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THE STRUCTURE OF FRACTIONAL SPACES GENERATED BY THE TWO-DIMENSIONAL DIFFERENCE OPERATOR ON THE HALF PLANE*

СТРУКТУРА ДРОБОВИХ ПРОСТОРІВ, ПОРОДЖЕНИХ ДВОВИМІРНИМ ДИФЕРЕНЦІАЛЬНИМ ОПЕРАТОРОМ НА ПІВПЛОЩИНІ

We consider a difference operator approximation A_h^x of the differential operator $A^x u(x) = -a_{11}(x)u_{x_1x_1}(x) - a_{22}(x)u_{x_2x_2}(x) + \sigma u(x)$, $x = (x_1, x_2)$ defined in the region $\mathbb{R}^+ \times \mathbb{R}$ with the boundary condition $u(0, x_2) = 0$, $x_2 \in \mathbb{R}$. Here, the coefficients $a_{ii}(x)$, $i = 1, 2$, are continuously differentiable, satisfy the uniform ellipticity condition $a_{11}^2(x) + a_{22}^2(x) \geq \delta > 0$, and $\sigma > 0$. We investigate the structure of the fractional spaces generated by the analyzed difference operator. Theorems on well-posedness in a Hölder space of difference elliptic problems are obtained as applications.

Розглянуто апроксимацію різницевими операторами A_h^x диференціального оператора $A^x u(x) = -a_{11}(x)u_{x_1x_1}(x) - a_{22}(x)u_{x_2x_2}(x) + \sigma u(x)$, $x = (x_1, x_2)$, що визначений у області $\mathbb{R}^+ \times \mathbb{R}$, з граничною умовою $u(0, x_2) = 0$, $x_2 \in \mathbb{R}$. У даному випадку коефіцієнти $a_{ii}(x)$, $i = 1, 2$, є неперервно диференційовними та задовольняють рівномірну умову еліптичності $a_{11}^2(x) + a_{22}^2(x) \geq \delta > 0$ і, крім того, $\sigma > 0$. Теореми про коректність різницевих еліптичних задач у просторах Гьольдера одержані як застосування.

1. Introduction. The importance of the positivity property of the differential operators in a Banach space in the study of various properties for partial differential equations is well-known (see, for example, [1 – 8] and the references therein). The positivity of wider class of differential and difference operators in Banach spaces have been investigated by many scientists (see [9 – 14] and the references therein).

The structure of fractional spaces generated by positive multidimensional differential and difference operators on space $\mathbb{R}^n$ and one dimensional differential and difference operators in Banach spaces has been well investigated (see [16 – 21] and the references therein). Note that the structure of fractional spaces generated by positive multidimensional differential and difference operators with local and nonlocal conditions on $\Omega \subset \mathbb{R}^n$ in Banach spaces $C(\Omega)$ has not been well studied.

It is well-known that (see, for example, [16]) the operator A is said to be positive in E if its spectrum $\sigma(A)$ lies inside of the sector S of the angle ϕ, $0 < 2\phi < 2\pi$, symmetric with respect to the real axis, and the estimate

$$\left\|(A - \lambda)^{-1}\right\|_{E \to E} \leq \frac{M(\phi)}{1 + |\lambda|}$$

holds on the edges $S_1(\phi) = \left\{\rho e^{i\phi} : 0 \leq \rho < \infty\right\}$, $S_2(\phi) = \left\{\rho e^{-i\phi} : 0 \leq \rho < \infty\right\}$ of S, and outside of the sector S. The infimum of all such angles ϕ is called the spectral angle of the positive operator A and is denoted by $\phi(A, E)$.

Throughout the article, M indicates positive constants which may differ from time to time and we are not interested to precise. If the constant depends only on $\alpha, \beta, \ldots$, then we will write $M(\alpha, \beta, \ldots)$.

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With the help of the positive operator A we introduce the fractional space $E_\beta = E_\beta(E,A)$, $0<\beta<1$, consisting of all elements $v\in E$ for which the norm

$$\|v\|_{E_\beta} = \sup_{\lambda>0} \lambda^\beta \left\|A(\lambda+A)^{-1}v\right\|_E + \|v\|_E$$

is finite.

In the paper [21] the structure of fractional spaces generated by the two-dimensional differential operator A^x defined by

$$A^x u(x) = -a_{11}(x)u_{x_1x_1}(x) - a_{22}(x)u_{x_2x_2}(x) + \sigma u(x) \tag{1}$$

over the region $\mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}$ with the boundary condition $u(0,x_2)=0,\ x_2\in\mathbb{R}$. Here, the coefficients $a_{ii}(t,x),\ i=1,2$, are continuously differentiable and satisfy the uniform ellipticity

$$a^2_{11}(x) + a^2_{22}(x) \geq \delta > 0 \tag{2}$$

and $\sigma>0$.

In the present paper, we will study the structure of fractional spaces generated by the two-dimensional difference operator A^x_h defined by formula

$$A^x_h u^h(x) = -b_{11}(x)\Delta^{x_1}_{1+}\Delta^{x_1}_{1-}u^h(x)h^{-2} - b_{22}(x)\Delta^{x_2}_{1+}\Delta^{x_2}_{1-}u^h(x)h^{-2} + \sigma u^h(x), \quad x=(x_1,x_2), \tag{3}$$

approximates of the differential operator A^x defined by (1). Here

$$\Delta^{x_1}_{1\pm}u(x) = \pm\big(u^h(x_1\pm h, x_2) - u^h(x_1,x_2)\big),$$

$$x_1, x_1+h\in\mathbb{R}^+_h, \quad x_2\in\mathbb{R}_h, \qquad \Delta^{x_2}_{1\pm}u(x) = \pm\big(u^h(x_1, x_2\pm h) - u^h(x_1,x_2)\big),$$

$$x_1\in\mathbb{R}^+_h, \quad x_2, x_2+h\in\mathbb{R}_h, \qquad \mathbb{R}_h = \big\{(x_2)_k = kh,\ k=0,\pm1,\pm2,\ldots\big\},$$

$$\mathbb{R}^+_h = \big\{(x_1)_m = mh,\ m=0,1,2,\ldots\big\}.$$

Assume that $b_{ii}(x)$ satisfy the uniform ellipticity condition

$$b^2_{11}(x) + b^2_{22}(x) \geq \delta > 0.$$

Next, to formulate our result we need to introduce the Hölder space $C^{2\alpha}_h = C^{2\alpha}\big(\mathbb{R}^2_{h,+}\big)$ of all bounded grid functions f^h defined on $\mathbb{R}^2_{h,+} = \mathbb{R}^+_h\times\mathbb{R}_h$ satisfying a Hölder condition with the indicator $\alpha\in(0,1/2)$ with the norm

$$\left\|f^h\right\|_{C^{2\alpha}_h} = \left\|f^h\right\|_{C\left(\mathbb{R}^2_{h,+}\right)} + \sup_{\substack{(x_1,x_2),(x'_1,x'_2)\in\mathbb{R}^2_{h,+}\\ (x_1,x_2)\neq(x'_1,x'_2)}} \frac{\left|f^h(x_1,x_2) - f^h(x'_1,x'_2)\right|}{\left(\sqrt{|x_1-x'_1|^2 + |x_2-x'_2|^2}\right)^{2\alpha}}.$$

Here $C_h = C\big(\mathbb{R}^2_{h,+}\big)$ denotes the Banach space of all bounded grid functions f^h defined on $\mathbb{R}^2_{h,+}$ with the norm

$$\left\|f^h\right\|_{C_h} = \sup_{(x_1,x_2)\in\mathbb{R}^2_{h,+}} \left|f^h(x_1,x_2)\right|.$$

Our goal in this paper is to study the structure of the fractional spaces $E_\alpha\big(A^x_h, C\big(\mathbb{R}^2_{h,+}\big)\big)$. Namelly, the following main theorem is proved.

Theorem 1. *Suppose* $0 < \alpha < 1/2$. *Then the norms of the spaces* $E_\alpha\big(A_h^x, C\big(\mathbb{R}^2_{h,+}\big)\big)$ *and* $C^{2\alpha}\big(\mathbb{R}^2_{h,+}\big)$ *are equivalent uniformly in* h, $0 < h \le h_0$.

The paper is organized as follows. In Section 2, auxiliary results are given. In Section 3, the proof of main Theorem 1 is presented. In Section 4, theorems on well-posedness in a Hölder space of difference elliptic problems are established. Finally, Section 5 is conclusion.

2. Auxiliary results. In this section, we give some auxiliary lemmas which will be useful in the sequel.

Lemma 1. *For all* $n, m \in Z^+$ *and* $h_1 > 0$, $b > 0$ *the following estimate holds*:

$$e^{-b(m+n)h_1}\Big(1+\ln\Big\{1+((m+n)h_1+h_1)^{-1}\Big\}\Big)(m+n)^{2\alpha}h_1^{2\alpha} \le M(\alpha), \quad n, m \in Z^+. \tag{4}$$

Lemma 2 [15]. *Let* p *and* q *be mutually conjugate exponents, that is,* $\dfrac{1}{p}+\dfrac{1}{q}=1$, $p>1$, *and let* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be any two sequences of nonnegative real numbers such that* $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. *Then, the following Hilbert's inequality holds*:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\csc\left(\frac{\pi}{p}\right)\left[\sum_{m=1}^{\infty} a_m^p\right]^{1/p}\left[\sum_{n=1}^{\infty} b_n^q\right]^{1/q}.$$

S. I. Danelich in [12] considered the positivity of a difference analog A_h^x of the $2m$th order multidimensional elliptic operator A^x with dependent coefficients on semispaces $\mathbb{R}^+ \times \mathbb{R}^{n-1}$. Following the paper [12], in the special case $m=1$ and $n=2$, we consider the problem of finding the resolvent equation of the operator $-A^x$

$$A_h^x u^h(x) + \lambda u^h(x) = f^h(x), \quad x \in \mathbb{R}^2_{h,+}. \tag{5}$$

Let $G_h(x_1, x_2, p, s, \lambda)$ be a Green function of the resolvent equation (5) for the difference operator A_h^x. There exists the inverse operator $(A_h^x+\lambda)^{-1}$ for all $\lambda \ge 0$ and the formula

$$(A_h^x+\lambda)^{-1} f^h(x_1,x_2) = \sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h} G_h\left(x_1,x_2,p,s,\lambda\right) f^h(p,s)h^2 \tag{6}$$

holds. Moreover, we have the following lemma.

Lemma 3 [12]. *The estimates for* $0 \le \lambda \le Lh^{-2}$ $(L>0)$

$$\begin{aligned}\big|G_h(x_1,x_2,p,s,\lambda)\big| &\le \\ \le C\exp\Big\{-a(\lambda+1)^{1/2}\big(|x_1-p|+|x_2-s|\big)\Big\} &\times \\ \times\left(1+\ln\left\{1+\Big((\lambda+1)^{1/2}\big(|x_1-p|+|x_2-s|\big)\Big)^{-1}\right\}\right)&,\end{aligned} \tag{7}$$

$$\begin{aligned}\big|G_{h_{x_1}}(x_1,x_2,p,s,\lambda)\big|,\ \big|G_{h_{x_2}}(x_1,x_2,p,s,\lambda)\big| &\le \\ \le C\exp\Big\{-a(\lambda+1)^{1/2}\big(|x_1-p|+|x_2-s|\big)\Big\}\big(|x_1-p|+|x_2-s|+h\big)^{-1}&,\end{aligned} \tag{8}$$

and for $\lambda \ge Lh^{-2}$ $(L>0)$

$$|G_h(x_1, x_2, p, s, \lambda)| \leq C \exp\left\{-ah^{-1}\left(|x_1 - p| + |x_2 - s|\right)\right\} \left| \left((\lambda + 1)h^2 + 1\right)^{-1} \right|, \qquad (9)$$

$$\left|G_{h_{x_1}}(x_1, x_2, p, s, \lambda)\right|, \left|G_{h_{x_2}}(x_1, x_2, p, s, \lambda)\right| \leq$$

$$\leq C \exp\left\{-ah^{-1}\bigl(|x_1 - p| + |x_2 - s|\bigr)\right\} \bigl((\lambda + 1)h^2 + 1\bigr)^{-1} h^{-1} \qquad (10)$$

hold. Here $a = a(\sigma)$.

Lemma 4 [12]. *Let* $\lambda \geq Lh^{-2}$, *where* $L > 0$ *is large enough. Then*

$$|G_h(x_1, x_2, p, s, \lambda)| \leq$$

$$\leq Ch \exp\left\{-ah^{-1}\left(|x_1 - p| + |x_2 - s|\right)\right\} \left((\lambda + 1) h^2\right)^{-2} \left(|x_1 - p| + |x_2 - s|\right)^{-1}. \qquad (11)$$

From (9) and (11) we obtain the estimate $(0 < \alpha < 1/2)$

$$|G_h(x_1, x_2, p, s, \lambda)| \leq$$

$$\leq Ch^{2\alpha} \exp\left\{-ah^{-1}\bigl(|x_1 - p| + |x_2 - s|\bigr)\right\} \bigl((\lambda + 1)h^2\bigr)^{-2} \left(|x_1 - p| + |x_2 - s|\right)^{-2\alpha}. \qquad (12)$$

Clearly, from estimates (7)–(10) it follows that A_h^x is a positive operator in $C\bigl(\mathbb{R}^2_{h,+}\bigr)$. Moreover, from that and the commutativity of A_h^x and its resolvent $\bigl(A_h^x + \lambda\bigr)^{-1}$ it follows that A_h^x is a positive operator in $E_\alpha\bigl(A, C\bigl(R^2_{h,+}\bigr)\bigr)$.

3. Proof of Theorem 1. In this section, we will prove the main Theorem 1 on structure of fractional spaces $E_\alpha\bigl(A_h^x, C\bigl(\mathbb{R}^2_{h,+}\bigr)\bigr)$. Namely, let $(x_1, x_2) \in \mathbb{R}^2_{h,+}$ and $\lambda > 0$ be fixed. From formula (6) it follows that

$$A_h^x\bigl(\lambda + A_h^x\bigr)^{-1} f^h(x_1, x_2) =$$

$$= \frac{\lambda^\alpha}{1+\lambda} f^h(x_1, x_2) + \lambda^{\alpha+1} \sum_{p \in \mathbb{R}_h^+} \sum_{s \in \mathbb{R}_h} G^h(x_1, x_2, p, s, \lambda)\bigl(f^h(x_1, x_2) - f^h(p, s)\bigr)h^2. \qquad (13)$$

By using equation (13), the triangle inequality, and the definition of $C_h^{2\alpha}$-norm, we obtain

$$\left|\lambda^\alpha A_h^x\bigl(\lambda + A_h^x\bigr)^{-1} f^h(x_1, x_2)\right| \leq$$

$$\leq \frac{\lambda^\alpha}{1+\lambda} \left|f^h(x_1, x_2)\right| + \lambda^{\alpha+1} \sum_{p \in \mathbb{R}_h^+} \sum_{s \in \mathbb{R}_h} \left|G^h\left(x_1, x_2, p, s, \lambda\right)\right| \left|f^h(x_1, x_2) - f^h(p, s)\right| h^2 \leq$$

$$\leq \left\|f^h\right\|_{C_h^{2\alpha}} \left[\frac{\lambda^\alpha}{1+\lambda} + \lambda^{\alpha+1} \sum_{p \in \mathbb{R}_h^+} \sum_{s \in \mathbb{R}_h} \left|G^h\bigl(x_1, x_2, p, s, \lambda\bigr)\right| \left(|x_1 - p|^2 + |x_2 - s|^2\right)^\alpha h^2\right] \leq$$

$$\leq \left\|f^h\right\|_{C_h^{2\alpha}} \bigl[I_1 + I_2\bigr].$$

Here

$$I_1 = \frac{\lambda^\alpha}{1+\lambda}, \qquad I_2 = \lambda^{\alpha+1} \sum_{p \in \mathbb{R}_h^+} \sum_{s \in \mathbb{R}_h} \left|G^h(x_1, x_2, p, s, \lambda)\right| \bigl(|x_1 - p|^2 + |x_2 - s|^2\bigr)^\alpha h^2.$$

It is clear that for any $\lambda > 0$

$$I_1 \leq 1. \tag{14}$$

Therefore, we will estimate I_2. Consider two cases: $|\lambda| \leq Lh^{-2}$ and $|\lambda| \geq Lh^{-2}$, respectively. First, assume that $|\lambda| \leq Lh^{-2}$. Applying estimate (7), we have

$$I_2 \leq \lambda^{\alpha+1} M \sum_{p\in\mathbb{R}_h^+} \sum_{s\in\mathbb{R}_h} e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)} \times$$

$$\times \left(1 + \ln\left\{1 + \left((1+\lambda)^{1/2}\big(|x_1-p| + |x_2-s| + h\big)\right)^{-1}\right\}\right)\left(|x_1-p|^2 + |x_2-s|^2\right)^{\alpha} h^2.$$

The change of variables $x_1 - p = mh$ and $x_2 - s = nh$ yields

$$I_2 \leq \lambda^{\alpha+1} M \sum_{m\in Z^+} \sum_{n\in Z^+} e^{-a(1+\lambda)^{1/2}(m+n)h} \times$$

$$\times \left(1 + \ln\left\{1 + \left((1+\lambda)^{1/2}((m+n)h + h)\right)^{-1}\right\}\right)(m+n)^{2\alpha} h^{2\alpha} h^2.$$

Letting $(1+\lambda)^{1/2} h = h_1$, we get

$$I_2 \leq \frac{\lambda^{\alpha+1}}{(1+\lambda)^{\alpha+1}} M \sum_{m\in Z^+} \sum_{n\in Z^+} e^{-a(m+n)h_1} \times$$

$$\times \left(1 + \ln\left\{1 + ((m+n)h_1 + h_1)^{-1}\right\}\right)(m+n)^{2\alpha} h_1^{2\alpha} h_1^2.$$

Applying estimate (4) for $b = \dfrac{a}{2}$, we obtain

$$I_2 \leq M_1 \sum_{m\in Z^+} \sum_{n\in Z^+} e^{-\frac{a}{2}(m+n)h_1} h_1^2 \leq M_2 \int\limits_0^\infty \int\limits_0^\infty e^{-\frac{a}{2}(q+y)}\, dq\, dy = M_3. \tag{15}$$

Then from estimates (14), (15) it follows

$$\sup_{(x_1,x_2)\in\mathbb{R}_{h,+}^2} \left|\lambda^{\alpha} A_h^x \left(A_h^x + \lambda\right)^{-1} f^h(x_1,x_2)\right| \leq M \left\|f^h\right\|_{C_h^{2\alpha}}. \tag{16}$$

Second, assume that $|\lambda| \geq Lh^{-2}$. From estimates (9) and (12) it follows

$$I_2 \leq M\lambda^{\alpha+1} \sum_{p\in\mathbb{R}_h^+} \sum_{s\in\mathbb{R}_h} \frac{e^{-ah^{-1}(|x_1-p|+|x_2-s)}}{\left((1+\lambda))h^2\right)^{1+2\alpha}} \times$$

$$\times \left(|x_1-p| + |x_2-s|\right)^{-2\alpha} \left(|x_1-p|^2 + |x_2-s|^2\right)^{\alpha} h^{2\alpha} h^2.$$

The change of variables $x_1 - p = mh$ and $x_2 - s = nh$ $m, n = 0, 1, \ldots$ yields

$$I_2 \leq M_1 \lambda^{\alpha+1} \sum_{m\in Z^+} \sum_{n\in Z^+} e^{-a(m+n)} \frac{h^{2\alpha+2}}{\left((1+\lambda)\, h^2\right)^{1+2\alpha}} \leq$$

$$\leq M_2 \frac{\left(\lambda h^2\right)^{1+\alpha}}{\left((1+\lambda))h^2\right)^{1+2\alpha}} \leq M_2 \frac{1}{\left(\lambda h^2\right)^{\alpha}} \leq M_2 \frac{1}{L^{\alpha}} \leq M_3. \tag{17}$$

Thus, from estimates (14), (17) it follows that

$$\sup_{(x_1,x_2)\in\mathbb{R}^2_{h,+}} \left|\lambda^{\alpha} A_h^x \left(A_h^x+\lambda\right)^{-1} f^h(x_1,x_2)\right| \leq M \left\|f^h\right\|_{C_h^{2\alpha}}. \tag{18}$$

Combining estimates (16) and (18), we obtain estimate

$$\left\|f^h\right\|_{E_\alpha\left(A_h^x, C(\mathbb{R}^2_{h,+})\right)} \leq M \left\|f^h\right\|_{C_h^{2\alpha}}.$$

Now, we will prove opposite estimate. Applying the triangle inequality and the definition of E_α-norm, we get

$$\left|\frac{f^h(x_1+\tau,x_2+k)-f^h(x_1,x_2)}{\left(\tau^2+k^2\right)^{\alpha}}\right| \leq 2\left\|f^h\right\|_{E_\alpha\left(A_h^x, C(\mathbb{R}^2_{h,+})\right)} \tag{19}$$

for any $(x_1,x_2)\in\mathbb{R}^2_{h,+}$ and $\tau^2+k^2\geq 1$. Now, we will prove the estimate

$$\left|\frac{f^h(x_1+\tau,x_2+k)-f^h(x_1,x_2)}{\left(\tau^2+k^2\right)^{\alpha}}\right| \leq \frac{M}{\alpha(1-\alpha)}\left\|f^h\right\|_{E_\alpha\left(A_h^x, C(\mathbb{R}^2_{h,+})\right)} \tag{20}$$

for any $(x_1,x_2)\in\mathbb{R}^2_{h,+}$ and $\tau^2+k^2< 1$. For any positive operator A we can write

$$v = \int_0^\infty A(\lambda+A)^{-2} v\, d\lambda.$$

Noted that A_h^x is a positive operator in the Banach space $E_\alpha\left(A_h^x, C\left(\mathbb{R}^2_{h,+}\right)\right)$. Hence, from this identity and formula (6) it follows that

$$f^h(x_1,x_2) = \int_0^\infty A_h^x\left(\lambda+A_h^x\right)^{-2} f^h(x_1,x_2)\, d\lambda =$$

$$= \int_0^\infty \sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h} G^h(x_1,x_2,p,s,\lambda) A_h^x\left(\lambda+A_h^x\right)^{-1} f^h(p,s) h^2\, d\lambda.$$

Consequently,

$$f^h(x_1+\tau,x_2+k)-f^h(x_1,x_2) =$$

$$= \int_0^\infty \lambda^{-\alpha} \sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h} \left[G^h\left(x_1+\tau,x_2+k,p,s,\lambda\right)-G^h(x_1,x_2,p,s,\lambda)\right]\times$$

$$\times \lambda^{\alpha} A_h\left(A_h+\lambda\right)^{-1} f^h(p,s) h^2\, d\lambda,$$

whence

$$\left|\frac{f^h(x_1+\tau,x_2+k)-f^h(x_1,x_2)}{(\tau^2+k^2)^\alpha}\right|\leq$$

$$\leq \left\|f^h\right\|_{E_\alpha\left(A_h^x,C(\mathbb{R}^2_{h,+})\right)}\int\limits_0^\infty \lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{\left|G^h\left(x_1+\tau,x_2+k,p,s,\lambda\right)-G^h\left(x_1,x_2,p,s,\lambda\right)\right|}{(\tau^2+k^2)^\alpha}h^2d\lambda.$$

Let

$$P=\int\limits_0^\infty \lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{\left|G^h\left(x_1+\tau,x_2+k,p,s,\lambda\right)-G^h\left(x_1,x_2,p,s,\lambda\right)\right|}{(\tau^2+k^2)^\alpha}h^2\,d\lambda.$$

Then

$$\left|\frac{f^h(x_1+\tau,x_2+k)-f^h(x_1,x_2)}{(\tau^2+k^2)^\alpha}\right|\leq P\left\|f^h\right\|_{E_\alpha\left(A_h^x,C(\mathbb{R}^2_{h,+})\right)}. \tag{21}$$

To estimate P, we will consider two cases: $|\lambda|\leq Lh^{-2}$ and $|\lambda|\geq Lh^{-2}$, respectively. We denote that $P=L_1+L_2$, where

$$L_1=\int\limits_0^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{\left|G^h\left(x_1+\tau,x_2+k,p,s,\lambda\right)-G^h\left(x_1,x_2,p,s,\lambda\right)\right|}{(\tau^2+k^2)^\alpha}h^2\,d\lambda,$$

$$L_2=\int\limits_{1/(\tau^2+k^2)}^\infty\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{\left|G^h\left(x_1+\tau,x_2+k,p,s,\lambda\right)-G^h(x_1,x_2,p,s,\lambda)\right|}{(\tau^2+k^2)^\alpha}h^2\,d\lambda.$$

We will estimate L_1 and L_2. Let us first assume that $|\lambda|\leq Lh^{-2}$. By Lemma 2 for $p=q=2$ and using the triangle inequality, estimates (7), (8), we have, for some x_1^* between $x_1,\ x_1+\tau$, and x_2^* between $x_2,\ x_2+k$,

$$L_1\leq M\frac{1}{(\tau^2+k^2)^\alpha}\times$$

$$\times\int\limits_0^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\left|\tau G^h_{x_1}\left(x_1^*,x_2^*,p,s,\lambda\right)+kG^h_{x_2}\left(x_1^*,x_2^*,p,s,\lambda\right)\right|h^2\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^\alpha}\int\limits_0^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\left|G^h_{x_1}\left(x_1^*,x_2^*,p,s,\lambda\right)\right|h^2\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^\alpha}\int\limits_0^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{e^{-a(1+\lambda)^{1/2}\left(\left|x_1^*-p\right|+\left|x^*-s\right|\right)}}{h+|x_1^*-p|+|x_2^*-s|}h^2\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\sum_{m\in\mathbb{R}_h^+}\sum_{n\in\mathbb{R}_h^+}\lambda^{-\alpha}\frac{e^{-a(1+\lambda)^{1/2}(mh+nh)}}{mh+nh}h^2\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha}\left(\sum_{m\in Z^+}e^{-2a(1+\lambda)^{1/2}mh}h\right)^{1/2}\left(\sum_{n\in Z^+}e^{-2a(1+\lambda)^{1/2}nh}h\right)^{1/2}d\lambda\leq$$

$$\leq M_1\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{m\in Z^+}e^{-2a(1+\lambda)^{1/2}mh}h\,d\lambda\leq$$

$$\leq M_2\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha}\int\limits_{0}^{\infty}e^{-2a(1+\lambda)^{1/2}x}dx\,d\lambda\leq$$

$$\leq M_3\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\frac{1}{\lambda^{\alpha}}\frac{1}{(1+\lambda)^{1/2}}d\lambda\leq\frac{M_4}{1-2\alpha},\tag{22}$$

$$L_2\leq\frac{M}{(\tau^2+k^2)^{\alpha}}\int\limits_{1/(\tau^2+k^2)}^{\infty}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\lambda^{-\alpha}e^{-a(1+\lambda)^{1/2}(|x_1-p|+|x_2-s|)}\times$$

$$\times\left(1+\ln\left\{1+\Big((1+\lambda)^{1/2}\big(|x_1-p|+|x_2-s|+h\big)\Big)^{-1}\right\}\right)h^2d\lambda+$$

$$+\frac{M}{(\tau^2+k^2)^{\alpha}}\int\limits_{1/(\tau^2+k^2)}^{\infty}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\lambda^{-\alpha}e^{-a(1+\lambda))^{1/2}(|x_1+\tau-p|+|x_2+k-s|)}\times$$

$$\times\left(1+\ln\left\{1+\Big((1+\lambda)^{1/2}\big(|x_1+\tau-p|+|x_2+k-s|+h\big)\Big)^{-1}\right\}\right)h^2d\lambda.$$

The change of variables $x_1-p=mh$ and $x_2-s=nh,\ m,n=0,1,\ldots,$ yields

$$L_2\leq\frac{M}{(\tau^2+k^2)^{\alpha}}\int\limits_{1/(\tau^2+k^2)}^{\infty}\sum_{m\in Z^+}\sum_{n\in Z^+}\lambda^{-\alpha}e^{-a(1+\lambda)^{1/2}(m+n)h}\times$$

$$\times\left(1+\Big((1+\lambda)^{1/2}(mh+nh+h)\Big)^{-1}\right)h^2d\lambda\leq$$

$$\leq M_1\int\limits_{1/(\tau^2+k^2)}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\lambda^{-\alpha}e^{-a(1+\lambda)^{1/2}(q+y)}\left(1+\Big((1+\lambda)^{1/2}(q+y)\Big)^{-1}\right)dq\,dy\,d\lambda\leq$$

$$\leq \frac{M_2}{(\tau^2+k^2)^{\alpha}}\left[\int\limits_{1/(\tau^2+k^2)}^{\infty}\frac{1}{\lambda^{\alpha}}\frac{1}{1+\lambda}\,d\lambda+\int\limits_{1/(\tau^2+k^2)}^{\infty}\frac{1}{\lambda^{\alpha}}\frac{1}{(1+\lambda))^{1/2}}\left(\int\limits_{0}^{\infty}e^{-2a(1+\lambda))^{1/2}q}dq\right)d\lambda\right]\leq$$

$$\leq\frac{M_3}{\alpha(1-\alpha)}. \tag{23}$$

From estimates (22) and (23) it follows that

$$P\leq\frac{M_5}{\alpha(1-\alpha)}. \tag{24}$$

Second, let us $|\lambda|\geq Lh^{-2}$. By using the triangle inequality, estimates (9), (10), we have, for some x_1^* between $x_1,\ x_1+\tau$, and x_2^* between $x_2,\ x_2+k$,

$$L_1\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\left|G_{x_1}^h\left(x_1^*,x_2^*,p,s,\lambda\right)\right|h^2d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}e^{-ah^{-1}(|x_1^*-p|+|x_2^*-s|)}h^{-1}\left(1+(1+\lambda)h^2\right)^{-1}h^2\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha}\sum_{m\in Z^+}\sum_{n\in Z^+}\frac{e^{-a(m+n)}}{(1+\lambda)h^2}\frac{\lambda^{-1/2}}{\lambda^{-1/2}}h^{-1}h^2\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\frac{\lambda^{-\alpha+1/2}}{1+\lambda}\sum_{m\in Z^+}\sum_{n\in Z^+}e^{-a(m+n)}\,d\lambda\leq$$

$$\leq M\frac{(\tau+k)}{(\tau^2+k^2)^{\alpha}}\int\limits_{0}^{1/(\tau^2+k^2)}\lambda^{-\alpha-1/2}d\lambda\leq\frac{M_1}{1-2\alpha}, \tag{25}$$

$$L_2\leq\frac{1}{(\tau^2+k^2)^{\alpha}}\int\limits_{1/(\tau^2+k^2)}^{\infty}\lambda^{-\alpha}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{\left|G^h\left(x_1+\tau,x_2+k,p,s,\lambda\right)-G^h\left(x_1,x_2,p,s,\lambda\right)\right|}{(\tau^2+k^2)^{\alpha}}h^2\,d\lambda\leq$$

$$\leq\frac{M}{(\tau^2+k^2)^{\alpha}}\int\limits_{1/(\tau^2+k^2)}^{\infty}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{e^{-ah^{-1}\left(|x_1-p|+|x_2-s|\right)}}{h^2(1+\lambda)}h^2\,d\lambda+$$

$$+\frac{M}{(\tau^2+k^2)^{\alpha}}\int\limits_{1/(\tau^2+k^2)}^{\infty}\sum_{p\in\mathbb{R}_h^+}\sum_{s\in\mathbb{R}_h}\frac{e^{-ah^{-1}(|x_1+\tau-p|+|x_2+k-s|)}}{h^2(1+\lambda)}h^2\,d\lambda\leq$$

$$\leq \frac{M_1}{(\tau^2+k^2)^{\alpha}} \int\limits_{1/(\tau^2+k^2)}^{\infty} \sum_{m\in Z^+}\sum_{n\in Z^+} \frac{\lambda^{-\alpha}}{(1+\lambda)} e^{-a(m+n)}\, d\lambda \leq$$

$$\leq \frac{M_2}{(\tau^2+k^2)^{\alpha}} \int\limits_{1/(\tau^2+k^2)}^{\infty} \frac{\lambda^{-\alpha}}{1+\lambda}\, d\lambda \leq \frac{M_1}{\alpha(1-\alpha)}. \tag{26}$$

By using estimates (25) and (26), we have

$$P \leq \frac{M}{\alpha\,(1-2\alpha)}. \tag{27}$$

From estimates (21), (24) and (27) it follows estimate (20). Combining estimates (19) and (20), we get

$$\sup_{\substack{(x_1+\tau,x_2+k),(x_1,x_2)\in\mathbb{R}^2_+ \\ (\tau,h)\neq(0,0)}} \left| \frac{f^h(x_1+\tau,x_2+k)-f^h(x_1,x_2)}{(\tau^2+k^2)^{\alpha}} \right| \leq$$

$$\leq \frac{M}{\alpha\,(1-2\alpha)} \left\| f^h \right\|_{E_\alpha\left(A_h^x, C(\mathbb{R}^2_{h,+})\right)}.$$

This means that the following estimate holds:

$$\left\| f^h \right\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})} \leq \frac{M}{\alpha\,(1-2\alpha)} \left\| f^h \right\|_{E_\alpha\left(A_h^x, C(\mathbb{R}^2_{h,+})\right)}.$$

Theorem 1 is proved.

4. Applications of Theorem 1. First, the boundary-value problem

$$-\frac{\partial^2 u(y,x)}{\partial y^2} - a_{11}(x)\frac{\partial^2 u(y,x)}{\partial x_1^2} - a_{22}(x)\frac{\partial^2 u(y,x)}{\partial x_2^2} + \sigma u(y,x) = f(y,x),$$

$$0<y<T, \quad x\in\mathbb{R}^2_+, \tag{28}$$

$$u(0,x)=\varphi(x), \quad u(T,x)=\psi(x), \quad x\in\mathbb{R}^2_+; \quad u(y,0,x_2)=0, \quad 0\leq y\leq T, \quad x_2\in\mathbb{R},$$

for the elliptic equation is considered. Here $a_{11}(x)$, $a_{22}(x)$, $\varphi(x)$, $\psi(x)$, and $f(y,x)$ are sufficiently smooth functions, and satisfy the uniform ellipticity (2) and $\sigma>0$.

The discretization of problem (28) is carried out in two steps. In the first step, let us use the discretization in x. To the differential operator A_h^x generated by the problem (28), we assign the difference operator A_h^x defined by formula (3).

Theorem 2. *For the solution of elliptic problem*

$$A_h^x u^h(x) = \Psi^h(x), \quad x\in\mathbb{R}^2_{h,+},$$

$$u^h(0,x_2)=0, \quad x_2\in\mathbb{R}_h, \tag{29}$$

the following coercive inequality holds:

$$\left\| \Delta_{1+}^{x_1}\Delta_{1-}^{x_1} u^h h^{-2} \right\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})} + \left\| \Delta_{1+}^{x}\Delta_{1-}^{x} u^h h^{-2} \right\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})} \leq M(\alpha)\|\Psi^h\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})},$$

where $M(\alpha)$, $0<\alpha<1/2$, *does not depend on* Ψ^h.

The proof of Theorem 2 uses the techniques introduced in [22] (Chapter 5) and it is based on estimates (7)–(10).

With the help of A_h^x, we arrive at the boundary-value problem

$$-\frac{d^2u^h(y,x)}{dy^2}+A_h^x u^h(y,x)=f^h(y,x),\qquad 0<y<T,\quad x\in\mathbb{R}^2_{h,+},$$
$$u^h(0,x)=\varphi(x),\qquad u^h(T,x)=\psi(x),\quad x\in\mathbb{R}^2_{h,+}, \tag{30}$$

for the system of ordinary differential equations. In the second step, problem (30) is replaced by the second order of accuracy difference scheme in y, we get the following difference problem:

$$-\frac{1}{\tau^2}\Big(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\Big)+A_h^x u_k^h(x)=\varphi_k^h(x),$$
$$\varphi_k^h(x)=f^h(y_k,x),\qquad y_k=k\tau,\quad 1\le k\le N-1,\quad N\tau=T,\quad x\in\mathbb{R}^2_{h,+}, \tag{31}$$
$$u_0^h(x)=\varphi^h(x),\qquad u_N^h(x)=\psi^h(x),\quad x\in\mathbb{R}^2_{h,+},$$

for the approximate solution of problem (30).

Theorem 3. *For the solution of the difference problem* (31) *the following inequalities are valid*:

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\big(u_{k+1}^h-2u_k^h+u_{k-1}^h\big)\right\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})}+\max_{1\le k\le N-1}\left\|u_k^h\right\|_{C^{2\alpha+2}(\mathbb{R}^2_{h,+})}\le$$
$$\le M(\alpha)\left[\max_{1\le k\le N-1}\left\|\varphi_k^h\right\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})}+\left\|\varphi^h\right\|_{C^{2\alpha+2}(\mathbb{R}^2_{h,+})}+\left\|\psi^h\right\|_{C^{2\alpha+2}(\mathbb{R}^2_{h,+})}\right],$$

where $M(\alpha)$ *does not depend on* $\{\varphi_k^h(x)\}_1^{N-1}$, $\varphi^h(x)$, $\psi^h(x)$, h *and* τ.

The proof of Theorem 3 is based on the Theorem 1 on the structure of the fractional spaces $E_\alpha(A_h^x,C_h)$ and on the positivity of the operator A_h^x and on Theorem 2 coercive inequality on for the difference elliptic problem (29) and on the theorems on the structure of the fractional spaces $E'_\alpha=E_\alpha\big((A_h^x)^{1/2},C_h\big)$ of paper [22] and on the coercive inequalities in $C_\tau(E_\alpha)$ for the solution of the second order of accuracy difference scheme

$$-\frac{1}{\tau^2}(u_{k+1}-2u_k+u_{k-1})+Au_k=f_k,\qquad f_k=f(t_k),\quad t_k=k\tau,$$
$$1\le k\le N-1,\quad N\tau=T,\quad u_0=\varphi,\quad u_N=\psi, \tag{32}$$

for the approximate solution of the boundary-value problem

$$-u''(t)+Au(t)=f(t),\quad 0<t<T,$$
$$u(0)=\varphi,\quad u(T)=\psi, \tag{33}$$

in a Banach space E with positive operator A.

Second, the nonlocal boundary-value problem

$$\frac{\partial^2 u(y,x)}{\partial y^2} - a_{11}(x)\frac{\partial^2 u(y,x)}{\partial x_1^2} - a_{22}(x)\frac{\partial^2 u(y,x)}{\partial x_2^2} + \sigma u(y,x) = f(y,x),$$

$$0 < y < T, \quad x \in \mathbb{R}^2_+, \tag{34}$$

$$u(0,x) = u(T,x), \quad u_y(0,x) = u_y(T,x), \quad x \in \mathbb{R}^2_+;$$

$$u(y,0,x_2) = 0, \quad 0 \le y \le T, \quad x_2 \in \mathbb{R},$$

for the elliptic equation is considered.

The discretization of problem (34) is carried out in two steps. In the first step, let us use the discretization in (x_1, x_2). To the differential operator A_h^x generated by the problem (34), we assign the difference operator A_h^x defined by formula (3), too. With the help of A_h^x, we arrive at the nonlocal boundary-value problem

$$-\frac{d^2 u^h(y,x)}{dy^2} + A_h^x u^h(y,x) = f^h(y,x), \qquad 0 < y < T, \quad x \in \mathbb{R}^2_+,$$

$$u^h(0,x) = u^h(T,x), \qquad u_y^h(0,x) = u_y^h(T,x), \quad x \in \mathbb{R}^2_+, \tag{35}$$

for the system of ordinary differential equations. In the second step, problem (35) is replaced by the second order of accuracy difference scheme in y, we get the following difference problem:

$$-\frac{1}{\tau^2}\big(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\big) + A_h^x u_k^h(x) = \varphi_k^h(x), \quad x \in \mathbb{R}^2_{h,+},$$

$$\varphi_k^h(x) = f^h(y_k, x), \quad y_k = k\tau, \quad 1 \le k \le N-1, \quad N\tau = T, \quad x \in \mathbb{R}^2_{h,+}, \tag{36}$$

$$u_0^h(x) = u_N^h(x), \quad -u_2^h(x) + 4u_1^h(x) - 3u_0^h(x) = u_{N-2}^h(x) - 4u_{N-1}^h(x) + 3u_N^h(x), \quad x \in \mathbb{R}^2_{h,+},$$

for the approximate solution of problem (35).

Theorem 4. *For the solution of the difference problem* (36) *the following inequalities are valid*:

$$\max_{1 \le k \le N-1} \big\| \tau^{-2}\big(u_{k+1}^h - 2u_k^h + u_{k-1}^h\big)\big\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})} + \max_{1 \le k \le N-1} \big\| u_k^h \big\|_{C^{2\alpha+2}(\mathbb{R}^2_{h,+})} \le$$

$$\le M(\alpha) \max_{1 \le k \le N-1} \big\| \varphi_k^h \big\|_{C^{2\alpha}(\mathbb{R}^2_{h,+})},$$

where $M(\alpha)$ *does not depend on* $\{\varphi_k^h(x)\}_1^{N-1}$, h *and* τ.

The proof of Theorem 4 is based on the Theorem 1 on the structure of the fractional spaces $E_\alpha(A_h^x, C_h)$ and on the positivity of the operator A_h^x and on the theorem on the structure of the fractional spaces $E'_\alpha = E_\alpha\big((A_h^x)^{1/2}, C_h\big)$ of paper [22] and on Theorem 2 coercive inequality on for the difference elliptic problem (29) and on the theorem on coercive inequalities in $C_\tau(E_\alpha)$ (see [22]) for the solution of the second order of accuracy difference scheme

$$-\frac{1}{\tau^2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = f_k, \quad f_k = f(t_k), \quad t_k = k\tau,$$

$$1 \le k \le N-1, \quad N\tau = T, \quad u_0 = u_N, \quad -u_2 + 4u_1 - 3u_0 = u_{N-2} - 4u_{N-1} + 3u_N,$$

for the approximate solution of the nonlocal boundary-value problem

$$-u''(t)+Au(t)=f(t),\quad 0<t<T,$$
$$u(0)=u(T),\quad u'(0)=u'(T),$$

in a Banach space E with positive operator A.

5. Conclusion. Banach fixed-point theorem and methods of the present paper and [24] enable us to establish the existence and uniqueness results which hold under the some sufficient conditions on nonlinear term for the solution of second order of approximation difference schemes for the approximate solution of the following mixed problem:

$$-\frac{\partial^2 u(y,x)}{\partial y^2}-a_{11}(y,x)\frac{\partial^2 u(y,x)}{\partial x_1^2}-a_{22}(y,x)\frac{\partial^2 u(y,x)}{\partial x_2^2}+\sigma u(y,x)=f(y,x,u,u_{x_1},u_{x_2},u_y),$$
$$0<y<T,\quad x\in\mathbb{R}^2_+,$$
$$u(0,x)=\varphi(x),\quad u(T,x)=\psi(x),\quad x\in\mathbb{R}^2_+;\qquad u(y,0,x_2)=0,\quad 0<y<T,\quad x_2\in\mathbb{R}.$$

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