}^{\infty} \left| \prod_{k=0}^{n-1} \frac{\lambda - a_k}{b_k} \right| + \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} \frac{y_{k-1}}{b_k} \prod_{i=k}^{n-1} \frac{\lambda - a_i}{b_i} \right| + \sum_{n=2}^{\infty} \left| \frac{y_{n-1}}{b_{n-1}} \right|.$$

Let

$$\sum\nolimits_1 = \sum_{n=2}^{\infty} \left| \prod_{k=0}^{n-1} \frac{\lambda - a_k}{b_k} \right|, \qquad \sum\nolimits_2 = \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} \frac{y_{k-1}}{b_k} \prod_{i=k}^{n-1} \frac{\lambda - a_i}{b_i} \right|, \qquad \sum\nolimits_3 = \sum_{n=2}^{\infty} \left| \frac{y_{n-1}}{b_{n-1}} \right|.$$

Since $\lim_{k\to\infty} b_k = b \neq 0$ from (2), $\lim_{k\to\infty} \dfrac{1}{b_k} = \dfrac{1}{b}$ and so $\left(\dfrac{1}{b_k} \right)$ is bounded. Hence, since

$$\text{there exists} \quad M > 0 \quad \text{such that} \quad \left| \frac{1}{b_n} \right| \leq M \quad \text{for all} \quad n \in \mathbb{N}, \tag{3}$$

the series

$$\sum\nolimits_3 = \sum_{n=2}^{\infty} \left| \frac{y_{n-1}}{b_{n-1}} \right| \leq M \sum_{n=2}^{\infty} |y_{n-1}| \leq M \|y\|_{\ell_1}$$

is convergent.

If $|\lambda - a| < |b|$, then $\lim_{k\to\infty} \left| \dfrac{\lambda - a_k}{b_k} \right| = \left| \dfrac{\lambda - a}{b} \right| < 1 \neq 1$ and the product $\prod_k \dfrac{\lambda - a_k}{b_k}$ is divergent from Lemma 3. Hence for $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$, the series $\sum_1$ is covergent if and only if $\lambda \in$ $\in (H \backslash K) \cup \{a_k : k \in \mathbb{N}, |a_k - a| < |b|\}$. Now, let we investigate the series $\sum_2$ to be convergent. If $\lambda \in \{a_k : k \in \mathbb{N}, |a_k - a| < |b|\}$, then it is clear that the series $\sum_2$ is convergent. Let $\lambda \in (H \backslash K)$. Then, from (3) and triangle inequality, we get

$$\sum\nolimits_2 = \sum_{n=2}^{\infty} \left| \sum_{k=1}^{n-1} \frac{y_{k-1}}{b_k} \prod_{i=k}^{n-1} \frac{\lambda - a_i}{b_i} \right| \leq M \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n-1} |y_{k-1}| \left| \prod_{i=k}^{n-1} \frac{\lambda - a_i}{b_i} \right| \right].$$

Therefore, if we take $p = 2$, $r = 1$, $a_k = |y_k|$ and $b_{nk} = \left| \prod_{i=k}^{n-1} \dfrac{\lambda - a_i}{b_i} \right|$ in Lemma 4, we have

$$\sum\nolimits_2 \leq M \sum_{k=2}^{\infty} |y_{k-1}| \sum_{n=1}^{\infty} \left| \prod_{i=k}^{n-1} \frac{\lambda - a_i}{b_i} \right|.$$

Since $\lambda \in H$, $\sum_{n=1}^{\infty} \left| \prod_{i=k}^{n-1} \dfrac{\lambda - a_i}{b_i} \right|$ is covergent. Setting $L := \sum_{n=1}^{\infty} \left| \prod_{i=k}^{n-1} \dfrac{\lambda - a_i}{b_i} \right|$, we obtain

$$\sum\nolimits_2 \leq ML \sum_{k=2}^{\infty} |y_k| \leq ML\|y\|_{\ell_1}$$

and so $\sum_2$ is covergent. That is, for $\lambda \in \sigma_r(\Delta_{a,b}, c_0)$, the operator $(\lambda I - \Delta_{a,b})^*$ is surjective if and only if $\lambda \in \{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\} \cup (H \backslash K)$. Hence from Lemma 1, $\lambda I - \Delta_{a,b}$ has bounded inverse.

***Corollary* 1.** $(\mathrm{III}_2)\,\sigma(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \setminus \{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\}$.

Proof. It is clear from Theorems 4 and 6, since

$$(\mathrm{III}_2)\,\sigma(\Delta_{a,b}, c_0) = \sigma_r(\Delta_{a,b}, c_0) \setminus (\mathrm{III}_1)\,\sigma(\Delta_{a,b}, c_0).$$

Theorem 7 ([10], Theorem 2.8)**.** $(\mathrm{III}_3)\,\sigma(\Delta_{a,b}, \ell_p) = E \cup K$.

***Corollary* 2.** $(\mathrm{I}_3)\,\sigma(\Delta_{a,b}, c_0) = (\mathrm{II}_3)\,\sigma(\Delta_{a,b}, c_0) = \varnothing$ *dir.*

Proof. Since from Table 1, $\sigma_p(\Delta_{a,b}, c_0) = (\mathrm{I}_3)\,\sigma(\Delta_{a,b}, c_0) \cup (\mathrm{II}_3)\,\sigma(\Delta_{a,b}, c_0) \cup (\mathrm{III}_3)\,\sigma(\Delta_{a,b}, c_0) = E \cup K$ and $(\mathrm{I}_3)\,\sigma(\Delta_{a,b}, c_0) \cap (\mathrm{II}_3)\,\sigma(\Delta_{a,b}, c_0) \cap (\mathrm{III}_3)\,\sigma(\Delta_{a,b}, c_0) = \varnothing$ the proof is finished from Theorem 7.

Theorem 8. (a) $\sigma_{ap}(\Delta_{a,b}, c_0) = (D \cup E) \setminus [\{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\} \cup H]$,

(b) $\sigma_\delta(\Delta_{a,b}, c_0) = D \cup E$,

(c) $\sigma_{co}(\Delta_{a,b}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup H \cup E$.

Proof. (a) It is clear from Theorems 1 and 6.

(b) It is clear from Theorem 1 and Conclusion 2, since from Table 1, $\sigma_\delta(\Delta_{a,b}, c_0) = \sigma(\Delta_{a,b}, c_0) \setminus (\mathrm{I}_3)\,\sigma(\Delta_{a,b}, c_0)$.

(c) Since from Table 1, $\sigma_{co}(\Delta_{a,b}, c_0) = (\mathrm{III}_1)\,\sigma(\Delta_{a,b}, c_0) \cup (\mathrm{III}_2)\,\sigma(\Delta_{a,b}, c_0) \cup (\mathrm{III}_3)\,\sigma(\Delta_{a,b}, c_0) = \sigma_r(\Delta_{a,b}, c_0) \cup (\mathrm{III}_3)\,\sigma(\Delta_{a,b}, c_0)$, the proof is finished from Theorems 4 and 7.

***Corollary* 3.** (a) $\sigma_{ap}\left(\Delta^*_{a,b}, \ell_1\right) = D \cup E$,

(b) $\sigma_\delta\left(\Delta^*_{a,b}, \ell_1\right) = (D \cup E) \setminus \left[\{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\} \cup H\right]$.

Proof. It is clear from Theorem 8 and Proposition 1 (c) and (d).

3. Remarks and some special cases. In recent years, some special cases of the operator $\Delta_{a,b}$ has been studied. These special cases are related to sequences (a_k) and (b_k). In here, we give some cases:

If we take $a_k = 1$ and $b_k = -1$ for all $k \in \mathbb{N}$ in the operator $\Delta_{a,b}$, then it reduces to the backward difference operator Δ which has been studied in [7].

If we take $a_k = r$ and $b_k = s$ for all $k \in \mathbb{N}$ in the operator $\Delta_{a,b}$, then it reduces to the generalized difference operator $B(r, s)$ which has been studied in [6].

If we take $a_k = -b_k = v_k$ for all $k \in \mathbb{N}$ in the operator $\Delta_{a,b}$, then it reduces to the generalized difference operator Δ_v which has been studied in [5].

If (a_k) is a sequence of positive real numbers such that $a_k \neq 0$ for all $k \in \mathbb{N}$ with $\lim_{k\to\infty} a_k = U \neq 0$ and (b_k) is either constant or strictly decreasing sequence of positive real numbers with $\lim_{k\to\infty} b_k = V \neq 0$ and $\sup_k a_k < U + V$, then the operator $\Delta_{a,b}$ reduces to the generalized difference operator Δ_{uv} which has been studied in [11].

***Remark* 1.** If (a_k) and (b_k) are convergent sequences of nonzero real numbers such that

$$\lim_{k\to\infty} a_k = a > 0 \qquad \text{and} \qquad \lim_{k\to\infty} b_k = b, |b| = a,$$

and

$$\sup_k a_k \le a \qquad \text{and} \qquad b_k^2 \le a_k^2 \quad \text{for all} \quad k \in \mathbb{N},$$

then we can prove that $H = \varnothing$ and so we have

$$\sigma_{ap}\left(\Delta_{a,b}, c_0\right) =$$

$$= \left(\{\lambda \in \mathbb{C} : |\lambda - a| \le |b|\} \cup \{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\}\right) \setminus \{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\},$$

$$\sigma_{\delta}\left(\Delta_{a,b}, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - a| \le |b|\} \cup \{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\},$$

$$\sigma_{co}\left(\Delta_{a,b}, c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup \{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\},$$

$$\sigma_{ap}\left(\Delta_{a,b}^*, \ell_1\right) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup \{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\},$$

$$\sigma_{\delta}\left(\Delta_{a,b}^*, \ell_1\right) =$$

$$= \left(\{\lambda \in \mathbb{C} : |\lambda - a| \le |b|\} \cup \{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\}\right) \setminus \{a_k : k \in \mathbb{N},\ |a_k - a| < |b|\}.$$

4. Conclusion. Many researchers have determined the spectrum and the fine spectrum of a matrix operator in some sequence spaces. Although the fine spectrum with respect to the Goldberg's classification of the generalized difference operator $\Delta_{a,b}$ over the sequence space c_0 were studied by El-Shabrawy [10], in the present paper, the concepts of the approximate point spectrum, defect spectrum and compression spectrum are introduced, and given the subdivisions of the spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence space c_0, as the new subdivisions of spectrum. It is immediate that our new results cover a wider class of linear operators which are represented by infinite lower triangular double-band matrices on the sequence space c_0. For this reason, our study is more general and more comprehensive than the previous work. We note that our new results in this paper improve and generalize the results which have been stated in [5 – 7].

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