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SIMPSON-TYPE INEQUALITIES FOR GEOMETRICALLY RELATIVE CONVEX FUNCTIONS

НЕРІВНОСТІ ТИПУ СІМПСОНА ДЛЯ ГЕОМЕТРИЧНО ВІДНОСНИХ ОПУКЛИХ ФУНКЦІЙ

We consider a class of geometrically relative convex functions and deduce several new integral inequalities of Simpson's type via geometrically relative convex functions. The ideas and techniques used in the paper may stimulate further research in this area.

Розглянуто клас геометрично відносних опуклих функцій та отримано кілька нових інтегральних нерівностей типу Сімпсона в термінах геометрично відносних опуклих функцій. Ідеї і техніка, що використовуються в роботі, можуть стимулювати подальші дослідження в даній області.

1. Introduction. Let $f\colon I=[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be a four time continuously differentiable on I°, where I° is the interior of I and $\|f^{(4)}\|_{\infty}<\infty$. Then following inequality is known as Simpson's inequality in the literature:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\leq\frac{1}{2880}\|f^{(4)}\|_{\infty}(b-a)^4. \qquad (1.1)$$

For useful details on Simpson's type of integral inequalities (see [1–3, 5, 13, 16, 17]). Convexity plays an important role in different fields of pure and applied sciences. Due to its significance, many researchers have paid much attention to this subject by investigating its various properties. Consequently, the concept of convexity has been extended and generalized in different directions using novel and innovative ideas (see [4, 6–12, 14, 15, 17]).

Noor et al. [10] introduced and investigated the concept of geometrically relative convex functions, which also contains the class of relative convex functions as special case.

In this paper, we consider the class of geometrically relative convex functions and derive several new Simpson's type of integral inequalities. This is the main motivation of this paper.

2. Preliminaries. In this section, we recall some previously known concepts.

Definition 2.1 [10]. *Let $\mathcal{G}\subseteq(0,\infty)$. Then $\mathcal{G}$ is said to be geometrically relative convex set, if there exists an arbitrary function $g\colon\mathbb{R}^n\to\mathbb{R}^n$ such that*

$$(g(x))^t(g(y))^{1-t}\in\mathcal{G}\quad\forall g(x),g(y)\in\mathcal{G},\ t\in[0,1].$$

Using $AM-GM$ inequality, we have

$$(g(x))^t(g(y))^{1-t}\leq tg(x)+(1-t)g(y)\quad\forall g(x),g(y)\in\mathcal{G},\ t\in[0,1].$$

Definition 2.2 [10]. *A function $f\colon\mathcal{G}\to\mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be geometrically relative convex function (GG-relative convex function) if there exists an arbitrary function $g\colon\mathbb{R}^n\to\mathbb{R}^n$ such that*

$$f((g(x))^t(g(y))^{1-t})\leq(f(g(x)))^t(f(g(y)))^{1-t}\quad\forall g(x),g(y)\in\mathcal{G},\qquad t\in[0,1]. \qquad (2.1)$$

From (2.1), it follows that

$$\log f((g(x))^t(g(y))^{1-t}) \leq t \log f(g(x)) + (1-t) \log f(g(y)) \quad \forall g(x), g(y) \in \mathcal{G}, \quad t \in [0,1].$$

Using $AM - GM$ inequality, we have

$$f((g(x))^t(g(y))^{1-t}) \leq (f(g(x)))^t(f(g(y)))^{1-t} \leq$$
$$\leq tf(g(x)) + (1-t)f(g(y)).$$

Thus it follows that every geometrically relative convex function (GG-relative convex function) is also GA-relative convex function, but the converse is not true.

For $t = \frac{1}{2}$ in (2.1), we have Jensen type of geometrically relative convex functions. That is

$$f\left(\sqrt{g(x)g(y)}\right) \leq \sqrt{f(g(x))f(g(y))}.$$

Definition 2.3 [10]. *Let I be a subinterval of $(0,\infty)$. Then f is geometrically relative convex function if and only if*

$$\begin{vmatrix} 1 & 1 & 1 \\ \log g(a) & \log g(x) & \log g(b) \\ \log f(g(a)) & \log f(g(x)) & \log f(g(b)) \end{vmatrix} \geq 0,$$

where $g(a) \leq g(x) \leq g(b)$.

One can easily show that the following are equivalent:

(1) f is geometrically relative convex function on geometrically relative convex set;

(2) $f(g(a))^{\log(g(b))} f(g(x))^{\log(g(a))} f(g(b))^{\log(g(x))} \geq f(g(a))^{\log(g(x))} f(g(x))^{\log(g(b))} \times$ $\times f(g(b))^{\log(g(a))}$, where $g(x) = g(a)^t g(b)^{1-t}$ and $t \in [0,1]$.

Definition 2.4 [10]. *A function $f\colon \mathcal{G} \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be GA-relative convex function, if there exists an arbitrary function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f((g(x))^t(g(y))^{1-t}) \leq tf(g(x)) + (1-t)f(g(y)) \quad \forall g(x), g(y) \in \mathcal{G},\ t \in [0,1]. \tag{2.2}$$

From Definitions 2.3 and 2.4, it follows that $GG \Longrightarrow GA$, but the converse is not true.

3. Main results. In this section, we prove our main results.

Lemma 3.1. *For $g(a)$, $g(b) \in \mathcal{G}$ and $t \in [0,1]$, if $g(a) < g(b)$, then*

$$(g(a))^{1-t}(g(b))^t \leq (1-t)g(a) + tg(b).$$

Essentially using the technique of [2], one can prove following result.

Lemma 3.2. *Let $f\colon I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I°, where $g(a)$, $g(b) \in I$ with $g(a) < g(b)$. Then following equality holds*:

$$\left| \frac{1}{6}\left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b))\right] - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) \right| =$$

$$= (g(b) - g(a)) \int_0^1 \mu(t) f'((1-t)g(a) + tg(b))dt,$$

where

$$\mu(t) = \begin{cases} t - \dfrac{1}{6}, & if \quad t \in \left[0, \dfrac{1}{2}\right), \\ t - \dfrac{5}{6}, & if \quad t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Now using Lemmas 3.1 and 3.2 we prove our main results.

Theorem 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $g(a)$, $g(b) \in I$ with $g(a) < g(b)$. If $f' \in L[g(a), g(b)]$ and $|f'|$ is monotonically decreasing and geometrically relative convex function, then following inequality holds*:

$$\left| \frac{1}{6}\left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b))\right] - \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f(g(x))dg(x) \right| \leq$$

$$\leq (g(b) - g(a))|f'(g(a))| \left\{ \mathcal{F}_1(t,w) + \mathcal{F}_2(t,w) \right\},$$

where

$$\mathcal{F}_1(t,w) = \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| w^t dt = \frac{-6 + 12w^{\frac{1}{6}} - 6\sqrt{w} - \ln w + 2\sqrt{w}\ln w}{6 \ln w^2}$$

and

$$\mathcal{F}_2(t,w) = \int_{\frac{1}{2}}^1 \left| \frac{1}{6} - t \right| w^t dt = \frac{6\sqrt{w} - 6w - 2\sqrt{w}\ln w + 5w \ln w}{6 \ln w^2},$$

respectively.

Proof. Using Lemma 3.2 and the fact that $|f'|$ is monotonically decreasing and geometrically relative convex function, we have

$$\left| \frac{1}{6}\left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b))\right] - \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f(g(x))dg(x) \right| \leq$$

$$\leq (g(b) - g(a)) \left\{ \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'((1-t)g(a) + tg(b))|dt + \right.$$

$$\left. + \int_{\frac{1}{2}}^1 \left| \frac{1}{6} - t \right| |f'((1-t)g(a) + tg(b))|dt \right\} \leq$$

$$\le (g(b)-g(a))\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||f'((g(a))^{1-t}(g(b)^t))|dt+\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|\frac{1}{6}-t\right||f'((g(a))^{1-t}(g(b)^t))|dt\right\}\le$$

$$\le (g(b)-g(a))|f'(g(a))|\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^t dt+\int_{\frac{1}{2}}^1\left|\frac{1}{6}-t\right|\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^t dt\right\}=$$

$$=(g(b)-g(a))|f'(g(a))|\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|w^t dt+\int_{\frac{1}{2}}^1\left|\frac{1}{6}-t\right|w^t dt\right\}=$$

$$=(g(b)-g(a))|f'(g(a))|\left\{\mathcal{F}_1(t,w)+\mathcal{F}_2(t,w)\right\}.$$

Theorem 3.1 is proved.

Theorem 3.2. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° where $g(a),g(b)\in I$ with $g(a)<g(b)$. If $f'\in L[g(a),g(b)]$ and $|f'|^q$ is monotonically decreasing and geometrically relative convex function, then for $\frac{1}{p}+\frac{1}{q}=1$ following inequality holds*:

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{g(a)+g(b)}{2}\right)+f(g(b))\right]-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|\le$$

$$\le (g(b)-g(a))|f'(g(a))|\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[(\mathcal{H}_1(t,q,w))^{\frac{1}{q}}+(\mathcal{H}_2(t,q,w))^{\frac{1}{q}}\right],$$

where

$$\mathcal{H}_1(t,q,w)=\int_0^{\frac{1}{2}}w^{qt}dt,$$

and

$$\mathcal{H}_2(t,q,w)=\int_{\frac{1}{2}}^1 w^{qt}dt,$$

respectively.

Proof. Using Lemma 3.2, Hölder's inequality and the fact that $|f'|^q$ is monotonically decreasing and geometrically relative convex function, we have

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{g(a)+g(b)}{2}\right)+f(g(b))\right]-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|\le$$

$$\leq (g(b)-g(a))\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'((1-t)g(a)+tg(b))|^q dt\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|f'((1-t)g(a)+tg(b))|^q dt\right)^{\frac{1}{q}}\right\}\leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^{qt} dt\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^{qt} dt\right)^{\frac{1}{q}}\right\}\leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[(\mathcal{H}_1(t,q,w))^{\frac{1}{q}}+(\mathcal{H}_2(t,q,w))^{\frac{1}{q}}\right].$$

Theorem 3.2 is proved.

Theorem 3.3. *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* I°, *where* $g(a),g(b)\in I$ *with* $g(a)<g(b)$. *If* $f'\in L[g(a),g(b)]$ *and* $|f'|^q$ *is monotonically decreasing and geometrically relative convex function, then for* $\frac{1}{p}+\frac{1}{q}=1$ *following inequality holds*:

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{g(a)+g(b)}{2}\right)+f(g(b))\right]-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|\leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[(\mathcal{H}(t,q,w))^{\frac{1}{q}}\right],$$

where

$$\mathcal{H}(t,q,w)=\int_0^1 w^{qt}dt.$$

Proof. Using Lemma 3.2, Hölder's inequality and the fact that $|f'|^q$ is monotonically decreasing and geometrically relative convex function, we have

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{g(a)+g(b)}{2}\right)+f(g(b))\right]-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|\leq$$

$$\leq (g(b)-g(a))\left(\int_0^1 |\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)g(a)+tg(b))|^q dt\right)^{\frac{1}{q}} \leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^{qt}dt\right)^{\frac{1}{q}}=$$

$$=(g(b)-g(a))|f'(g(a))|\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[(\mathcal{H}(t,q,w))^{\frac{1}{q}}\right].$$

Theorem 3.3 is proved.

Theorem 3.4. *Let $f\colon I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, where $g(a),g(b)\in I$ with $g(a)<g(b)$. If $f'\in L[g(a),g(b)]$ and $|f'|^q$ is monotonically decreasing and geometrically relative convex function, then for $\dfrac{1}{p}+\dfrac{1}{q}=1$ following inequality holds*:

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{g(a)+g(b)}{2}\right)+f(g(b))\right]-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|\leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[(\mathcal{H}(t,q,w))^{\frac{1}{q}}\right],$$

where

$$\mathcal{H}(t,q,w)=\int_0^1 w^{qt}dt.$$

Proof. Using Lemma 3.2, Holder's inequality and the fact that $|f'|^q$ is monotonically decreasing and geometrically relative convex function, we have

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{g(a)+g(b)}{2}\right)+f(g(b))\right]-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|\leq$$

$$\leq (g(b)-g(a))\left(\int_0^1 |\mu(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)g(a)+tg(b))|^q dt\right)^{\frac{1}{q}} \leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^{qt}dt\right)^{\frac{1}{q}}=$$

$$=(g(b)-g(a))|f'(g(a))|\left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[(\mathcal{H}(t,q,w))^{\frac{1}{q}}\right].$$

Theorem 3.4 is proved.

Theorem 3.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $g(a), g(b) \in I$ with $g(a) < g(b)$. If $f' \in L[g(a), g(b)]$ and $|f'|^q$ is monotonically decreasing and geometrically relative convex function, then for $\frac{1}{p} + \frac{1}{q} = 1$ following inequality holds*:

$$\left| \frac{1}{6} \left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b)) \right] - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) \right| \leq$$

$$\leq (g(b)-g(a))|f'(g(a))| \left(\frac{2(q-1)(1+2^{\frac{2q-3}{q-1}})}{(2q-3)6^{\frac{2q-3}{q-1}}} \right)^{1-\frac{1}{q}} \left[(\mathcal{L}(t,q,w))^{\frac{1}{q}} \right],$$

where

$$\mathcal{L}(t,q,w) = \int\limits_0^1 |\mu(t)|^2 \, w^{qt} dt.$$

Proof. Using Lemma 3.2, Hölder's inequality and the fact that $|f'|^q$ is monotonically decreasing and geometrically relative convex function, we have

$$\left| \frac{1}{6} \left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b)) \right] - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) \right| \leq$$

$$\leq (g(b)-g(a)) \left(\int\limits_0^1 |\mu(t)|^{\frac{q-2}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int\limits_0^1 |\mu(t)|^2 \, |f'((1-t)g(a)+tg(b))|^q dt \right)^{\frac{1}{q}} \leq$$

$$\leq (g(b)-g(a))|f'(g(a))| \left(\int\limits_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^{\frac{q-2}{q-1}} dt + \int\limits_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^{\frac{q-2}{q-1}} dt \right)^{1-\frac{1}{q}} \times$$

$$\times \left(\int\limits_0^1 |\mu(t)|^2 \left(\frac{|f'(g(b))|}{|f'(g(a))|} \right)^{qt} dt \right)^{\frac{1}{q}} =$$

$$= (g(b)-g(a))|f'(g(a))| \left(\frac{2(q-1)(1+2^{\frac{2q-3}{q-1}})}{(2q-3)6^{\frac{2q-3}{q-1}}} \right)^{1-\frac{1}{q}} \left[(\mathcal{L}(t,q,w))^{\frac{1}{q}} \right].$$

Theorem 3.5 is proved.

Theorem 3.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $g(a), g(b) \in I$ with $g(a) < g(b)$. If $f' \in L[g(a), g(b)]$ and $|f'|^q$ is monotonically decreasing and geometrically relative convex function, then for $q > 1$ following inequality holds*:

$$\left| \frac{1}{6} \left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b)) \right] - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) \right| \leq$$

$$\leq (g(b)-g(a))|f'(g(a))| \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \left[(\mathcal{H}_1(t,q,w))^{\frac{1}{q}} + (\mathcal{H}_2(t,q,w))^{\frac{1}{q}}\right],$$

where

$$\mathcal{H}_1(t,q,w) = \int_0^{\frac{1}{2}} w^{qt}dt$$

and

$$\mathcal{H}_2(t,q,w) = \int_{\frac{1}{2}}^1 w^{qt}dt,$$

respectively.

Proof. Using Lemma 3.2, power mean inequality and the fact that $|f'|^q$ is monotonically decreasing and geometrically relative convex function, we have

$$\left|\frac{1}{6}\left[f(g(a)) + 4f\left(\frac{g(a)+g(b)}{2}\right) + f(g(b))\right] - \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right| \leq$$

$$\leq (g(b)-g(a))\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right||f'((1-t)g(a)+tg(b))|^q dt\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\left|t-\frac{5}{6}\right|\int_{\frac{1}{2}}^1 |f'((1-t)g(a)+tg(b))|^q dt\right)^{\frac{1}{q}}\right\} \leq$$

$$\leq (g(b)-g(a))|f'(g(a))|\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^{qt}dt\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(\frac{|f'(g(b))|}{|f'(g(a))|}\right)^{qt}dt\right)^{\frac{1}{q}}\right\} \leq$$

$$\leq (g(b)-g(a))|f'(g(a))| \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \left[(\mathcal{H}_1(t,q,w))^{\frac{1}{q}} + (\mathcal{H}_2(t,q,w))^{\frac{1}{q}}\right].$$

Theorem 3.6 is proved.

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