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ENTIRE FUNCTIONS SHARE TWO HALF SMALL FUNCTIONS
ЦІЛІ ФУНКЦІЇ ПОДІЛЯЮТЬ ДВІ НАПІВМАЛІ ФУНКЦІЇ

The paper generalizes a result by P. Li and C. C. Yang [Illinois J. Math. – 2000. – **44**. – P. 349 – 362] and extends the previous work of G. Qiu [Kodai Math. J. – 2000. – **23**. – P. 1 – 11].
У роботі узагальнено результат П. Лі та Ц. Ц. Янга [Illinois J. Math. – 2000. – **44**. – P. 349 – 362] та розширено результати попередньої роботи Г. Кіу [Kodai Math. J. – 2000. – **23**. – P. 1 – 11].

1. Introduction and main results. Throughout f denotes an entire function, i.e., a function that is analytic in the whole complex plane, and f' denotes its derivative. We use the same signs as given in Nevanlinna theory (see [3, 4]). In particular $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, except possibly on a set of finite linear measure. A meromorphic function α is said to be a small function of f if $T(r, \alpha) = S(r, f)$. We say that two nonconstant meromorphic functions f and g share the value or small function α IM (ignoring multiplicities), if $f - \alpha$ and $g - \alpha$ have the same zeros. Let k be a positive integer, we denote by $N_{k)}\left(r, \dfrac{1}{f-\alpha}\right)$ the counting function of zeros of $f - \alpha$ with multiplicity $\leq k$ and by $N_{(k}\left(r, \dfrac{1}{f-\alpha}\right)$ the counting function of zeros of $f - \alpha$ with multiplicity $> k$. We denote by $N_{=k}\left(r, \dfrac{1}{f-\alpha}\right)$ counting function of zeros of $f - \alpha$ which have the multiplicity k . In the same manner we define

$$\bar{N}_{k)}\left(r, \frac{1}{f-\alpha}\right), \qquad \bar{N}_{(k}\left(r, \frac{1}{f-\alpha}\right) \quad \text{and} \quad \bar{N}_{=k}\left(r, \frac{1}{f-\alpha}\right),$$

where in counting the zeros of $f - \alpha$ we ignore the multiplicities.

If $g(z) - \alpha(z) = 0$ whenever $f(z) - \alpha(z) = 0$, then we write $f = \alpha \Rightarrow g = \alpha$ (some times we say f and g share half α IM). Thus f and g share α IM if and only if $f = \alpha \Leftrightarrow g = \alpha$, where $f = \alpha \Leftrightarrow g = \alpha$ means $f = \alpha \Rightarrow g = \alpha$ and $g = \alpha \Rightarrow f = \alpha$.

On the problems of uniqueness of an entire and its first derivative that share some values. E. Mues and N. Steinmets (see [5]) proved the following:

Theorem A. *If a nonconstant entire function f and its derivative f' share two distinct finite values IM, then $f \equiv f'$.*

Li and Yang (see [1]) extended this result as follows:

Theorem B. *Let f be a nonconstant entire function and a, b be two distinct complex numbers. If $f = a \Rightarrow f' = a$ and $f = b \Rightarrow f' = b$, then only one of the following cases holds*:

(I) $f \equiv f'$;

(II) *if* $ab \neq 0$, *then* $f(z) = a + ce^{\frac{b}{b-a}z}$ *or* $f(z) = b + ce^{\frac{a}{a-b}z}$;

(III) *if* $ab = 0$, *then* $f(z) = (a+b)(ce^{\frac{1}{4}z} - 1)^2$, *where* c *is a nonzero constant.*

On the other hand G. Qiu (see [2]) generalized Theorem A to the following:

Theorem C. *Let* f *be a nonconstant entire function,* α *and* β *be two distinct small functions of* f *with* $\alpha \not\equiv \infty$ *and* $\beta \not\equiv \infty$ *. If* f *and* f' *share* α *and* β *IM, then* $f \equiv f'$.

In this paper, we will generalize and extends the above results to obtain the following results:

Theorem 1. *Let* f *be a nonconstant entire function,* α *and* β *be two distinct small functions of* f *with* $\alpha \not\equiv \infty$ *and* $\beta \not\equiv \infty$. *If* $f = \alpha \Rightarrow f' = \alpha$ *and* $f = \beta \Rightarrow f' = \beta$, *then exactly one of the following four cases must occur*:

(i) $f \equiv f'$;

(ii) *if* $\alpha \not\equiv \alpha'$ *and* $\beta \not\equiv \beta'$, *then* $f(z) = \beta + c(\beta - \alpha)e^{\int_0^z (\frac{\alpha - \alpha'}{\alpha - \beta})(t)dt}$ *or* $f(z) = \alpha + c(\beta - \alpha) \times \times e^{\int_0^z (\frac{\beta - \beta'}{\beta - \alpha})(t)dt}$;

(iii) *if* $\alpha \equiv \alpha'$ *and* $\beta \not\equiv \beta'$, *then* $f(z) = \alpha + (\beta - \alpha)\left(1 + ce^{\frac{1}{4}\int_0^z (\frac{\beta - \beta'}{\beta - \alpha})(t)dt}\right)^2$ *or if* $\alpha \not\equiv \alpha'$ *and* $\beta \equiv \beta'$, *then* $f(z) = \beta + (\alpha - \beta)\left(1 + ce^{\frac{1}{4}\int_0^z (\frac{\alpha - \alpha'}{\alpha - \beta})(t)dt}\right)^2$, *where* c *is a nonzero constant*;

(iv) *if* $\alpha \equiv \alpha'$ *and* $\beta \equiv \beta'$, *then* $T(r,f) = N_{=2}\left(r, \dfrac{1}{f - \alpha}\right) + S(r,f) = N_{1)}\left(r, \dfrac{1}{f' - \alpha}\right) + + S(r,f) = T(r,f') + S(r,f) = N_{1)}\left(r, \dfrac{1}{f' - \beta}\right) + S(r,f) = N_{=2}\left(r, \dfrac{1}{f - \beta}\right) + S(r,f)$.

From Theorem 1, we deduce the following corollaries:

***Corollary* 1.** *Let* f *be a nonconstant entire function,* α *and* β *be two distinct small functions of* f *with* $\alpha \not\equiv \infty$ *and* $\beta \not\equiv \infty$. *If* $f = \alpha \Rightarrow f' = \alpha$ *and* $f = \beta \Rightarrow f' = \beta$, *and if* $\alpha \not\equiv \alpha'$ *or* $\beta \not\equiv \beta'$, *then* f *as in Theorem* 1 (i)–(iii).

***Corollary* 2.** *Let* f *be a nonconstant entire function,* α *and* β *be two distinct small functions of* f *with* $\alpha \not\equiv \infty$ *and* $\beta \not\equiv \infty$. *If* $f = \alpha \Rightarrow f' = \alpha$ *and* $f = \beta \Rightarrow f' = \beta$, *and if* $\bar{N}\left(r, \dfrac{1}{f - \alpha}\right) = = \bar{N}\left(r, \dfrac{1}{f' - \alpha}\right) + S(r,f)$, *then* f *as in Theorem* 1 (i)–(iii).

***Corollary* 3.** *Let* f *be a nonconstant entire function,* α *and* β *be two distinct small functions of* f *with* $\alpha \not\equiv \infty$ *and* $\beta \not\equiv \infty$. *If* $f = \alpha \Leftrightarrow f' = \alpha$ *and* $f = \beta \Rightarrow f' = \beta$, *then only one of the following cases holds*:

(i) $f \equiv f'$;

(ii) *if* $\alpha \not\equiv \alpha'$ *and* $\beta \not\equiv \beta'$, *then* $f(z) = \beta + c(\beta - \alpha)e^{\int_0^z (\frac{\alpha - \alpha'}{\alpha - \beta})(t)dt}$;

(iii) *if* $\alpha \equiv \alpha'$ *and* $\beta \not\equiv \beta'$, *then* $\alpha \equiv \beta'$ *and* $f(z) = \alpha + (\beta - \beta')(1 + ce^{\frac{1}{4}z})^2$ *or if* $\alpha \not\equiv \alpha'$ *and* $\beta \equiv \beta'$, *then* $f(z) = \beta + (\alpha - \beta)(1 + ce^{\frac{1}{4}\int_0^z (\frac{\alpha - \alpha'}{\alpha - \beta})(t)dt})^2$, *where* c *is a nonzero constant.*

***Remark* 1.** If $\alpha \equiv a$ and $\beta \equiv b$ are constants, then Corollary 1 becomes Theorem B. Therefore, Corollary 1 is generalization Theorem B.

2. If f and f' share α IM, then $\bar{N}\left(r, \dfrac{1}{f - \alpha}\right) = \bar{N}\left(r, \dfrac{1}{f' - \alpha}\right)$ and hence the case (iv) in Theorem 1 is impossible. Therefore Corollary 2 is extension of Theorem C.

3. Also in Theorem 1, if $\alpha(z) \equiv z$ and $\beta(z) \equiv 1$, then $f(z) = 1 + (z-1)e^{z-1}$. That is Theorem 1 is strictly an extension and generalization of Theorems B and C.

4. It is obvious that Corollary 3 is an extension of Theorem C.

2. Some lemmas. For the proof of our results we need the following lemmas:

Lemma 1 [2]. *Let f be a nonconstant entire function, α_1 and α_2 be two distinct small functions of f with $\alpha_1 \not\equiv \infty$ and $\alpha_2 \not\equiv \infty$. Set*

$$\Delta(f) = \begin{vmatrix} f-\alpha_1 & \alpha_1-\alpha_2 \\ f'-\alpha_1' & \alpha_1'-\alpha_2' \end{vmatrix} = \begin{vmatrix} f-\alpha_2 & \alpha_1-\alpha_2 \\ f'-\alpha_2' & \alpha_1'-\alpha_2' \end{vmatrix}. \tag{2.1}$$

Then

$$\Delta(f) \not\equiv 0, \tag{2.2}$$

$$m\left(r, \frac{\Delta(f)f}{(f-\alpha_1)(f-\alpha_2)}\right) = S(r,f), \tag{2.3}$$

$$m\left(r, \frac{\Delta(f)}{f-\alpha_i}\right) = S(r,f), \quad i=1,2, \tag{2.4}$$

$$\sum_{j=1}^{2} N\left(r, \frac{1}{f-\alpha_j}\right) - N\left(r, \frac{1}{\Delta(f)}\right) \le \sum_{j=1}^{2} \bar{N}\left(r, \frac{1}{f-\alpha_j}\right) + S(r,f). \tag{2.5}$$

Lemma 2 [1]. *Let f be a nonconstant meromorphic function and α, β, γ be small functions of f with $\alpha \not\equiv 0$ or $\gamma \not\equiv 0$. Furthermore, let $g = \alpha f^2 + \beta f + \gamma$. If $\bar{N}(r,f) + \bar{N}\left(r, \frac{1}{f}\right) = S(r,f)$ and $N_{1)}\left(r, \frac{1}{g}\right) = S(r,f)$, then $\beta^2 - 4\alpha\gamma \equiv 0$.*

Lemma 3 [3, p. 47]. *Let f be a nonconstant meromorphic function and a_1, a_2, a_3 be distinct small functions of f, then*

$$T(r,f) \le \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f).$$

3. Proof of Theorem 1. Suppose that $f \not\equiv f'$ and that the auxiliary function

$$\omega = \frac{\Delta(f)(f-f')}{(f-\alpha)(f-\beta)}, \tag{3.1}$$

where $\Delta(f)$ is defined by (2.1), $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. From (2.2) we know that $\Delta(f) \not\equiv 0$. Therefore it follows that $\omega \not\equiv 0$. It is easy to see from (2.5) that $N(r,\omega) = S(r,f)$. By (2.3) we obtain

$$m(r,\omega) \le m\left(r, \frac{\Delta(f)f}{(f-\alpha)(f-\beta)}\right) + m\left(r, 1-\frac{f'}{f}\right) = S(r,f).$$

Thus

$$T(r,\omega) = S(r,f). \tag{3.2}$$

From the fact $f=\alpha \Rightarrow f'=\alpha,\ f=\beta \Rightarrow f'=\beta$ and Lemma 3 we know that

$$T(r,f) \le \bar{N}\left(r, \frac{1}{f-\alpha}\right) + \bar{N}\left(r, \frac{1}{f-\beta}\right) + \bar{N}(r,f) + S(r,f) \le$$

$$\leq \bar{N}\left(r, \frac{1}{f' - \alpha}\right) + \bar{N}\left(r, \frac{1}{f' - \beta}\right) + S(r, f) \leq$$

$$\leq 2T(r, f') + S(r, f) \leq 2T(r, f) + S(r, f).$$

It follows that every $S(r, f)$ is also an $S(r, f')$ and vice versa. From now on we will write $S(r)$ for the common error term. Let $F = \dfrac{f - \alpha}{\beta - \alpha}$. Then from (2.1) we get

$$\Delta(f) = (\beta - \alpha)^2 F'. \tag{3.3}$$

Substituting F and $\Delta(f)$ into (3.1), ω is expressed to

$$\omega = \frac{F'[\alpha - \alpha' + (\beta - \beta' - \alpha + \alpha')F - (\beta - \alpha)F']}{F(F - 1)}, \tag{3.4}$$

which may also be written $F^2 = a_1F + a_2F' + a_3FF' + a_4F'^2$, where $T(r, a_j) = S(r),\ j = 1,\ 2,\ 3,\ 4$. From the definition of F and the last formula we see that

$$2T(r, f) + S(r) = 2T(r, F) = 2m(r, F) + 2N(r, F) =$$

$$= 2m(r, F) + S(r) \leq$$

$$\leq m(r, F) + m(r, F') + 2m\left(r, \frac{F'}{F}\right) + S(r) \leq$$

$$\leq m(r, F) + m(r, F') + S(r) \leq$$

$$\leq 2m(r, F) + S(r).$$

That is $T(r, F') = T(r, F) + S(r) = T(r, f) + S(r)$. We rewrite (3.4) in the form

$$\left[F - \frac{1}{2} - \frac{\beta - \beta' - \alpha + \alpha'}{2\omega}F'\right]^2 =$$

$$= \left[\left(\frac{\beta - \beta' - \alpha + \alpha'}{2\omega}\right)^2 - \frac{\beta - \alpha}{\omega}\right]F'^2 + \frac{\beta - \beta' + \alpha - \alpha'}{2\omega}F' + \frac{1}{4}. \tag{3.5}$$

In the following we shall treat three cases:

***Case* 1:** $\alpha \not\equiv \alpha'$ and $\beta \not\equiv \beta'$. Since $f = \alpha \Rightarrow f' = \alpha$ and $f = \beta \Rightarrow f' = \beta$, so the zeros of $f - \alpha$ and $f - \beta$ with multiplicities longer than one are zeros of $\alpha - \alpha'$ and $\beta - \beta'$ respectively. It follows that

$$\bar{N}_{(2}\left(r, \frac{1}{f - \alpha}\right) + \bar{N}_{(2}\left(r, \frac{1}{f - \beta}\right) = S(r).$$

From this, (3.1) and (3.2) we deduce that

$$\bar{N}\left(r, \frac{1}{\Delta(f)}\right) \leq N\left(r, \frac{1}{\omega}\right) + \bar{N}_{(2}\left(r, \frac{1}{f - \alpha}\right) + \bar{N}_{(2}\left(r, \frac{1}{f - \beta}\right) = S(r),$$

and so from (3.3),

$$\bar{N}\left(r, \frac{1}{F'}\right) + \bar{N}\left(r, F'\right) = S(r).$$

Appling Lemma 2 to equation (3.5) we find that

$$\left(\frac{\beta - \beta' + \alpha - \alpha'}{2\omega}\right)^2 - \left[\left(\frac{\beta - \beta' - \alpha + \alpha'}{2\omega}\right)^2 - \frac{\beta - \alpha}{\omega}\right] \equiv 0.$$

That is $\omega \equiv \dfrac{(\alpha - \alpha')(\beta - \beta')}{\alpha - \beta}$. Substituting this into (3.5) gives

$$\left[F - 1 - \left(\frac{\alpha - \beta}{\alpha - \alpha'}\right) F'\right] \left[F + \left(\frac{\alpha - \beta}{\beta - \beta'}\right) F'\right] \equiv 0,$$

which implies that

$$f(z) = \beta + c(\beta - \alpha) e^{\int_0^z \left(\frac{\alpha - \alpha'}{\alpha - \beta}\right)(t)dt} \qquad \text{or} \qquad f(z) = \alpha + c(\beta - \alpha) e^{\int_0^z \left(\frac{\beta - \beta'}{\beta - \alpha}\right)(t)dt},$$

where c is a nonzero constant.

***Case* 2:** $\alpha \equiv \alpha'$ and $\beta \not\equiv \beta'$ or $\alpha \not\equiv \alpha'$ and $\beta \equiv \beta'$. Without loss of generality, we can assume that $\alpha \equiv \alpha'$ and $\beta \not\equiv \beta'$. According to the discussion in Case 1 we know that $\bar{N}_{(2}\left(r, \dfrac{1}{f - \beta}\right) = S(r)$, and so from the definition of F we obtain $\bar{N}_{(2}\left(r, \dfrac{1}{F - 1}\right) = S(r)$. Since the zeros of $f - \alpha$ are all the zeros of $f' - \alpha = f' - \alpha'$, it follows that $N_{1)}\left(r, \dfrac{1}{f - \alpha}\right) = S(r)$. Further, we can conclude from (3.1) that the zeros of $f - \alpha$ which multiplicity p (≥ 3) are the zeros of ω. Thus, from (3.2) we get

$$\bar{N}_{(3}\left(r, \frac{1}{f - \alpha}\right) \leq N\left(r, \frac{1}{\omega}\right) + S(r) = S(r).$$

Thus

$$\bar{N}\left(r, \frac{1}{f - \alpha}\right) = \bar{N}_{=2}\left(r, \frac{1}{f - \alpha}\right) + S(r).$$

From this and the definition of F we get

$$\bar{N}\left(r, \frac{1}{F}\right) = \bar{N}_{=2}\left(r, \frac{1}{F}\right) + S(r).$$

From (3.4) we easily see that the zero of F' must be the zero of F with multiplicity 2 if it is not zero of ω. Let $h = \dfrac{F}{F'^2}$. Then we have

$$\bar{N}\left(r, \frac{1}{h}\right) + \bar{N}(r, h) = S(r). \tag{3.6}$$

Equation (3.4) can be written as

$$(\beta - \alpha)(F' - \delta F)^2 = F\left[\left((\beta - \alpha)\delta^2 - \omega\right) F + \omega\right], \tag{3.7}$$

where $\delta = \dfrac{\beta - \beta'}{2(\beta - \alpha)}$. If $(\beta - \alpha)\delta^2 - \omega \not\equiv 0$, then from (3.7), we find that

$$F(z_0) = \left(\frac{-\omega}{(\beta-\alpha)\delta^2 - \omega}\right)(z_0) \quad \Rightarrow \quad F'(z_0) = \left(\frac{-\omega\delta}{(\beta-\alpha)\delta^2 - \omega}\right)(z_0),$$

and thus $h(z_0) = \left(\dfrac{\omega - (\beta-\alpha)\delta^2}{\omega\delta^2}\right)(z_0)$. Noting that $F(z_1) = 1$ implies that $F'(z_1) = \left(\dfrac{\beta-\beta'}{\beta-\alpha}\right)(z_1)$ and thus $h(z_1) = \left(\dfrac{\beta-\alpha}{\beta-\beta'}\right)^2(z_1)$, by Lemma 3, we get

$$T(r,F) \le \bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{F + \frac{\omega}{(\beta-\alpha)\delta^2-\omega}}\right) + S(r) \le$$

$$\le \bar{N}\left(r, \frac{1}{h - (\frac{\beta-\alpha}{\beta-\beta'})^2}\right) + \bar{N}\left(r, \frac{1}{h - \frac{\omega-(\beta-\alpha)\delta^2}{\omega\delta^2}}\right) + S(r) \le$$

$$\le 2T(r,h) + S(r).$$

Therefore ω, α and β are small functions of h. From the definition of h and equation (3.7), we obtain

$$\left(hF' - \frac{\beta-\beta'}{2\omega}\right)^2 = h + \frac{(\beta-\beta')^2}{4\omega^2} - \frac{\beta-\alpha}{\omega}.$$

Therefore $h + \dfrac{(\beta-\beta')^2}{4\omega^2} - \dfrac{\beta-\alpha}{\omega}$ has no simple zero. Hence by Lemma 2, we get $\dfrac{(\beta-\beta')^2}{4\omega^2} - \dfrac{\beta-\alpha}{\omega} \equiv 0$. That is $\omega \equiv \dfrac{(\beta-\beta')^2}{4(\beta-\alpha)}$. Thus (3.7) becomes

$$\left(\frac{1}{\delta}F' - F\right)^2 = F. \tag{3.8}$$

Let $G = \dfrac{1}{\delta}F' - F$. We get $F = G^2$ and thus $F' = 2GG'$. From (3.8) we have $\left(\dfrac{2}{\delta}G' - G\right)^2 \equiv 1$. Hence either $\dfrac{2}{\delta}G' - G \equiv 1$ or $\dfrac{2}{\delta}G' - G \equiv -1$. If $\dfrac{2}{\delta}G' - G \equiv -1$, then we find that $f = \alpha - (\beta-\alpha)(1 + ce^{\frac{1}{2}s})$, where $s = \int_0^z \delta(t)dt$ and c is a nonzero constant. From this and $f = \beta \Rightarrow f' = \beta$ we arrive at a contradiction. Therefore $\dfrac{2}{\delta}G' - G \equiv 1$. From this it is easy to see that $f = \alpha + (\beta-\alpha)(1 + ce^{\frac{1}{2}s})^2$, where $s = \int_0^z \delta(t)dt$ and c is a nonzero constant.

***Case* 3:** $\alpha \equiv \alpha'$ and $\beta \equiv \beta'$. By the discussion in Case 2 we know that

$$N_{1)}\left(r, \frac{1}{f-\alpha}\right) + N_{1)}\left(r, \frac{1}{f-\beta}\right) = S(r) \tag{3.9}$$

and

$$N_{(3}\left(r, \frac{1}{f-\alpha}\right) + N_{(3}\left(r, \frac{1}{f-\beta}\right) \le 3N\left(r, \frac{1}{\omega}\right) \le 3T(r,\omega) + O(1) = S(r). \tag{3.10}$$

From $\alpha \equiv \alpha'$, $\beta \equiv \beta'$ and (3.4) we deduce that

$$m\left(r, \frac{1}{F'}\right) \leq m\left(r, \frac{1}{\omega}\right) + m\left(r, \frac{F'}{F(F-1)}\right) + S(r) \leq$$
$$\leq T(r,\omega) + m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{F'}{F-1}\right) + S(r) = S(r),$$

so that

$$m\left(r, \frac{1}{f-\alpha}\right) + m\left(r, \frac{1}{f-\beta}\right) = m\left(r, \frac{1}{F}\right) + m\left(r, \frac{1}{F-1}\right) + S(r) =$$
$$= m\left(r, \frac{1}{F(F-1)}\right) + S(r) \leq$$
$$\leq m\left(r, \frac{1}{F'}\right) + S(r) = S(r).$$

Combining this, (3.9) and (3.10) we obtain

$$T(r,f) = N_{=2}\left(r, \frac{1}{f-\alpha}\right) + S(r) = N_{=2}\left(r, \frac{1}{f-\beta}\right) + S(r). \tag{3.11}$$

Set

$$\Gamma = 2\frac{f''-\beta}{f'-\beta} - \frac{f'-\beta}{f-\beta}. \tag{3.12}$$

Since $\beta \equiv \beta'$, $m(r,\Gamma) = S(r)$. It follows from (3.12) that if z_β is a zero of $f-\beta$ with multiplicity 2, then $\Gamma(z_\beta) = O(1)$. Thus, from (3.11) we get

$$N(r,\Gamma) \leq \bar{N}\left(r, \frac{1}{f'-\beta}\right) - \bar{N}_{=2}\left(r, \frac{1}{f-\beta}\right) + S(r). \tag{3.13}$$

Also, if z_α is a zero of $f-\alpha$ with multiplicity 2, then

$$\left[2(f''-\beta) - (\Gamma+1)(\alpha-\beta)\right](z_\alpha) = 0. \tag{3.14}$$

On the other hand, differentiating (3.1) twice and then using $f(z_\alpha) = \alpha$, we arrive at $[2(f''-\alpha) - \omega](z_\alpha) = 0$. If we now eliminate $f''(z_\alpha)$ between this and (3.14) we obtain

$$\left[\omega - (\Gamma-1)(\alpha-\beta)\right](z_\alpha) = 0. \tag{3.15}$$

Set

$$\Omega = 2\frac{f''-\alpha}{f'-\alpha} - \frac{f'-\alpha}{f-\alpha}.$$

Similarly as the above, we have $m(r,\Omega) = S(r)$,

$$N(r,\Omega) \leq \bar{N}\left(r, \frac{1}{f'-\alpha}\right) - \bar{N}_{=2}\left(r, \frac{1}{f-\alpha}\right) + S(r)$$

and

$$[\omega - (\Omega - 1)(\alpha - \beta)]\,(z_\beta) = 0.$$

We discuss the following four subcases:

***Subcase* 3.1:** $\omega - (\Gamma - 1)(\alpha - \beta) \equiv 0$ and $\omega - (\Omega - 1)(\alpha - \beta) \equiv 0$. Then $\Gamma \equiv \Omega$. Hence $\left(\dfrac{f' - \beta}{f' - \alpha}\right)^2 = c\left(\dfrac{f - \beta}{f - \alpha}\right)$, where c is a nonzero constant. Therefore $2T(r, f') + S(r) = T(r, f)$. This is impossible because f is an entire function.

***Subcase* 3.2:** $\omega - (\Gamma - 1)(\alpha - \beta) \not\equiv 0$ and $\omega - (\Omega - 1)(\alpha - \beta) \not\equiv 0$. Then from (3.15), (3.1) and (3.13) we deduce that

$$\bar{N}_{=2}\left(r, \frac{1}{f - \alpha}\right) \leq N\left(r, \frac{1}{\omega - (\Gamma - 1)(\alpha - \beta)}\right) + S(r) \leq$$

$$\leq T(r, \omega) + T(r, \Gamma) + S(r) =$$

$$= N(r, \Gamma) + S(r) \leq$$

$$\leq \bar{N}\left(r, \frac{1}{f' - \beta}\right) - \bar{N}_{=2}\left(r, \frac{1}{f - \beta}\right) + S(r).$$

Together with (3.11) we have

$$T(r, f) \leq \bar{N}\left(r, \frac{1}{f' - \beta}\right) + S(r) \leq T(r, f') + S(r) \leq T(r, f) + S(r). \tag{3.16}$$

Consequently,

$$N_{(2}\left(r, \frac{1}{f' - \beta}\right) + m\left(r, \frac{1}{f' - \beta}\right) = S(r). \tag{3.17}$$

Similarly, from $\omega - (\Omega - 1)(\alpha - \beta) \not\equiv 0$ we get

$$N_{(2}\left(r, \frac{1}{f' - \alpha}\right) + m\left(r, \frac{1}{f' - \alpha}\right) = S(r).$$

From this, (3.11), (3.16) and (3.17) we arrive at the conclusion (iv).

***Subcase* 3.3:** $\omega - (\Gamma - 1)(\alpha - \beta) \not\equiv 0$ and $\omega - (\Omega - 1)(\alpha - \beta) \equiv 0$. Since $\omega - (\Gamma - 1)(\alpha - \beta) \not\equiv 0$, by the discussion in Subcase 3.2 we have (3.16) and (3.17). From $\omega - (\Omega - 1)(\alpha - \beta) \equiv 0$, we find that $\bar{N}_{(2}\left(r, \dfrac{1}{f' - \alpha}\right) = S(r)$ and

$$f' = \alpha \Rightarrow f = \alpha. \tag{3.18}$$

Otherwise $f'(z_0) = \alpha \Rightarrow f(z_0) \neq \alpha$ holds for a sequence z_0 whose counting function is an $S(r)$.

We set

$$\nu = \frac{\Delta(f')(f - f')}{(f' - \alpha)(f' - \beta)}. \tag{3.19}$$

From (2.1) we conclude that $\nu \not\equiv 0$. Again by (2.4), (3.11), (3.17) and (3.16) we see that

$$m(r,\nu) = m\left(r, \frac{\Delta(f')\left(\frac{f-\beta}{f'-\beta}-1\right)}{f'-\alpha}\right) \leq m\left(r, \frac{\Delta(f')}{f'-\alpha}\right) + m\left(r, \frac{f-\beta}{f'-\beta}\right) + S(r) \leq$$

$$\leq m\left(r, \frac{f-\beta}{f'-\beta}\right) + S(r) = N\left(r, \frac{1}{f-\beta}\right) - N\left(r, \frac{1}{f'-\beta}\right) + S(r) =$$

$$= T(r,f) - T(r,f') + S(r) = S(r). \tag{3.20}$$

From (3.19), (2.5), (3.18), (3.11), (3.17) we deduce that

$$N(r,\nu) \leq \bar{N}\left(r, \frac{1}{f'-\alpha}\right) - \bar{N}\left(r, \frac{1}{f-\alpha}\right) + \bar{N}\left(r, \frac{1}{f'-\beta}\right) - \bar{N}\left(r, \frac{1}{f-\beta}\right) =$$

$$= N_{1)}\left(r, \frac{1}{f'-\alpha}\right) - \bar{N}_{=2}\left(r, \frac{1}{f-\alpha}\right) + N_{1)}\left(r, \frac{1}{f'-\beta}\right) - \bar{N}_{=2}\left(r, \frac{1}{f-\beta}\right) +$$

$$+S(r) = N_{1)}\left(r, \frac{1}{f'-\beta}\right) - \bar{N}_{=2}\left(r, \frac{1}{f-\beta}\right) + S(r). \tag{3.21}$$

Let z_1 be a common zero of $f-\alpha$ (or $f-\beta$) and $f'-\alpha$ (or $f'-\beta$) with multiplicities 2 and 1 respectively. From (3.1) and (3.19) it follows that $(\omega - 2\nu)(z_1) = 0$. If $\omega - 2\nu \not\equiv 0$, then from (3.2), (3.20), (3.21), (3.17), (3.11) and (3.16) we conclude that

$$\bar{N}_{=2}\left(r, \frac{1}{f-\alpha}\right) + \bar{N}_{=2}\left(r, \frac{1}{f-\beta}\right) \leq N\left(r, \frac{1}{\omega - 2\nu}\right) \leq T(r,\omega) + T(r,\nu) + O(1) =$$

$$= m(r,\nu) + N(r,\nu) + S(r) =$$

$$= N_{1)}\left(r, \frac{1}{f'-\beta}\right) - \bar{N}_{=2}\left(r, \frac{1}{f-\beta}\right) + S(r) \leq$$

$$\leq T(r,f') - \frac{1}{2}T(r,f) + S(r) =$$

$$= \frac{1}{2}T(r,f) + S(r).$$

That is $2T(r,f) \leq T(r,f) + S(r)$, a contradiction. Therefore we have $\omega - 2\nu \equiv 0$. From this it is easy to arrive at the contradiction.

***Subcase* 3.4:** $\omega - (\Gamma - 1)(\alpha - \beta) \equiv 0$ and $\omega - (\Omega - 1)(\alpha - \beta) \not\equiv 0$. Similarly as the Subcase 3.3, we will arrive at the same contradiction.

Theorem 1 is proved.

4. Proof of corollaries. We proof only Corollary 3; proofs of the remaining corollaries are easy. If $\alpha \not\equiv \alpha'$ and $\beta \not\equiv \beta'$, then $f(z) = \alpha + c(\beta - \alpha)e^{\int_0^z (\frac{\beta-\beta'}{\beta-\alpha})(t)dt}$. By differentiating both sides of this last function with respect to z, we obtain

$$f'(z) - \alpha = c(\beta - \alpha')\left(e^{\int_0^z (\frac{\beta-\beta'}{\beta-\alpha})(t)dt} + \frac{\alpha' - \alpha}{c(\beta - \alpha')}\right).$$

But this is a contradiction to our assumption that $f = \alpha \Leftrightarrow f' = \alpha$. If $\alpha \equiv \alpha'$ and $\beta \not\equiv \beta'$, then

$$f(z) = \alpha + (\beta - \alpha)\left(1 + ce^{\frac{1}{4}\int_0^z (\frac{\beta-\beta'}{\beta-\alpha})(t)dt}\right)^2.$$

Differentiating once gives

$$f'(z) - \alpha = \left(1 + ce^{\frac{1}{4}\int_0^z (\frac{\beta-\beta'}{\beta-\alpha})(t)dt}\right)\left(\beta' - \alpha + c[\beta' - \alpha + 1/2(\beta - \beta')]e^{\frac{1}{4}\int_0^z (\frac{\beta-\beta'}{\beta-\alpha})(t)dt}\right).$$

Since $f = \alpha \Leftrightarrow f' = \alpha$, we have either $\beta' - \alpha + 1/2(\beta - \beta') \equiv 0$ or $\beta' - \alpha \equiv 0$. If $\beta' - \alpha + 1/2(\beta - \beta') \equiv 0$, then we can write

$$f(z) - \alpha = (\beta - \alpha)\left(1 + \frac{c}{\sqrt{\beta - \alpha}}\right)^2.$$

This is impossible. Therefore $\beta' - \alpha \equiv 0$, in this case $f(z) = \alpha + (\beta - \beta')\left(1 + ce^{\frac{1}{4}z}\right)^2$. Finally, if $f = \alpha \Leftrightarrow f' = \alpha$, then it is clear that $\bar{N}\left(r, \dfrac{1}{f - \alpha}\right) = \bar{N}\left(r, \dfrac{1}{f' - \alpha}\right)$. Thus the case (iv) in Theorem 1 does not appear. Now complete the proof of Corollary 3.

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