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SUFFICIENT CONDITIONS FOR BOUNDED TURNING OF ANALYTIC FUNCTIONS

ДОСТАТНІ УМОВИ ДЛЯ ОБМЕЖЕНОГО ПОВОРОТУ АНАЛІТИЧНИХ ФУНКЦІЙ

Let function f be analytic in the open unit disk and be normalized such that $f(0) = f'(0) - 1 = 0$. In this paper methods from the theory of first order differential subordinations are used for obtaining sufficient conditions for f to be with bounded turning, i.e., the read part of its first derivative to map the unit disk onto the right half plane. In addition, several open problems are posed.

Нехай f — функція, аналітична у відкритому одиничному крузі, нормована так, що $f(0) = f'(0) - 1 = 0$. Методи теорії диференціальних підпорядкувань першого порядку застосовано, щоб отримати достатні умови того, що функція f має обмежений поворот, тобто дійсна частина її першої похідної відображає одиничний круг на праву півплощину. Крім того, сформульовано кілька відкритих проблем.

1. Introduction and preliminaries. Let $\mathcal{H}(\mathbb{D})$ be the class of functions that are analytic in unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}(\mathbb{D})$ of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots,\ z \in \mathbb{D}$.

The class of *starlike functions*, which is a subclass of the class of univalent functions, is defined by

$$S^* = \left\{ f \in \mathcal{A} \colon \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D} \right\}.$$

The functions $f \in S^*$ map the unit disk onto a starlike region, i.e., if $w \in f(\mathbb{D})$, then $tw \in f(\mathbb{D})$ for all $t \in [0, 1]$. More details can be found in [2].

Another subclasses of univalent functions are

$$R_\alpha = \left\{ f \in \mathcal{A} \colon \operatorname{Re} f'(z) > \alpha,\ z \in \mathbb{D} \right\}, \quad 0 \le \alpha < 1,$$

and

$$\mathcal{R}(\alpha) = \left\{ f \in \mathcal{A} \colon \left| \arg f'(z) \right| < \frac{\alpha\pi}{2},\ z \in \mathbb{D} \right\}, \quad 0 < \alpha \le 1,$$

which are subclasses of the *class of functions with bounded turning*, $R = R_0 = \mathcal{R}(1)$. The name of the class R follows from the fact that $\operatorname{Re} f'(z) > 0$ is equivalent with $\left| \arg f'(z) \right| < \dfrac{\pi}{2}$ and $\arg f'(z)$ is the angle of rotation of the image of a line segment starting from z under the mapping f. It is well known that S^* does not contain R and R does not contain S^* [7], which brings big interest for the class R [6–8].

In this paper we will study the expression

$$z \frac{f'(z) - 1}{f(z) - z} = \frac{f'(z) - 1}{f(z)/z - 1}, \tag{1}$$

for receiving some results that will lead to necessary conditions for a function $f \in \mathcal{A}_n,\ n \in \mathbb{N}$, $n \geq 2$, to be with bounded turning. Here, the class $\mathcal{A}_n,\ n \in \mathbb{N},\ n \geq 2$, is defined by

$$\mathcal{A}_n = \left\{ f \in \mathcal{A} \colon f(z) = z + a_n z^n + \ldots,\ z \in \mathbb{D},\ a_n \neq 0 \right\}.$$

The study will involve a method from the theory of differential subordinations, while valuable references on this topic are [1] and [3]. Using a similar techniques as in this paper, in [8] the expression

$$\frac{f'(z) - 1}{f(z)/z}$$

is studied and results concerning the univalence and the starlikeness of f from $\mathcal{A}$ are given.

First we introduce the notion of subordination. If $f,\ g \in \mathcal{A}$, then we say that f is *subordinate* to g, and write $f(z) \prec g(z)$, if there exists a function w, analytic in the unit disc $\mathbb{D}$, such that $w(0) = 0$, $\left|w(z)\right| < 1$ and $f(z) = g\big(w(z)\big)$ for all $z \in \mathbb{D}$. Specially, if g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

For obtaining the main result we will use the method of differential subordinations [3]. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. Namely, if $\varphi \colon \mathbb{C}^2 \to \mathbb{C}$ is analytic in a domain D, if h is univalent in $\mathbb{D}$, and if p is analytic in $\mathbb{D}$ with $\big(p(z), zp'(z)\big) \in D$ when $z \in \mathbb{D}$, then p is said to satisfy a first-order differential subordination if

$$\varphi\big(p(z), zp'(z)\big) \prec h(z). \tag{2}$$

The univalent function q is said to be a *dominant* of the differential subordination (2) if $p(z) \prec q(z)$ for all the functions p satisfying (2). If $\widetilde{q}$ is a dominant of (2) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (2), then we say that $\widetilde{q}$ is the *best dominant* of the differential subordination (2).

From the theory of first-order differential subordinations we will use the following lemma.

Lemma 1 [5]. *Let q be univalent in the unit disk $\mathbb{D}$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi\big(q(z)\big)$, $h(z) = \theta\big(q(z)\big) + Q(z)$, and suppose that*:

(i) *Q is starlike in the unit disk $\mathbb{D}$*,

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re} \left[\dfrac{\theta'\big(q(z)\big)}{\phi\big(q(z)\big)} + \dfrac{zQ'(z)}{Q(z)} \right] > 0,\ z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta\big(p(z)\big) + zp'(z)\phi\big(p(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\phi\big(q(z)\big) = h(z), \tag{3}$$

then $p(z) \prec q(z)$, and q is the best dominant of (3).

Using Lemma 1 we will prove the following result that will be used in next section.

Lemma 2. *Let $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $n \geq 2$, such that $f(z) \neq z$ for all $z \in \mathbb{D} \setminus \{0\}$. Also, let q be univalent in the unit disk $\mathbb{D}$, with $q(0) = a_n = \dfrac{f^{(n)}(0)}{n!}$, and*

$$\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right] > 0, \quad z \in \mathbb{D}. \tag{4}$$

If

$$z\,\frac{f'(z)-1}{f(z)-z}-n\prec\frac{zq'(z)}{q(z)},\tag{5}$$

then

$$\frac{f(z)-z}{z^n}\prec q(z),$$

and q *is the best dominant of* (5).

Proof. If we choose $\theta(w)=0$ and $\phi(w)=\dfrac{1}{w}$, then $\theta,\ \phi\in\mathcal{H}(D)$, where $D=\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$. The condition $D\supset q(\mathbb{D})$ from Lemma 1 is equivalent to $q(z)\neq 0$ for all $z\in\mathbb{D}$, and we will prove that this last relation holds under our assumptions. Also, let note that $\phi(w)=1/w\neq 0$ for all $w\in q(\mathbb{D})$, and let define

$$Q(z):=zq'(z)\phi\big(q(z)\big)=\frac{zq'(z)}{q(z)}.$$

Denoting

$$\Phi(z):=1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)},$$

first we will show that the assumption (4), which is equivalent to

$$\operatorname{Re}\Phi(z)>0,\quad z\in\mathbb{D},$$

implies $Q\in\mathcal{H}(\mathbb{D})$, i.e., $q(z)\neq 0$ for all $z\in\mathbb{D}$. In the beginning, from $q(0)=a_n\neq 0$ we receive that Q is regular in $z_0=0$. Further, let suppose that there exists $z_0\in\mathbb{D}\setminus\{0\}$, such that $q(z_0)=0$. It means that q has the form

$$q(z)=(z-z_0)^m g(z),\qquad z\in\mathbb{D},\quad m\in\mathbb{N}^*,$$

where $g\in\mathcal{H}(\mathbb{D})$, with $g(z_0)\neq 0$. It follows that there exists $r>0$, such that $g(z)\neq 0$ for all $z\in U(z_0;r):=\big\{z\in\mathbb{C}\colon |z-z_0|<r\big\}\subset\mathbb{D}$. Now, a simple computation shows that for all $z\in U(z_0;r)\setminus\{z_0\}$,

$$\frac{zq'(z)}{q(z)}=\frac{mz}{z-z_0}+\frac{zg'(z)}{g(z)},$$

hence

$$\Phi(z)=\frac{-\dfrac{mzz_0}{(z-z_0)^2}+\dfrac{zg'(z)}{g(z)}+\dfrac{z^2g''(z)}{g(z)}-\left(\dfrac{zg'(z)}{g(z)}\right)^2}{\dfrac{mz}{z-z_0}+\dfrac{zg'(z)}{g(z)}}=$$

$$=\frac{1}{z-z_0}\,\frac{-mzz_0+(z-z_0)^2\left[\dfrac{zg'(z)}{g(z)}+\dfrac{z^2g''(z)}{g(z)}-\left(\dfrac{zg'(z)}{g(z)}\right)^2\right]}{mz+(z-z_0)\dfrac{zg'(z)}{g(z)}}.$$

Finally, having in mind that $g(z_0) \neq 0$, from the above relation we receive that $z_0 \in \mathbb{D}$ is a pole of the function Φ, which contradicts the assumption (4). Thus, we obtain that $q(z) \neq 0$ for all $z \in \mathbb{D}$, hence the function Q is analytic on $\mathbb{D}$.

Further, q is an univalent function, implying $q'(z) \neq 0$ for all $z \in \mathbb{D}$,

$$Q'(0) = \frac{q'(0)}{q(0)} = \frac{q'(0)}{a_n} \neq 0$$

and

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right] > 0, \quad z \in \mathbb{D},$$

meaning that Q is a starlike function. In addition, for the function $h(z) = \theta\big(q(z)\big) + Q(z) = Q(z)$ we have

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0, \quad z \in \mathbb{D}.$$

After choosing $p(z) = \dfrac{f(z) - z}{z^n}$, then $p \in \mathcal{H}(\mathbb{D})$, $p(0) = a_n$ and $p(z) \neq 0$ for all $z \in \mathbb{D}$, i.e., $p(\mathbb{D}) \subset D$, and all the conditions of Lemma 1 are satisfied. Concerning that the subordinations (3) and (5) are equivalent, we receive the conclusion of Lemma 2.

2. Main results and consequences. Using Lemma 2 we will study the modulus of (1) and will receive conclusions that will later lead to criteria for a function f to be in the class R.

Theorem 1. *Let* $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $n \geq 2$, *such that* $f(z) \neq z$ *for all* $z \in \mathbb{D} \setminus \{0\}$, *and let* $a_n = \dfrac{f^{(n)}(0)}{n!}$. *If*

$$z\,\frac{f'(z) - 1}{f(z) - z} - n \prec \frac{\lambda z}{a_n + \lambda z} =: h_1(z), \tag{6}$$

where $0 < |\lambda| \leq |a_n|$, *then*

$$\frac{f(z) - z}{z^n} \prec a_n + \lambda z \tag{7}$$

and the function $a_n + \lambda z$ *is the best dominant of* (6). *Even more*,

$$\left| \frac{f(z) - z}{z^n} - a_n \right| < \lambda, \quad z \in \mathbb{D}, \tag{8}$$

and this conclusion is sharp, *i.e.*, *in the inequality* (8) *the parameter* $|\lambda|$ *can not be replaced by a smaller number so that the implication holds.*

Proof. The function $q(z) = a_n + \lambda z$ satisfies all the conditions of Lemma 2, since

$$\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right] = \operatorname{Re} \frac{1}{1 + \lambda/a_n z} > 0, \quad z \in \mathbb{D},$$

whenever of $0 < |\lambda| \leq |a_n|$. Further, the subordinations (5) and (6) are equivalent, and therefore (7) follows directly from Lemma 2.

For the sharpness of our result, let assume that subordination (6) and inequality $\left| \dfrac{f(z) - z}{z^n} - a_n \right| <$ $< |\lambda_1|$, $z \in \mathbb{D}$, holds, i.e., $\dfrac{f(z) - z}{z^n} \prec a_n + \lambda_1 z$. But, the function $a_n + \lambda z$ is the best dominant of (6), meaning that $a_n + \lambda z \prec a_n + \lambda_1 z$, i.e., $|\lambda| \leq |\lambda_1|$.

***Remark* 1.** It is easy to verify the following:

(i) If $0 < |\lambda| < |a_n|$, then $h_1(\mathbb{D})$ (where h_1 was defined in (6)) is an open disk with the center

$$c = \frac{1}{2}\left[h_1\left(e^{i\arg(a_n/\lambda)}\right) + h\left(-e^{i\arg(a_n/\lambda)}\right)\right] = \frac{|\lambda|^2}{|\lambda|^2 - |a_n|^2},$$

and radius

$$r = \left|h_1\left(e^{i\arg(a_n/\lambda)}\right) - c\right| = \frac{|\lambda||a_n|}{|a_n|^2 - |\lambda|^2}.$$

(ii) If $|\lambda| = |a_n|$, then $h_1(z) = \dfrac{z}{z + e^{i\arg(a_n/\lambda)}}$, and

$$h_1(\mathbb{D}) = \left\{z \in \mathbb{C} : \operatorname{Re} z < \frac{1}{2}\right\}.$$

Therefore, Theorem 1 can be written in the following equivalent form.

Theorem 1′. *Let $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $n \geq 2$, such that $f(z) \neq z$ for all $z \in \mathbb{D} \setminus \{0\}$, and let $a_n = \dfrac{f^{(n)}(0)}{n!}$.*

(i) *If $0 < |\lambda| < |a_n|$ and*

$$\left|z\,\frac{f'(z) - 1}{f(z) - z} - n - \frac{|\lambda|^2}{|\lambda|^2 - |a_n|^2}\right| < \frac{|\lambda|\,|a_n|}{|a_n|^2 - |\lambda|^2}, \quad z \in \mathbb{D},$$

then

$$\left|\frac{f(z) - z}{z^n} - a_n\right| < |\lambda|, \quad z \in \mathbb{D}.$$

(ii) *If*

$$\operatorname{Re}\left[z\,\frac{f'(z) - 1}{f(z) - z}\right] < n + \frac{1}{2}, \quad z \in \mathbb{D},$$

then

$$\left|\frac{f(z) - z}{a_n z^n} - 1\right| < 1, \quad z \in \mathbb{D}.$$

These implications are sharp, i.e., in both cases the radius of the open disk from the conclusion is the smallest possible so that the corresponding implication holds.

***Remark* 2.** The sharpness of Theorems 1 and 1′ can be verified by using function $f(z) = z + a_n z^n + \lambda z^{n+1}$, with $a_n \neq 0$, for which

$$z\,\frac{f'(z) - 1}{f(z) - z} - n = \frac{\lambda z}{a_n + \lambda z}.$$

Thus, in the case $0 < |\lambda| < |a_n|$ we have

$$\left|z\,\frac{f'(z) - 1}{f(z) - z} - n - \frac{\lambda^2}{\lambda^2 - |a_n|^2}\right| = \frac{|\lambda|\,|a_n|}{|a_n|^2 - |\lambda|^2} \quad \text{for} \quad z = e^{i\arg(a_n/\lambda)},$$

while in the case $|a_n| = |\lambda|$ we get

$$\operatorname{Re}\left[z\,\frac{f'(z) - 1}{f(z) - z}\right] = n + \frac{1}{2} \quad \text{for} \quad z = e^{i\arg(a_n/\lambda)}.$$

Now we will give several corollaries and examples of Theorem $1'$, in the case when $n = 2$. We start with part (i) of Theorem $1'$.

***Corollary* 1.** *Let* $f \in \mathcal{A}_2$, *and* $\lambda \in \mathbb{C}$ *with* $\frac{4}{5}|a_2| \leq |\lambda| < |a_2|$, *where* $a_2 = \frac{f''(0)}{2}$. *Also, let denote*

$$\mu := \begin{cases} -2 + \dfrac{|\lambda|}{|a_2| - |\lambda|}, & \text{if } \dfrac{4}{5}|a_2| \leq |\lambda| \leq \sqrt{\dfrac{2}{3}}|a_2|, \\ 2 + \dfrac{|\lambda|}{|a_2| + |\lambda|}, & \text{if } \sqrt{\dfrac{2}{3}}|a_2| \leq |\lambda| < |a_2|. \end{cases}$$

If

$$\left|f'(z) - 1\right| < \mu \left|\frac{f(z)}{z} - 1\right|, \quad z \in \mathbb{D} \setminus \{0\}, \tag{9}$$

then

$$\left|\frac{f(z) - z}{z^2} - a_2\right| < \eta_1 := |\lambda|, \quad z \in \mathbb{D}, \tag{10}$$

and

$$\left|f'(z) - 1\right| < \eta_2, \quad z \in \mathbb{D}, \tag{11}$$

where

$$\eta_2 := \begin{cases} \dfrac{\left(|a_2| + |\lambda|\right)\left(2|a_2| - 3|\lambda|\right)}{|\lambda| - |a_2|}, & \text{if } \dfrac{4}{5}|a_2| \leq |\lambda| \leq \sqrt{\dfrac{2}{3}}|a_2|, \\ 2|a_2| + 3|\lambda|, & \text{if } \sqrt{\dfrac{2}{3}}|a_2| \leq |\lambda| < |a_2|. \end{cases}$$

Moreover, *the implication* (9) $\Rightarrow$ (10) *is sharp for* $\sqrt{\frac{2}{3}}|a_2| \leq |\lambda| < |a_2|$, *and the implication* (9) $\Rightarrow$ (11) *is sharp for* $\sqrt{\frac{2}{3}}|a_2| \leq |\lambda| < |a_2|$, *i.e., for these ranges of* $|\lambda|$, *the values* η_1 *and* η_2 *are the smallest ones so that the corresponding implications hold.*

Also, if $\eta_2 < 1$, *then* f *is univalent with bounded turning, i.e.,* $f \in R_{\alpha_1}$ *and* $f \in \mathcal{R}(\alpha_2)$, *where* $\alpha_1 = 1 - \eta_2$ *and* $\alpha_2 = \arcsin \eta_2$.

Proof. First we will prove inequality (10). The assumption (9) leads to

$$\left|f'(z) - 1\right| < \mu \left|\frac{f(z)}{z} - 1\right| = \mu \left|\frac{f(z) - z}{z}\right|, \quad z \in \mathbb{D} \setminus \{0\},$$

meaning that $\frac{f(z) - z}{z} \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, hence $f(z) \neq z$ for all $z \in \mathbb{D} \setminus \{0\}$. Also, the inequality (9) implies

$$\left|z \frac{f'(z) - 1}{f(z) - z}\right| < \mu, \quad z \in \mathbb{D} \setminus \{0\},$$

and letting $z \to 0$ in the above inequality we obtain that $\mu \geq 2$ is a necessary condition for the above inequality to hold in the case $z = 0$.

It is easy to check that

$$\mu = \frac{|\lambda|\,|a_2| - \left|3|\lambda|^2 - 2|a_2|^2\right|}{|a_2|^2 - |\lambda|^2} = r - |2 + c|,$$

where r and c are defined as in the Remark 1, and that $\mu \ge 2$ whenever $|\lambda| \ge \dfrac{4}{5}|a_2|$.

Further, we can write

$$\left| z\,\frac{f'(z)-1}{f(z)-z} - (2+c) + (2+c) \right| < \mu, \quad z \in \mathbb{D} \setminus \{0\},$$

and it follows that

$$\left| z\,\frac{f'(z)-1}{f(z)-z} - (2+c) \right| < \mu + |2+c| = r, \quad z \in \mathbb{D} \setminus \{0\}.$$

The above inequality holds for $z=0$, since $|c| < \mu + |2+c| = r$ for $0 < |\lambda| < |a_2|$, and thus, from the first part of the Theorem $1'$(i) for the special case $n=2$ we have (10).

From the assumption (9) we get

$$\left|f'(z)-1\right| < \mu \left| \frac{f(z)}{z} - 1 \right| < \mu \left| \frac{f(z)-z}{z^2} \right|, \quad z \in \mathbb{D} \setminus \{0\}, \tag{12}$$

and the inequality (11) follows from (10) and (12), having in mind that $\eta_2 = \mu\left(|a_2| + \lambda\right)$.

The implication (9) $\Rightarrow$ (10) is sharp for $\sqrt{\dfrac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$, and the implication (9) $\Rightarrow$ (11) is sharp for $\sqrt{\dfrac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$, since for the function $f(z) = z + a_2 z^2 + \lambda z^3$ we obtain

$$\left|f'(z)-1\right| = |z|\,|2a_2 + 3\lambda z| < 2|a_2| + 3|\lambda|, \quad z \in \mathbb{D},$$

$$\left| \frac{f(z)}{z} - 1 \right| = |z|\,|a_2 + \lambda z|$$

and

$$\left| \frac{f(z)-z}{z^2} - a_2 \right| = |\lambda|\,|z| < |\lambda|, \quad z \in \mathbb{D}.$$

The assertion (9) is equivalent to

$$\mu > \left| \frac{3\lambda z + 2a_2}{\lambda z + a_2} \right|, \quad z \in \mathbb{D} \setminus \{0\},$$

and a simple computation shows that

$$\sup \left\{ \left| \frac{3\lambda z + 2a_2}{\lambda z + a_2} \right| \,:\, z \in \mathbb{D} \setminus \{0\} \right\} = 2 + \frac{|\lambda|}{|a_2| + |\lambda|},$$

whenever $|\lambda| < |a_2|$, hence

$$\mu \ge 2 + \frac{|\lambda|}{|a_2| + |\lambda|},$$

which holds for $\sqrt{\dfrac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$.

Since

$$-2+\frac{|\lambda|}{|a_2|-|\lambda|}<2+\frac{|\lambda|}{|a_2|+|\lambda|}, \quad \text{if} \quad |\lambda|<\sqrt{\frac{2}{3}}\,|a_2|,$$

the function $f(z)=z+a_2z^2+\lambda z^3$ shows that the implication (9) $\Rightarrow$ (10) is not sharp for $\frac{4}{5}\,|a_2|< < |\lambda| < \sqrt{\frac{2}{3}}\,|a_2|$.

Finally, from (10) and the definitions of the classes R_α and $R(\alpha)$ we receive $f\in R_{\alpha_1}$ and $f\in R(\alpha_2)$.

For $\eta_2=1$ the Corollary 1 reduces to the next example.

***Example* 1.** *Let $f\in\mathcal{A}_2$, with $\frac{1}{5}<|a_2|\le\frac{5}{18}$, where $a_2=\frac{f''(0)}{2}$. Also, let*

$$\mu_*:=\begin{cases}\dfrac{3}{1+|a_2|}, & \textit{if} \quad 0.2=\dfrac{1}{5}<|a_2|\le\dfrac{1}{2+\sqrt{6}}=0.22474\ldots,\\ \dfrac{1}{|\lambda|_*+|a_2|}, & \textit{if} \quad \dfrac{1}{2+\sqrt{6}}\le|a_2|\le\dfrac{5}{18}=0.27\ldots,\end{cases}$$

where

$$|\lambda_*|:=\frac{-\big(1+|a_2|\big)+\sqrt{25|a_2|^2+14|a_2|+1}}{6}.$$

If

$$\big|f'(z)-1\big|<\mu_*\left|\frac{f(z)}{z}-1\right|, \quad z\in\mathbb{D}\setminus\{0\}, \tag{13}$$

then

$$\big|f'(z)-1\big|<1, \quad z\in\mathbb{D}. \tag{14}$$

This implication is sharp for $\frac{1}{5}<|a_2|\le\frac{1}{2+\sqrt{6}}=0.22474\ldots$. Also, the function f is univalent with bounded turning, i.e., $f\in R$.

Proof. We need to prove that conditions of Corollary 1, in the case $\eta_2=1$, are equivalent to the assumptions of this example.

For the case when $\frac{4}{5}\,|a_2|\le|\lambda|\le\sqrt{\frac{2}{3}}\,|a_2|$, then $\eta_2=\mu_*\big(|a_2|+|\lambda|\big)=1$ if and only if $\mu_*=\frac{1}{|\lambda|+|a_2|}$, i.e.,

$$-2\big(|\lambda|+|a_2|\big)+\frac{\lambda\big(|a_2|+|\lambda|\big)}{|a_2|-|\lambda|}=1,$$

or in other words

$$|\lambda|=|\lambda_*|:=\frac{-\big(1+|a_2|\big)+\sqrt{25|a_2|^2+14|a_2|+1}}{6}.$$

Here, we considered only the positive sign of the square root since the negative one leads to negative values of $|\lambda|$. Further, the inequalities

$$\frac{4}{5}\,|a_2|\le|\lambda|=|\lambda_*|\le\sqrt{\frac{2}{3}}\,|a_2|$$

are equivalent to

$$0.22474\ldots = \frac{1}{2+\sqrt{6}} \le |a_2| \le \frac{5}{18} = 0.27\ldots.$$

In a similar way, for the case $\sqrt{\frac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$ we have $\eta_2 = 1$ if and only if $3|\lambda|+2|a_2| = 1$, i.e., $|\lambda| = \frac{1-2|a_2|}{3}$. A simple calculus shows that

$$\sqrt{\frac{2}{3}}\,|a_2| \le |\lambda| < |a_2|$$

is equivalent to

$$0.2 = \frac{1}{5} < |a_2| \le \frac{1}{2+\sqrt{6}} = 0.22474\ldots,$$

which completes the proof.

***Remark* 3.** Weather implications (9) $\Rightarrow$ (11) for $\frac{4}{5}\,|a_2| \le |\lambda| < \sqrt{\frac{2}{3}}\,|a_2|$ $\Big($Corollary 1 and (13) $\Rightarrow$ (14) for $\frac{1}{5} < |a_2| \le \frac{1}{2+\sqrt{6}} = 0.22474\ldots\Big)$. Example 1 are sharp are still **open problems**.

Part (ii) from Theorem $1'$ brings the following result.

***Corollary* 2.** *Let $f \in \mathcal{A}_2$ and $2 \le \mu < \frac{5}{2}$. If*

$$\left|f'(z)-1\right| < \mu\left|\frac{f(z)}{z}-1\right|, \quad z \in \mathbb{D}\setminus\{0\},$$

then

$$\left|\frac{f(z)-z}{a_2 z^2}-1\right| < 1, \quad z \in \mathbb{D}, \quad \textit{where} \quad a_2 = \frac{f''(0)}{2},$$

and

$$\left|f'(z)-1\right| < 2\,\mu\,|a_2| =: \eta_3, \quad z \in \mathbb{D}.$$

Even more, if $\eta_3 < 1$, then the function f is univalent with bounded turning, i.e., $f \in R_{\beta_1}$ and $f \in \mathcal{R}(\beta_2)$, where $\beta_1 = 1-\eta_3$ and $\beta_2 = \arcsin\eta_3$.

Proof. The assumption leads to

$$\left|f'(z)-1\right| < \mu\left|\frac{f(z)}{z}-1\right| = \mu\left|\frac{f(z)-z}{z}\right|, \quad z \in \mathbb{D}\setminus\{0\},$$

meaning that $\frac{f(z)-z}{z} \ne 0$ for all $z \in \mathbb{D}\setminus\{0\}$, hence $f(z) \ne z$, for all $z \in \mathbb{D}\setminus\{0\}$. It also implies that

$$\left|z\,\frac{f'(z)-1}{f(z)-z}\right| < \mu, \quad z \in \mathbb{D}\setminus\{0\},$$

and letting $z \to 0$ in the above inequality we obtain that $\mu \ge 2$, and thus

$$\left|z\,\frac{f'(z)-1}{f(z)-z}\right| \le \mu, \quad z \in \mathbb{D}.$$

From here

$$\operatorname{Re}\left[z\,\frac{f'(z)-1}{f(z)-z}\right] \le \mu < \frac{5}{2}, \quad z \in \mathbb{D},$$

and the rest of the proof follows from Theorem $1'$(ii) for $n = 2$.

Similarly as in Example 1, taking $\eta_3 = 1$ in the previous corollary we receive:

***Example* 2.** *Let $f \in \mathcal{A}_2$, with $\frac{1}{5} < |a_2| \le \frac{1}{4}$, where $a_2 = \frac{f''(0)}{2}$. If*

$$\left|f'(z)-1\right| < \frac{1}{2|a_2|}\left|\frac{f(z)}{z}-1\right|, \quad z \in \mathbb{D}\setminus\{0\}, \tag{15}$$

then

$$\left|f'(z)-1\right| < 1, \quad z \in \mathbb{D},$$

and further, the function f is univalent with bounded turning, i.e., $f \in R$.

***Remark* 4.** (i) It can be verified that $\frac{1}{2|a_2|} < \mu_*$ for $\frac{1}{5} < |a_2| \le \frac{5}{18}$, meaning that the condition (13) is weaker than condition (15), i.e., the result from Example 1 is better then the result from Example 2.

(ii) It is an **open problem** weather the result from Corollary 2 is sharp, i.e., weather η_3 is the smallest constant so the implication holds.

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