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L. K. Ardekani, B. Davvaz (Yazd Univ., Iran)

DERIVATIONS OF GAMMA (SEMI)HYPERRINGS

ПОХІДНІ ГАММА (НАПІВ)ГІПЕРКІЛЕЦЬ

Differential Γ-(semi)hyperrings are Γ-(semi)hyperrings equipped with derivation, which is a linear unary function satisfying the Leibniz product rule. We introduce the notions of derivation and weak derivation on Γ-hyperrings and Γ-semihyperrings and obtain some important results relating to them in a specific way.

Диференціальні Γ-(напів)гіперкільця — це Γ-(напів)гіперкільця з похідною, що є лінійною унарною функцією і задовольняє правило добутку Лейбніца. Введено поняття похідної та слабкої похідної на Γ-гіперкільцях та Γ-напівгіперкільцях та отримано деякі важливі результати, що стосуються цих понять у конкретному вигляді.

1. Introduction. The notion of Γ-ring was introduced by N. Nobusawa in 1964 [19]. After him in 1966, Barnes extended this notion and obtained more results [2]. Almost ten years later Kyuno in [15, 16] investigated new aspects of Γ-rings. The notion of Γ-semirings was introduced by Rao [21] as a generalization of Γ-ring as well as of semiring. In fact, every semiring $(R, +, \circ)$ can be considered as a Γ-semiring by putting $\Gamma = \{\circ\}$. In this way, many classical notions of semiring have been extended to Γ-semiring.

The theory of algebraic hyperstructures is a well established branch of classical algebraic theory. In 1934, at the 8th Congress of Scandinavian Mathematicians, Marty [18] has introduced, for the first time, the notion of hypergroup, using it in different contexts: algebraic functions, rational fractions and noncommutative groups. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. One of the first books, dedicated especially to hypergroups, is written by Corsini in 1993 [3]. Another book on this subject was published one year later [24]. A recent book on hyperstructures [4] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book is devoted especially to the study of hyperring theory, written by Davvaz and Leoreanu-Fotea [5]. It begins with some basic results cocerning ring theory and algebraic hyperstructures, which represent the most general algebraic context, in which the reality can be modelled. After these subjects, several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. Now, we recall some preliminary definitions from [5]. For more details and properties, we refer the readers to [3, 5, 24].

Let H be a nonempty set and $\circ\colon H \times H \to P^*(H)$ be a hyperoperation, where $P^*(H)$ is a family of all nonempty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two nonempty subsets A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $A \circ \{x\} = A \circ x$ and $\{x\} \circ A = x \circ A$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$. In addition, if for every $x \in H$, $x \circ H = H = H \circ x$, then $(H, \circ)$ is called a *hypergroup*. A nonempty subset K of a semihypergroup $(H, \circ)$ is called a *subsemihypergroup* if it is a semihypergroup. In other words, a nonempty subset K of a semihypergroup $(H, \circ)$ is a subsemihypergroup if $K \circ K \subseteq K$. We say that a hypergroup $(H, \circ)$ is *canonical* if (1) it is commutative; (2) it has a scalar identity (also called scalar unit), which means that there exists $e \in H$ such that for every $x \in H$, we have $x \circ e = \{x\}$; (3) every element has a unique inverse, which means that for all $x \in H$, there exists a unique $x^{-1} \in H$ such that $e \in x^{-1} \circ x$; (4) it is reversible, which means that if $x \in y \circ z$, then $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

As a generalization of semihypergroups, Davvaz et al. studied the notion of Γ-semihypergroups in [1, 12]: Let S and Γ be two nonempty sets. Then S is called a Γ-*semihypergroup* if $x\alpha y \subseteq S$ and $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $\alpha, \beta \in \Gamma$ and $x, y, z \in S$. Γ-semihypergroup S is called a Γ-*hypergroup* if (S, γ) is a hypergroup, for all $\gamma \in \Gamma$.

The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if "$+$" and "$\cdot$" are two hyperoperations such that $(R, +)$ is a hypergroup and "$\cdot$" is an associative hyperoperation which is distributive with respect to "$+$" . There are different notions of hyperrings. If only the addition "$+$" is a hyperoperation and the multiplication "$\cdot$" is a usual operation, then R is called Krasner hyperring. Exactly, a *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms: (1) $(R, +)$ is a canonical hypergroup (the scaler identity of $(R, +)$ is denoted by 0); (2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = x$; (3) the multiplication is distributive with respect to the hyperoperation $+$. In [9], Γ-hyperrings as a generalization of hyperrings are introduced: Let $(R, +)$ be a canonical hypergroup and Γ be a nonempty set. Then R is called a Γ-*hyperring* if there exists a mapping $R \times \Gamma \times R \longrightarrow P^*(R)$ (images denoted by $x\alpha y$, for all $x, y \in R$ and $\alpha \in \Gamma$), where $P^*(R)$ is the set of all nonempty subsets of R, satisfying the following conditions: (1) there exists a zero element that a bilaterally absorbing element, i.e., $x\alpha 0 = 0\alpha x = 0$ and $x + 0 = x$, for all $\alpha \in \Gamma$ and $x \in R$; (2) $x\alpha(y + z) = x\alpha y + x\alpha z$; (3) $(x + y)\alpha z = x\alpha z + y\alpha z$; (4) $x\alpha(y\beta z) = (x\alpha y)\beta z$. Let $(R, +)$ be a commutative semihypergroup and $(\Gamma, +)$ be a commutative group. Then R is called a Γ-*semihyperring* if there exists a mapping $R \times \Gamma \times R \longrightarrow P^*(R)$ (images denoted by $x\alpha y$, for all $x, y \in R$ and $\alpha \in \Gamma$) satisfying the conditions (2), (3), (4) and (5) $x(\alpha + \beta)y = x\alpha y + x\beta y$ [8]. In the above definition if R is a semigroup, then R is called a *multiplicative* Γ-*semihyperring* [8]. We recall the following definitions and notions of [9]: A Γ-semihyperring (Γ-hyperring) R is called *commutative* (*weak commutative*, respectively) if $x\alpha y = y\alpha x$ ($x\alpha y \cap y\alpha x \neq \varnothing$, respectively), for all $x, y \in R$ and $\alpha \in \Gamma$. The meaning of *center* (*weak center*, respectively) of R is $Z(R) = \{x \in R \mid x\alpha y = y\alpha x, \text{ for all } y \in R \text{ and } \alpha \in \Gamma\}$ $\big(Z^\circ(R) = \big\{x \in R \mid x\alpha y \cap y\alpha x \neq \varnothing, \text{ for all } y \in R \text{ and } \alpha \in \Gamma\big\}$, respectively$\big)$. We say that R is a Γ-semihyperring with zero, if there exists $0 \in R$ such that $x \in x + 0$, $0 \in 0\alpha x$ and $0 \in x\alpha 0$, for all $x \in R$ and $\alpha \in \Gamma$. Let A and B be two nonempty subsets of Γ-semihyperring (Γ-hyperring) R. We define

$$A + B = \{t \in R \mid t \in a + b, \; a \in A, b \in B\},$$

$$A\Gamma B = \{t \in R \mid t \in a\alpha b, \; a \in A, b \in B, \alpha \in \Gamma\},$$

$$A\Gamma^{\Sigma} B = \Big\{t \in R \mid t \in \sum\nolimits_{i=1}^{n} a_i\alpha_i b_i, \; a_i \in A, b_i \in B, \alpha_i \in \Gamma, n \in \mathbb{N}\Big\},$$

$$\mathbb{N}X = \Big\{t \in R \mid t \in \sum\nolimits_{i=1}^{n} n_i x_i, \; x_i \in X, \; n, n_i \in \mathbb{N}\Big\}.$$

A nonempty subset S of Γ-semihyperring (Γ-hyperring, respectively) R is called a Γ-*subsemihyperring* (Γ-*subhyperring*, respectively) if it is closed with respect to the hypermultiplication and hyperaddition. In other words, a nonempty subset S of Γ-semihyperring R is a Γ-subsemihyperring if $S + S \subseteq S$ and $S\Gamma S \subseteq S$. A *right ideal* (*left ideal*, respectively) I of a Γ-semihyperring R is an additive subsemihypergroup $(R, +)$ such that $I\Gamma R \subseteq I$ ($R\Gamma I \subseteq I$, respectively). If I is both right and left ideal of R, then we say that I is a *two-sided ideal* or simply an *ideal* of R. An ideal of Γ-hyperring R defines similarly. Let X be a nonempty subset of Γ-semihyperring (Γ-hyperring) R. By the term left ideal $\langle X\rangle_l$ (right ideal $\langle X\rangle_r$, respectively) of R generated by X, we mean the intersection of all left ideals (right ideals, respectively) of R contains X. Hence,

(1) $\langle X\rangle_l = \mathbb{N}X + R\Gamma^{\Sigma}X$;

(2) $\langle X\rangle_r = \mathbb{N}X + X\Gamma^{\Sigma}R$;

(3) $\langle X\rangle = \mathbb{N}X + R\Gamma^{\Sigma}X + X\Gamma^{\Sigma}R + R\Gamma^{\Sigma}X\Gamma^{\Sigma}R$.

2. Derivation on Γ-(semi)hyperrings. The concept of derivation on rings has been introduced by Posner [17]. In 1987, Jing introduced the notion of derivation on Γ-rings [14]. After him, many mathematicians studied derivation on rings and Γ-rings (see, for example, [6, 13, 20, 22, 23]). In this section, we introduce the notions of derivation and weak derivation on Γ-hyperrings and Γ-semihyperrings. Also, some properties of them are explored.

Definition 2.1. *Let $(R, +, \Gamma)$ be a Γ-semihyperring (Γ-hyperring). The function $d\colon R \longrightarrow R$ is called derivation if for all $x, y \in R$ and $\alpha \in \Gamma$,* (1) $d(x + y) = d(x) + d(y)$; (2) $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$. *The function $d\colon R \to R$ is called weak derivation if for all $x, y \in R$, it satisfies in* (1) *and* (3) $d(x\alpha y) \subseteq d(x)\alpha y + x\alpha d(y)$.

It is clear that every derivation is a weak derivation. Let $(R, +, \Gamma)$ be a Γ-hyperring and d be a derivation on R. Then by the second condition of definition of derivation we have

$$d(0) = d(0\alpha 0) = d(0)\alpha 0 + 0\alpha d(0) = 0, \quad \text{where} \quad \alpha \in \Gamma.$$

The set of all derivations (weak derivations, respectively) of Γ-semihyperring (Γ-hyperring) $(R, +, \Gamma)$ is denoted by $\Delta(R, +, \Gamma)$ ($D(R, +, \Gamma)$, respectively). Note that $\Delta(R, +, \Gamma) \subseteq D(R, +, \Gamma) \subseteq \mathrm{Hom}(R, +)$. The following examples illustrate the above concepts.

***Example* 2.1.** Let R be a Γ-semihyperring with zero. Then the function $d\colon R \longrightarrow R$ defined by $d(x) = 0$, for all $x \in R$, is a weak derivation. If R is a Γ-hyperring, then d is a derivation.

***Example* 2.2.** Let $(R, +, \circ)$ be a semihyperring such that $x \circ y = x \circ y + x \circ y$ and Γ be a commutative group. Define $x\alpha y = x \circ y$, for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is a Γ-semihyperring [8]. It is easily to check that the function $d\colon R \longrightarrow R$ defined by $d(x) = x$, for all $x \in R$, is a derivation on $(R, +, \Gamma)$.

***Example* 2.3.** Let $R = \mathbb{Z}_4$ and $\Gamma = \{\overline{0}, \overline{2}\}$. Then R is a multiplicative Γ-semihyperring with the hyperoperation $x\alpha y = \{\overline{0}, \overline{2}\}$, for all $x, y \in R$ and $\alpha \in \Gamma$ [8]. There exist only two derivations on R, they are as $d_1(x) = x$, for all $x \in R$ and

$$d_2(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad x = 3, \\ 2 & \text{if} \quad x = 2, \\ 3 & \text{if} \quad x = 1. \end{cases}$$

Also, there exist only four weak derivations on R, they are as $d_1,\ d_2,\ d_3(x)=0$ and

$$d_4(x)=\begin{cases}0 & \text{if} \quad x=0{,}2,\\ 2 & \text{if} \quad x=1{,}3.\end{cases}$$

Consequently, $\Delta(\mathbb{Z}_4,+_4,\Gamma)=\{d_1,d_2\}$ and $D(\mathbb{Z}_4,+_4,\Gamma)=\{d_1,d_2,d_3,d_4\}$.

***Example* 2.4.** Let $R=\mathbb{Z}_6$, $\Gamma=\{\overline{0},\overline{3}\}$ and $I=\{\overline{0},\overline{2},\overline{4}\}$. Then $(R,+_6,\Gamma)$ is a multiplicative Γ-semihyperring with respect to the hyperoperation $x\alpha y=x\times_6\alpha\times_6 y+_6 I$, for all $x,y\in\mathbb{Z}_6$ and $\alpha\in\Gamma$ [8]. It is clear that for every $\alpha\in\Gamma$, we have $x\alpha y=I$ or $\{\overline{1},\overline{3},\overline{5}\}$. The function $d\colon R\longrightarrow R$ defined by $d(x)=2\times_6 x$, for all $x\in R$, is a derivation.

***Example* 2.5.** Let $R=\{a,b,c\}$ and $\Gamma=\{\alpha,\beta\}$. Then, R is a Γ-hyperring with the following operations and hyperoperation [9]:

$\oplus$	a	b	c
a	a	b	c
b	b	$\{a,b\}$	c
c	c	c	$\{a,b,c\}$

α	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

β	a	b	c
a	a	a	a
b	a	c	b
c	a	b	c

It is clear that a is the scaler identity of $(R,\oplus)$. By the above tables and the conditions of Definition 2.1, we conclude that there exists only one derivation on R and it is as $d_1(x)=a$, for all $x\in R$. By direct calculation, there exist only two weak derivations on R, they are as d_1 and $d_2(x)=x$, for all $x\in R$. Hence, $\Delta(R,+,\Gamma)=\{d_1\}$ and $D(R,+,\Gamma)=\{d_1,d_2\}$.

***Example* 2.6.** Let $(R,+)$ be a group and $(\Gamma,+)$ be a subgroup of R. Then R is a multiplicative Γ-semihyperring with respect the following hyperoperation $x\alpha y=R$ [9]. It is clear that every homomorphism on $(R,+)$ is a weak derivation. Therefore, in this case $D(R,+,\Gamma)=\mathrm{Hom}(R,+)$ which implies that every isomorphism on $(R,+)$ is a derivation.

***Example* 2.7.** Let $(R,+,\cdot)$ be a commutative ring, $(\Gamma,+)$ be a subgroup of R and ρ be an equivalence relation defined as $x\rho y$ if and only if $x=y$ or $x=-y$. Then the set $R/\rho=\{\rho(x)\mid x\in R\}$ becomes a $\overline{\Gamma}$-hyperring with respect to the hyperoparation $\rho(x)\oplus\rho(y)=\{\rho(x+y),\rho(x-y)\}$ and multiplication $\rho(x)\rho(\alpha)\rho(y)=\rho(x\alpha y)$, where $\overline{\Gamma}=\{\rho(\alpha)\mid\alpha\in\Gamma\}$[9]. Suppose that d is a derivation on $(R,+,\Gamma)$. We define $\overline{d}$ on $(R/\rho,\oplus,\overline{\Gamma})$ as $\overline{d}(\rho(x))=\rho(d(x))$, for all $x\in R$. It is clear that $\overline{d}$ is well-defined. Also, $\overline{d}$ is a weak derivation because:

$$\overline{d}(\rho(x)\oplus\rho(y))=\overline{d}\{\rho(x+y),\rho(x-y)\}=\{\rho\big(d(x+y)\big),\rho\big(d(x-y)\big)\}=$$

$$=\Big\{\rho\big(d(x)+d(y)\big),\rho\big(d(x)-d(y)\big)\Big\}=\rho(d(x))\oplus\rho(d(y))=$$

$$=\overline{d}(\rho(x))\oplus\overline{d}(\rho(y))$$

and

$$\overline{d}\big(\rho(x)\rho(\alpha)\rho(y)\big)=\overline{d}\big(\rho(x\alpha y)\big)=\rho\big(d(x\alpha y)\big)=$$

$$=\rho\big(d(x)\alpha y+x\alpha d(y)\big)\subseteq\Big\{\rho(d(x)\alpha y+x\alpha d(y)),\rho\big(d(x)\alpha y-x\alpha d(y)\big)\Big\}=$$

$$=\rho\big(d(x)\alpha y\big)\oplus\rho\big(x\alpha d(y)\big)=\rho(d(x))\rho(\alpha)\rho(y)\oplus\rho(x)\rho(\alpha)\rho(d(y))=$$

$$= \overline{d}(\rho(x))\rho(\alpha)\rho(y) \oplus \rho(x)\rho(\alpha)\overline{d}(\rho(y)).$$

Note that if d is a weak derivation, then $\overline{d}$ is a weak derivation, too.

***Example* 2.8.** Let $\{A_n \mid n \in \mathbb{R}\}$ be a family of disjoint set such that

$$A_n = \begin{cases} \{0\} & \text{if} \quad n = 0, \\ (0,1) & \text{if} \quad 0 < n < 1, \\ [m, m+1) & \text{if} \quad m \leq n < m+1. \end{cases}$$

Then for every $x \in \mathbb{R}$ there exists $n \in \mathbb{R}$ such that $x \in A_n$. So, $\mathbb{R}$ is a $\mathbb{R}$-semihyperring with the following hyperoperation: $x \oplus y = A_{n+m}$ and $x\alpha y = A_{n\alpha m}$, for all $x \in A_n$, $y \in A_m$ and $\alpha \in \mathbb{R}$ [9]. The function $d(x) = 0$, for all $x \in \mathbb{R}$, is a derivation.

The following example shows that the identity function is not always derivation.

***Example* 2.9.** Let $R = \{A_n \mid n \in \mathbb{Z}\}$ be a family of disjoint set such that $A_n = [n, n+1)$. Then for every $x \in R$ there exists $n \in \mathbb{N}$ such that $x \in A_n$. So, R is a $\mathbb{Z}$-semihyperring with the hyperoperations $x \oplus y = A_{n+m}$ and $x\alpha y = A_{n\alpha m}$, for all $x \in A_n$, $y \in A_m$ and $\alpha \in \mathbb{Z}$ [8]. The function $d(x) = x$, for all $x \in R$, is not a derivation because for all $\alpha \in \mathbb{Z}$, $x \in A_n$ and $y \in A_m$, where $n, m \in \mathbb{N}$, we have $d(x\alpha y) = d(A_{n\alpha m}) = A_{n\alpha m} = [n\alpha m, n\alpha m + 1)$ but $d(x)\alpha y \oplus x\alpha d(y) = A_{n\alpha m} \oplus A_{n\alpha m} = A_{2(n\alpha m)} = [2(n\alpha m), 2(n\alpha m) + 1)$. For example, suppose that $x = 1$, $y = 3$ and $\alpha = 2$. Then $x \in A_1$ and $y \in A_3$. Consequently, we find that $d(x\alpha y) = A_6 = [6, 7)$ but $d(x)\alpha y \oplus x\alpha d(y) = A_6 + A_6 = A_{12} = [12, 13)$. Note that d is not a weak derivation, too.

Lemma 2.1. *Let $(R, +, \Gamma)$ be a Γ-semihyperring (Γ-hyperring) and d be a weak derivation on R. Then, for all $n \in \mathbb{N}$, $x, y \in R$ and $\alpha \in \Gamma$, we have*:

(1) *If R is commutative, then* $d(\underbrace{x\alpha x\alpha \ldots \alpha x}_{\text{the number of } x = n}) \subseteq n\big(x^{n-1}\alpha d(x)\big)$. *The equality holds when d is a derivation.*

(2) $d^n(x\alpha y) \subseteq \sum_{i=0}^{n} \binom{n}{i} \big(d^{n-i}(x)\alpha d^i(y)\big)$, *where d^n shows derivation of order n and $d^0(x) = x$, for all $x \in R$. The equality holds when d is derivation.*

Proof. The proof follows easily by induction.

Let $(R, +, \Gamma)$ be a Γ-semihyperring with zero (Γ-hyperring). Then R is said to be of *characteristic n*, if n is the smallest positive integer such that $0 \in nx = \underbrace{x + \ldots + x}_{n}$, for all $x \in R$. If no such of n exists, R is said to be of characteristic 0.

***Example* 2.10.** Γ-semihyperring R in Examples 2.3 and 2.4 are of characteristic 4 and 6, respectively.

Let R be a commutative Γ-semihyperring with zero (Γ-hyperring) and d be a derivation of R. If R is of characteristic n, then by the above lemma, $0 \in d(\underbrace{x\alpha x\alpha \ldots \alpha x}_{\text{the number of } x = n})$, for all $x \in R$.

Definition 2.2. *A Γ-semihyperring R with zero (Γ-hyperring) is called prime if $0 \in x\alpha r\beta y$, for all $r \in R$ and $\alpha, \beta \in \Gamma$, implies that either $x = 0$ or $y = 0$. R is called semiprime if $0 \in x\alpha r\beta x$, for all $r \in R$ and $\alpha, \beta \in R$, implies that $x = 0$. Obviously, every prime Γ-semihyperring (prime Γ-hyperring, respectively) is a semiprime Γ-semihyperring (semiprime Γ-hyperring, respectively) but the converse is not always true.*

***Example* 2.11.** In Example 2.2, $(R, +, \Gamma)$ is (semi)prime if and only if $(R, +, \circ)$ is (semi)prime. In Example 2.3, $(R, +, \Gamma)$ is not semiprime because $0 \in 3\alpha r\beta 3$, for all $r \in R$ and $\alpha, \beta \in \Gamma$, but $3 \neq 0$. So, $(R, +, \Gamma)$ is not prime. Similarly, one can show that in Example 2.4, $(R, +, \Gamma)$ is not semiprime and prime.

Lemma 2.2. *Let R be a prime Γ-semihyperring with zero* (Γ*-hyperring*), *I be a non-zero ideal of R and $x, y \in R$. Then we have*:

(1) *If $I\alpha x = 0$, for all $\alpha \in \Gamma$* (*or $x\alpha I = 0$, for all $\alpha \in \Gamma$*), *then $x = 0$.*

(2) *If $0 = x\alpha I\beta y$, for all $\alpha, \beta \in \Gamma$, then $x = 0$ or $y = 0$.*

(3) *If $x \in Z$ and $0 \in x\alpha y$, for all $\alpha \in \Gamma$, then $x = 0$ or $y = 0$.*

Proof. (1) Suppose that $I\alpha x = 0$, for all $\alpha \in \Gamma$. Then $u\beta r\alpha x \subseteq I\alpha x = \{0\}$, for all $r \in R$, $u \in I$ and $\beta \in \Gamma$. Hence, $x = 0$, since R is prime and $I \neq 0$. In the case $x\alpha I = 0$, the proof is similar.

(2) Assume that $x\alpha I\beta y = 0$, for all $\alpha, \beta \in \Gamma$. Then $x\alpha I\gamma r\beta y \subseteq x\alpha I\beta y = \{0\}$, for all $r \in R$ and $\gamma \in \Gamma$. This implies that $x\alpha I\gamma r\beta y = 0$, for all $r \in R$ and $\alpha, \beta, \gamma \in \Gamma$. Hence, $x\alpha I = 0$ or $y = 0$, since R is prime. So, by part (1), $x = 0$ or $y = 0$.

(3) Suppose that $x \in Z$ and $0 \in x\alpha y$, for all $\alpha \in \Gamma$. Then, for all $r \in R$ and $\beta \in \Gamma$, we get $0 \in r\beta 0 = r\beta x\alpha y = x\beta r\alpha y$. Therefore, $x = 0$ or $y = 0$, since R is prime.

If R is a Γ-hyperring, the proof is similar.

Theorem 2.1. *Let $(R, +, \Gamma)$ be a prime Γ-semihyperring with zero* (Γ*-hyperring*), *d be a derivation on R and I be a non-zero ideal of R. Then, for all $x \in R$ and $\alpha \in \Gamma$, we have*:

(1) *If $d(I) = 0$, then $d = 0$.*

(2) *If $d(I)\alpha x = 0$, for all $\alpha \in \Gamma$* (*or $x\alpha d(I) = 0$, for all $\alpha \in \Gamma$*), *then $x = 0$ or $d = 0$.*

(3) *If $d(R)\alpha x = 0$, for all $\alpha \in \Gamma$* (*or $x\alpha d(R) = 0$, for all $\alpha \in \Gamma$*), *then $x = 0$ or $d = 0$.*

Proof. Suppose that $(R, +, \Gamma)$ is a Γ-semihyperring with zero.

(1) For all $u \in I$, $x \in R$ and $\alpha \in \Gamma$, we have $0 = d(u\alpha x) = d(u)\alpha x + u\alpha d(x) \supseteq 0 + u\alpha d(x) \supseteq$ $\supseteq u\alpha d(x)$. Therefore, $u\alpha d(x) = 0$, for all $\alpha \in \Gamma$. Consequently, $I\alpha d(x) = 0$ which implies that $d = 0$, by Lemma 2.2 (1).

(2) Suppose that $d(I)\alpha x = 0$, for all $\alpha \in \Gamma$. Then $0 = d(y\beta u)\alpha x = d(y)\beta u\alpha x + y\beta d(u)\alpha x \supseteq$ $\supseteq d(y)\beta u\alpha x$, for all $u \in I$, $y \in R$ and $\beta \in \Gamma$. Thus, $d(y)\beta u\alpha x = 0$, for all $\alpha, \beta \in \Gamma$. Therefore, $d(y)\beta I\alpha x = 0$ which implies that $d = 0$ or $x = 0$, by Lemma 2.2 (2). In the case $x\alpha d(I) = 0$, the proof is similar.

(3) In (2), put R instead of I.

If R is a Γ-hyperring, the proof is similar.

Definition 2.3. *Let R be a Γ-semihyperring with zero* (Γ*-hyperring*). *Then R is called n-torsion free if $0 \in nx = \underbrace{x + \ldots + x}_{n}$, $x \in R$, implies that $x = 0$, where n is a positive integer number.*

***Example* 2.12.** In Example 2.3, $(R, +, \Gamma)$ is 3-torsion free but R is not a 2-torsion free Γ-semihyperring, since $2 + 2 = 0$ but $2 \neq 0$. In Example 2.4, $(R, +, \Gamma)$ is not 2-torsion, 3-torsion and 4-torsion free but it is 5-torsion.

Theorem 2.2. *Let $(R, +, \Gamma)$ be a prime 2-torsion free Γ-semihyperring with zero* (Γ*-hyperring*) *and I be a non-zero ideal of R. Then we have*:

(1) *If d is a derivation of R such that $d^{(2)}(I) = 0$, then $d = 0$.*

(2) *If d_1 and d_2 are derivations of R such that $d_1 d_2(I) = 0$ and $d_2(I) \subseteq I$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. (1) By Lemma 2.1 for all $u, v \in I$ and $\alpha \in \Gamma$, we have

$$0 = d^{(2)}(u\alpha v) = d^{(2)}(u)\alpha v + 2d(u)\alpha d(v) + u\alpha d^{(2)}(v) \supseteq 2d(u)\alpha d(v).$$

Hence, $d(u)\alpha d(v) = 0$, since R is 2-torsion free Γ-semihyperring. Therefore, $d = 0$, by Theorem 2.1 (1) and (2).

(2) For all $u, v \in I$ and $\alpha \in \Gamma$, we get

$$0 = d_1 d_2(u\alpha v) = d_1\big(d_2(u)\alpha v + u\alpha d_2(v)\big) =$$

$$= d_1 d_2(u)\alpha v + d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v) + u\alpha d_1 d_2(v) \supseteq$$

$$\supseteq d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v).$$

Consequently, $d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v) = 0$. By replacing u by $d_2(u)$ in the above equation, we find that $d_2^{(2)}(u)\alpha d_1(v) \subseteq d_2^{(2)}(u)\alpha d_1(v) + d_1 d_2(u)\alpha d_2(v) = 0$, that is $d_2^{(2)}(u)\alpha d_1(v) = 0$. Thus, $d_1 = 0$ or $d_2^{(2)}(I) = 0$, by Theorem 2.1 (1) and (2). Now, Part (1) implies that $d_1 = 0$ or $d_2 = 0$.

If R is a Γ-hyperring, the proof is similar.

Theorem 2.3. *Let R be a prime 2-torsion free Γ-hyperring, I be a non-zero right ideal of R and d be a non-zero derivation of R. Then the Γ-subhyperring of R generated by $d(I)$ contains no non-zero right ideal of R if and only if $d(I)\Gamma I = 0$.*

Proof. Suppose that A is a Γ-subsemihyperring of R generated by $d(I)$ i.e., $A = \big\langle d(I)\big\rangle$ and set $S = A \cap I$. For every $u \in I$, $s \in S$ and $\gamma \in \Gamma$, we have $d(s)\gamma u + s\gamma d(u) = d(s\gamma u) \subseteq A$. Consequently, $d(s)\gamma u \subseteq A,\ I$ which implies that $d(s)\gamma u \subseteq S$. It is clear that for every $s \in S$ and $\gamma \in \Gamma$, $d(s)\gamma I$ is a right ideal of R. So, $d(s)\gamma I = \big\langle 0\big\rangle = 0$, for all $s \in S$ and $\gamma \in \Gamma$, by hypothesis. In other hands, $d(u\beta a) = d(u)\beta a + u\beta d(a) \subseteq A$, for all $u \in I$, $a \in A$, $\beta \in \Gamma$. Thus, $u\beta d(a) \subseteq A$. This implies that $u\beta d(a) \subseteq S$. Therefore, $d\big(u\beta d(a)\big)\gamma v = 0$, for all $v \in I$. By Lemma 2.2, $0 = d\big(u\beta d(a)\big) = d(u)\beta d(a) + u\beta d^2(a)$, for all $u \in I$, $a \in A$, $\beta \in \Gamma$. Replace u by $u\gamma v$, where $v \in I$ and $\gamma \in \Gamma$. Then we get $d(u)\gamma v\beta d(a) = 0$ which means that $d(u)\gamma I\beta d(a) = 0$. This implies that $d(u)\gamma I\alpha R\beta d(a)\eta u = 0$, for all $\gamma, \alpha, \beta, \eta \in \Gamma$. Hence, $d(u)\alpha I = 0$ or $d(a)\eta u = 0$, since R is prime. Set $L = \big\{u \in I \mid d(u)\alpha I = 0, \text{ for all } \alpha \in \Gamma\big\}$ and $K = \big\{u \in I \mid d(a)\eta u = 0, \text{ for all } a \in A \text{ and } \eta \in \Gamma\big\}$. It is clear that L and K are canonical subgroups of I and $I = L \cup K$. But, I can not be union of proper canonical subgroups of itself. So, $I = L$ or $I = K$. If $I = K$, then we have $d(A)\Gamma I = 0$. Hence, $d^2(I)\Gamma I = 0$, since $A = \big\langle d(I)\big\rangle$. By Lemma 2.1, for all $u, v, w \in I$ and $\beta, \gamma \in \Gamma$, we obtain $0 = d^2(u\beta v)\gamma w = 2d(u)\beta d(v)\gamma w$. This implies that $d(u)\beta d(v)\gamma w = 0$, because R is 2-torsion free. Replacing u by $u\alpha x$, where $x \in I$ and $\alpha \in \Gamma$, we get $d(u)\alpha x\beta d(v)\gamma w = 0$. So, $d(u)\alpha x\eta R\beta d(v)\gamma w = 0$, for all $\alpha, \eta, \beta, \gamma \in \Gamma$. Hence $d(u)\alpha v = 0$, for all $u, v \in I$ and $\alpha \in \Gamma$, since R is prime. Therefore, in both of cases we have $d(I)\Gamma I = 0$.

Conversely, suppose that $d(I)\Gamma I = 0$. Then $A\Gamma I = 0$. Hence, A contains no non-zero right ideal, since R is prime.

A proper ideal P of Γ-hyperring R is called *prime* if for every $x, y \in R$, $x\Gamma R\Gamma y \subseteq P$ implies that $x \in P$ or $y \in P$. This condition is equivalent to $I\Gamma J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, where I and J are two ideals of R [5]. A Γ-hyperring with Δ (D, respectively) is called a *differential* Γ*-hyperring* (*weak differential* Γ*-hyperring*, respectively). An ideal I of (weak) differential Γ-hyperring R is called (*weak*) *differential ideal* if for all (weak) derivation d of R, we have $d(u) \in I$, for all $u \in I$. For every differential Γ-hyperring R, $\big\langle 0\big\rangle_R$ is a differential ideal.

Theorem 2.4. *Let R be a differential Γ-hyperring and P is a prime differential ideal of R. Then $J = \{a \in R \mid R\Gamma^{\Sigma}a \subseteq P\}$ is the largest differential ideal of R that is contained in P.*

Proof. It is easy to check that J is the largest ideal of R that is contained in P. We prove that J is differentiable. If $a \in J$, then $R\Gamma^{\Sigma}a \subseteq P$, i.e., $A = \Big\{t \,\Big|\, t \in \sum_{i=1}^{n} r_i\alpha_i a,\ r_i \in R, \alpha_i \in \Gamma, n \in \mathbb{N}\Big\} \subseteq P$. Therefore, for every $\sum_{i=1}^{n} r_i\alpha_i a \subseteq A$, we have $d\Big(\sum_{i=1}^{n} r_i\alpha_i a\Big) \subseteq d(P) \subseteq P$. Thus, $\sum_{i=1}^{n} \big(d(r_i)\alpha_i a + r_i\alpha_i d(a)\big) \subseteq P$ which leads to $\sum_{i=1}^{n} d(r_i)\alpha_i a + \sum_{i=1}^{n} r_i\alpha_i d(a) \subseteq P$. Then $\sum_{i=1}^{n} r_i\alpha_i d(a) \subseteq P$ which implies that $R\Gamma^{\Sigma}d(a) = \Big\{t \,\Big|\, t \in \sum_{i=1}^{n} r_i\alpha_i d(a),\ r_i \in R, \alpha_i \in \Gamma, n \in \mathbb{N}\Big\} \subseteq P$. Consequently, $d(a) \in J$.

References

1. *Anvariyeh S. M., Mirvakili S., Davvaz B.* On Γ-hyperideals in Γ-semihypergroups // Carpathian J. Math. – 2010. – **26**. – P. 11 – 23.
2. *Barnes W. E.* On the Γ-rings of Nobusawa // Pacif. J. Math. – 1966. – **18**. – P. 411 – 422.
3. *Corsini P.* Prolegomena of hypergroup theory. – Aviani Editor, Italy, 1993.
4. *Corsini P., Leoreanu-Fotea V.* Applications of hyperstructure theory // Adv. Math. – 2003. – **5**.
5. *Davvaz B., Leoreanu-Fotea V.* Hyperring theory and applications. – Int. Acad. Press, USA, 2007.
6. *Davvaz B., Zareyan L., Leoreanu-Fotea V.* $(3,3)$-Ary differential rings // Mediterr. J. Math. – 2012. – **9**. – P. 357 – 378.
7. *Dehkordi S. O., Davvaz B.* A strong regular relation on Γ-semihyperrings // J. Sci. Islam. Repub. Iran. – 2011. – **22**. – P. 257 – 266.
8. *Dehkordi S. O., Davvaz B.* Γ-semihyperrings: approximations and rough ideals // Bull. Malays. Math. Sci. Soc. – 2012. – **35**. – P. 1035 – 1047.
9. *Dehkordi S. O., Davvaz B.* Ideal theory in Γ-semihyperrings // Iran. J. Sci. Technol. Trans. A. Sci. – 2013. – **37**. – P. 251 – 263.
10. *Dehkordi S. O., Davvaz B.* Γ-semihyperrings: ideals, homomorphisms and regular relations // Afr. Mat. – 2015. – **26**. – P. 849 – 861.
11. *Hedayati H., Davvaz B.* Fundamental relation on Γ-hyperrings // Ars Combin. – 2011. – **100**. – P. 381 – 394.
12. *Heidari D., Dehkordi S. O., Davvaz B.* Γ-semihypergroups and their properties // Politehn. Univ. Bucharest Sci. Bull. Ser. A. Appl. Math. and Phys. – 2010. – **72**. – P. 195 – 208.
13. *Huang S., Davvaz B.* Generalized derivations of rings and Banach algebras // Communs Algebra. – 2013. – **41**. – P. 1188 – 1194.
14. *Jing F. J.* On derivations of Γ-rings // J. Qufu Norm. Univ. Nat. Sci. – 1987. – **13**. – P. 159 – 161.
15. *Kyuno S.* On prime Gamma rings // Pacif. J. Math. – 1978. – **75**. – P. 185 – 190.
16. *Kyuno S.* Prime ideals in Gamma rings // Pacif. J. Math. – 1982. – **98**. – P. 375 – 379.
17. *Posner E.* Derivations in prime rings // Proc. Amer. Math. Soc. – 1975. – **8**. – P. 1093 – 1100.
18. *Marty F.* Sur une generalization de la notion de group // 8th Congr. Math. Scandenaves, Stockholm. – 1934. – P. 45 – 49.
19. *Nobusawa N.* On a generalization of the ring theory // Osaka J. Math. – 1964. – **1**. – P. 81 – 89.
20. *Ozturk M. A., Jun Y. B., Kim K. H.* On derivations of prime Gamma rings // Tr. Inst. Mat. – 2002. – **26**. – P. 317 – 327.
21. *Rao M. K.* Γ-semiring // Southeast Asian Bull. Math. – 1995. – **19**. – P. 49 – 54.
22. *Sapanci M., Nakajima A.* Jordan derivations on completely prime gamma rings // Math. Japonica. – 1997. – **46**. – P. 47 – 51.
23. *Soyturk M.* The commutativity in prime gamma rings with derivation // Tr. Inst. Mat. – 1994. – **18**. – P. 149 – 155.
24. *Vougiouklis T.* Hyperstructures and their representations. – Florida: Hadronic Press, 1994.

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