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SOME RESULTS ON THE GLOBAL SOLVABILITY FOR STRUCTURALLY DAMPED MODELS WITH A SPECIAL NONLINEARITY*

ДЕЯКІ РЕЗУЛЬТАТИ ПРО ГЛОБАЛЬНУ РОЗВ'ЯЗНІСТЬ ДЛЯ МОДЕЛЕЙ ЗІ СТРУКТУРНИМ ЗАТУХАННЯМ ТА НЕЛІНІЙНІСТЮ СПЕЦІАЛЬНОГО ВИГЛЯДУ

The main purpose of this paper is to prove the global (in time) existence of solution for the semilinear Cauchy problem

$$u_{tt}+(-\Delta)^{\sigma}u+(-\Delta)^{\delta}u_t=|u_t|^p,\qquad u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x).$$

The parameter $\delta\in(0,\sigma]$ describes the structural damping in the model varying from the exterior damping $\delta=0$ up to the visco-elastic type damping $\delta=\sigma$. We will obtain the admissible sets of the parameter p for the global solvability of this semilinear Cauchy problem with arbitrary small initial data u_0,u_1 in the hyperbolic-like case $\delta\in\left(\dfrac{\sigma}{2},\sigma\right)$ and in the exceptional case $\delta=0$.

Основною метою цієї статті є доведення глобального (за часом) існування розв'язку напівлінійної задачі Коші

$$u_{tt}+(-\Delta)^{\sigma}u+(-\Delta)^{\delta}u_t=|u_t|^p,\qquad u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x).$$

Параметр $\delta\in(0,\sigma]$ описує структурне затухання в моделі, що змінюється від зовнішнього затухання $\delta=0$ до затухання в'язкоеластичного типу $\delta=\sigma$. Визначено множини параметра p, допустимі з точки зору глобальної розв'язності даної напівлінійної задачі Коші з як завгодно малими початковими даними u_0, u_1 у випадку гіперболічного типу $\delta\in\left(\dfrac{\sigma}{2},\sigma\right)$, а також у винятковому випадку $\delta=0$.

1. Introduction. The semilinear Cauchy problem

$$u_{tt}-\Delta u=|u|^p,\qquad u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x)$$

has been investigated in recent years in many papers in order to prove Strauss' conjecture about the critical exponent p_{crit} for the global solvability with arbitrarily small data (u_0,u_1). The next question of interest is to understand the influence of a damping term of the form $(-\Delta)^{\delta}u_t$ that will be included in the model. There are several papers that have introduced methods to deal with the nonlinear Cauchy problem of the form

$$u_{tt}-\Delta v+\mu(-\Delta)^{\delta}v_t=F(x,u,u_t),\qquad v(0,x)=v_0(x),\quad v_t(0,x)=v_1(x).$$

In [2] the authors have linear decay estimates to prove global (in time) existence results for this kind of nonlinear Cauchy problems. From these paper one can learn that several interesting intervals of δ generate either a parabolic or a hyperbolic behavior of the solution for the corresponding linear model from the point of decay estimates.

More precisely, the solutions to the linear model

$$v_{tt}-\Delta v+\mu(-\Delta)^{\delta}v_t=0,\qquad v(0,x)=v_0(x),\quad v_t(0,x)=v_1(x),$$

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for $\delta \in \left(0, \dfrac{1}{2}\right)$ behave themselves similarly to solutions to a corresponding parabolic model. Meanwhile for $\delta \in \left(\dfrac{1}{2}, 1\right)$ solutions behave more like those of a corresponding hyperbolic model. In [8] the authors have applied a strategy to study the following more general model:

$$v_{tt} + (-\Delta)^\sigma v + \mu(-\Delta)^\delta v_t = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x). \tag{1}$$

There were several difficulties which arose in the understanding of the last model. In order to estimate the nonlinear term by Duhamel's principle, some new results on the fractional Gagliardo – Nirenberg inequality from harmonic analysis have been applied successfully. These problems appeared mainly due to the fact that the fractional Laplacian is a nonlocal operator.

Classically there exist several equivalent definitions for this operator. Without any restriction on the parameter we can define the fractional Laplacian for all positive σ by the Fourier transform

$$\mathcal{F}\big((-\Delta)^\sigma f(\xi)\big) = |\xi|^{2\sigma}\mathcal{F}(f)(\xi)$$

for all $\sigma > 0$. In the last expression $\mathcal{F}(f)$ denotes the Fourier transform of the function f with respect to x variable.

The above formal definition is not so useful for practical applications. Therefore, for some special values of σ it is possible to introduce the fractional Laplacian as a singular integral. Indeed, for $\sigma \in (0,1)$ we can adopt the following more convenient integral representation of the fractional Laplacian:

$$(-\Delta)^\sigma u(x) = c_{n,\sigma} \int\limits_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\sigma}}\, dy$$

for sufficient smooth u with a normalization positive constant $c_{n,\sigma} = \dfrac{2^{2\sigma}\sigma\Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(1-\sigma)}$ depending on n and σ.

The nonlinear Cauchy problem with the right-hand side $|u_t|^p$ has some specific feature in the comparison with the cases of nonlinearity $|u|^p$ or $\big||D|^a u\big|^p$. We do not apply only the Gagliardo – Nirenberg inequality to obtain the global solvability for the last model, since the term $|u_t|^p$ is absent from the definition of the data spaces and the solution spaces.

The goal of this paper is to cover possible values of the parameter σ, δ which were missing in our previous paper [8] for the model with nonlinearity $|u_t|^p$. We will obtain global solvability results for the whole range $\delta \in \left(\dfrac{\sigma}{2}, \sigma\right)$, whereas such a result was obtained in [8] just only for the case $\delta = \dfrac{\sigma}{2}$. Unlike the proof in [8] in the case $\delta = \dfrac{\sigma}{2}$, the method we will use in this paper does not require large values of p for the global solvability of the nonlinear Cauchy problem with a class of nonlinearities $f = f(u_t)$ on the right-hand side of the equation, since we do not apply the fractional power rules (see [7]).

Finally, we will state and prove an interesting result in the case $\delta = 0$ with the same type of nonlinearity $|u_t|^p$. This case requires a certain attention due to the special formula of the characteristic roots of the corresponding parameter-dependent ordinary differential equation after applying the Fourier transform to the equation in the Cauchy problem (1).

Recall that by applying the partial Fourier transform with respect to the spatial variables x to the equation in (1) the following Cauchy problem is obtained for the Fourier transform $w = w(\xi, t) := := \hat{u}_{x\to\xi}(\xi, t) = F(u)(\xi, t)$:

$$w_{tt} + \mu|\xi|^{2\delta} w_t + |\xi|^{2\sigma} w = 0, \qquad w(0, \xi) = w_0(\xi), \quad w_t(0, \xi) = w_1(\xi).$$

The characteristic equation $\lambda^2 + \mu|\xi|^{2\delta}\lambda + |\xi|^{2\sigma} = 0$ has the roots

$$\lambda_{1,2}(\xi) = \frac{1}{2}\Big(-\mu|\xi|^{2\delta} \pm \sqrt{\mu^2|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big).$$

Using these characteristic roots we can write down directly the detailed formula for the solution of the linear Cauchy problem (1) by means of the fundamental solutions:

$$u = K_0 * u_0 + K_1 * u_1,$$

where $*$ denotes the convolution with respect to x variable. In the last expression the fundamental solutions K_i are given by the following formulas:

$$K_0 = \mathcal{F}^{-1}\left(\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\right), \qquad K_1 = \mathcal{F}^{-1}\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right).$$

Our main results are contained in the following theorems which will be formulated with different values of the damping parameter δ.

Theorem 1. *Consider the Cauchy problem for the structurally damped model*

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u_t|^p, \qquad u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x),$$

with $\sigma \in \left(1, \frac{n}{2}\right)$, $\mu > 0$ *and* $\delta \in \left(\frac{\sigma}{2}, \sigma\right)$. *The data* (u_0, u_1) *are supposed to belong to the function spaces* $(L^1 \cap H^s) \times (L^1 \cap H^{s-\sigma})$, *where the parameter* s *satisfies* $\left[\frac{n}{2}\right] + 1 + \left[\frac{[n/2] + 1}{2}\right] \le s \in \mathbb{N}$. *Then for all*

$$p \in \left(\max\left(1 + \frac{2s}{n}, s\right); \frac{n}{[n + 2\sigma - 2s]^+}\right)$$

there exists a uniquely determined global (in time) small data solution in $C([0, \infty), H^s) \cap C^1([0, \infty), H^{s-\sigma})$.

We see that in the case $s > \frac{n}{2} + \sigma$ the condition for p is simply reduced to $p > \max\left\{1 + \frac{2s}{n}, s\right\}$. This conditions on s, p in the above theorem may seem too strict, however as we will explain at the end of the proof of Theorem 1, the analogous result with the same restriction on s, p is also valid if we replace $|u_t|^p$ in the right-hand side by more general nonlinearity function $f(u_t)$ satisfying some growth of power type.

For the external damping model with $\sigma = 1$, $\delta = 0$ we have the following result.

Theorem 2. *Let us consider the Cauchy problem for the external damped model*

$$u_{tt} - \Delta u + u_t = |u_t|^p, \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x).$$

The data (u_0, u_1) *are assumed to belong to the function space* $(L^1 \cap H^s) \times (L^1 \cap H^{s-1})$ *with* $s > 1 + \dfrac{n}{2}$. *Then for any*

$$p > \max\left\{2; s; 1 + \frac{4(s-1)}{ns}\right\}$$

there exists a uniquely determined global (*in time*) *small data energy solution from* $C\big([0,\infty), H^s\big) \cap$ $\cap\, C^1\big([0,\infty), H^{s-1}\big)$.

It should be noted that we also can obtain a similar result with general $\sigma > 1$ with only minor changes on the conditions for s and p.

Throughout this paper the notation $f \lesssim g$ for two functions $f = f(t),\ g = g(t)$ means $|f(t)| \le$ $\le c|g(t)|$ for all t in the range that will be shown, with a positive constant c.

The paper is organized as follows. In Section 2 we recall some estimates for solutions to the homogeneous linear models corresponding to the problems stated in Theorems 1 and 2. In Section 3 we shall present the proofs of our main results. As conclusion, in the last Section 4 we will give some remarks and comments for our further study in the case of variable coefficients models.

2. Estimates of solution for linear Cauchy problems. In this section of the article we will present several estimates for solutions to homogeneous Cauchy problems which are useful in the study of the nonlinear Cauchy problems. These estimates were obtained already in [8] and [9] by using singular Bessel integrals for the Fourier transform of radial functions and exploiting the asymptotic profiles of the characteristic roots $\lambda_i,\ i = 1, 2$, that were given before. For the case $\delta = 0$ such estimates were obtained by comparing the solution of the damped model with those for the corresponding evolution model $u_t + (-\Delta)^\sigma u = 0$ by studying the diffusion phenomenon.

First, we have the following result (Proposition 22 in [8]) on $(L^1 \cap L^2) - L^2$ and $L^2 - L^2$ decay estimates of solutions for the linear (homogeneous) Cauchy problem in the case $\delta \in \left(\dfrac{\sigma}{2}, \sigma\right)$.

Proposition 1. *Let us consider the Cauchy problem*

$$v_{tt} + (-\Delta)^\sigma v + (-\Delta)^\delta v_t = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x),$$

for $\delta \in \left(\dfrac{\sigma}{2}, \sigma\right)$ *and data* $(v_0, v_1) \in (L^1 \cap H^{2\delta}) \times (L^1 \cap L^2)$. *Then the solutions and their derivatives satisfy the following* $(L^1 \cap L^2) - L^2$ *estimates*:

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\delta}} \|v_0\|_{L^1 \cap L^2} + (1+t)^{1-\frac{n}{4\delta}} \|v_1\|_{L^1 \cap L^2},$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}} \|v_0\|_{L^1 \cap H^{2(\sigma-\delta)}} + (1+t)^{1-\frac{n+2\sigma}{4\delta}} \|v_1\|_{L^1 \cap L^2},$$

$$\||D|^\sigma v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}} \|v_0\|_{L^1 \cap H^\sigma} + (1+t)^{1-\frac{n+2\sigma}{4\delta}} \|v_1\|_{L^1 \cap L^2},$$

$$\||D|^{2\delta} v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\delta}-1} \|v_0\|_{L^1 \cap H^{2\delta}} + (1+t)^{-\frac{n}{4\delta}} \|v_1\|_{L^1 \cap L^2},$$

and the $L^2 - L^2$ *estimates*:

$$\|v(t,\cdot)\|_{L^2} \lesssim \|v_0\|_{L^2} + (1+t)\|v_1\|_{L^2},$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\sigma}{2\delta}}\|v_0\|_{H^{2(\sigma-\delta)}} + \|v_1\|_{L^2},$$

$$\||D|^\sigma v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{\sigma}{2\delta}}\|v_0\|_{H^\sigma} + \|v_1\|_{L^2},$$

$$\||D|^{2\delta} v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\|v_0\|_{H^{2\delta}} + (1+t)^{-\frac{2\delta-\sigma}{2\delta}}\|v_1\|_{L^2},$$

in arbitrary space dimensions n.

Under special restrictions to the dimension n and the parameters σ, δ one can prove a sharper result than Proposition 1 by using the technique from [2] (see Proposition 24 in [8]).

Proposition 2. *Let us consider the Cauchy problem*

$$v_{tt} + (-\Delta)^\sigma v + (-\Delta)^\delta v_t = 0, \qquad v(0,x) = v_0(x), \qquad v_t(0,x) = v_1(x),$$

for $\delta \in \left(\dfrac{\sigma}{2}, \sigma\right)$ *and data* $(v_0, v_1) \in (L^1 \cap H^{2\delta}) \times (L^1 \cap L^2)$ *under the constrain condition* $n \geq 2\sigma$ *for the dimension* n. *Then the solution and its derivatives satisfy the following* $(L^1 \cap L^2) - L^2$ *estimates*:

$$\|v(t,\cdot)\|_{L^2} \lesssim \begin{cases} (1+t)^{-\frac{n}{4\delta}}\|v_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n-2\sigma}{4\delta}}\|v_1\|_{L^1\cap L^2} & \text{if} \quad n > 2\sigma, \\ (1+t)^{-\frac{\sigma}{2\delta}}\|v_0\|_{L^1\cap L^2} + \log(e+t)\|v_1\|_{L^1\cap L^2} & \text{if} \quad n = 2\sigma, \end{cases}$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}}\|v_0\|_{L^1\cap H^{2(\sigma-\delta)}} + (1+t)^{-\frac{n}{4\delta}}\|v_1\|_{L^1\cap L^2},$$

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}}\|v_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n}{4\delta}}\|v_1\|_{L^1\cap L^2},$$

$$\big\||D|^{2\delta} v(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\delta}-1}\|v_0\|_{L^1\cap H^{2\delta}} + (1+t)^{-\frac{n+4\delta-2\sigma}{4\delta}}\|v_1\|_{L^1\cap L^2}.$$

More generally, for arbitrary $\alpha > 0$ *it holds*

$$\big\|\partial_t^j |D|^\alpha v(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n-2\sigma}{4\delta}-\frac{\sigma j+\alpha}{2\delta}}\Big((1+t)^{-\frac{\sigma}{2\delta}}\|v_0\|_{L^1\cap H^{2j(\sigma-\delta)+|\alpha|}} + \|v_1\|_{L^1\cap H^{2(j-1)\delta+\alpha}}\Big). \quad (2)$$

In the exceptional case $\delta = 0$, the method to use the asymptotic profile of the characteristic roots does not bring any decay rate in t of the solution to the corresponding linear Cauchy problem. The study of the diffusion phenomenon in the abstract setting for the equation $u_{tt} + Bu + u_t = 0$ leads to the following result (see Proposition 2.1 in [9]).

Proposition 3. *The solution* $v = v(t,x)$ *of the linear Cauchy problem for external damped model or for the model with friction*

$$v_{tt} + (-\Delta)^\sigma v + v_t = 0, \qquad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x),$$

and its derivatives satisfy the following $(L^1 \cap L^2) - L^2$ *estimates*:

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}}\|v_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n}{4\sigma}}\|v_1\|_{L^1\cap H^{-\sigma}},$$

$$\|v(t,\cdot)\|_{\dot{H}^\sigma} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}}\|v_0\|_{L^1\cap \dot{H}^\sigma} + (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}}\|v_1\|_{L^1\cap L^2},$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-1}\|v_0\|_{L^1\cap \dot H^\sigma} + (1+t)^{-\frac{n}{4\sigma}-1}\|v_1\|_{L^1\cap L^2},$$

$$\|v(t,\cdot)\|_{\dot H^k} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{k}{2\sigma}}\|v_0\|_{L^1\cap \dot H^k} + (1+t)^{-\frac{n}{4\sigma}-\frac{k}{2\sigma}}\|v_1\|_{L^1\cap \dot H^{k-\sigma}}$$

for all $k \geq 0$, *and the* $L^2 - L^2$ *estimates*

$$\|v(t,\cdot)\|_{L^2} \lesssim \|v_0\|_{L^2} + (1+t)\|v_1\|_{L^2},$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\|v_0\|_{H^\sigma} + \|v_1\|_{L^2},$$

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|v_0\|_{H^\sigma} + (1+t)^{-\frac{1}{2}}\|v_1\|_{L^2}.$$

3. Proof of our main results. In this section we present the proofs of Theorems 1 and 2. Before going into details we would like to mention that, unlike the case of nonlinearity $\big||D|^a u\big|^p$, the case with $|u_t|^p$ in the right-hand side will be proceeded with some necessary changes, mainly due to the estimates of the nonlinear term in the integral $\displaystyle\int_0^t K_1(t-\tau,x)|u_t(\tau,x)|^p\,d\tau$. It is a quite different situation, when $|u_t|^p$ is replaced by $|u|^p$. In that case, the fractional Gagliardo – Nirenberg inequality allows us to derive the admissible range of p for the global (in time) existence of solutions.

***3.1. Proof of Theorem* 1.** The proof follows the Banach fixed point method for the solution mapping that will be shown to be Lipschitz for small data which are arbitrarily chosen. We introduce now the data space $A := (L^1 \cap H^s) \times (L^1 \cap H^{s-\sigma})$. The statements from Proposition 2 suggest us to use the auxiliary space $X(t) = C([0,t], H^s) \cap C^1([0,t], H^{s-\sigma})$ with the norm

$$\|u\|_{X(t)} = \sup_{\tau\in[0,t]}\Big((1+\tau)^{\frac{n-2\sigma}{4\delta}}\|u(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n}{4\delta}}\|u_t(\tau,\cdot)\|_{L^2} +$$

$$+(1+\tau)^{\frac{n+2(s-\sigma)}{4\delta}}\|u_t(\tau,\cdot)\|_{\dot H^{s-\sigma}} + (1+\tau)^{\frac{n-2\sigma+2s}{4\delta}}\|u(\tau,\cdot)\|_{\dot H^s}\Big).$$

We define the mapping N between the data and the solution in the following way:

$$N : u \in X(t) \to Nu = K_0 * u_0 + K_1 * u_1 + \int_0^t K_1(t-\tau,x) *_{(x)} |u_t(\tau,x)|^p d\tau \in X(t). \tag{3}$$

By standard arguments the uniqueness, local and global in time existence of solutions to the Cauchy problem will be implied from the following pair of inequalities:

$$\|Nu\|_{X(t)} \leq C\|(u_0,u_1)\|_A + C\|u\|^p_{X(t)}, \tag{4}$$

$$\|Nu - Nv\|_{X(t)} \leq C\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{5}$$

From the estimates in Proposition 2 it is obvious that

$$\|\partial_t^j Nu(t,\cdot)\|_{H^k} \lesssim \|(u_0,u_1)\|_A + \int_0^t (1+t-\tau)^{-\frac{n+2(\sigma j+k-\sigma)}{4\delta}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^1\cap H^{k-\sigma+2j(\sigma-\delta)}}$$

with $k,\ j$ such that $k+j=s$ and $j=0,1$. We estimate the norm $\big\||u_t(\tau,\cdot)|^p\big\|_{L^1\cap H^{k-\sigma+2j(\sigma-\delta)}}$ by the following result (see Lemma 3 in Matsumura [4] for the general nonlinearity $f = f(u,u_t)$).

Lemma 1 (cf. Lemma 3 in [4] for general case of $f(v)$)**.** *Suppose that* $f = f(v)$ *has the form* $f = |v|^p,\ \pm v|v|^{p-1}, p > s$ *with an integer* $s \geq [n/2] + 1 + \left[\dfrac{[n/2]+1}{2}\right]$. *Then there exists a nondecreasing nonnegative function* $h(\cdot)$ *with* $h(0) = 0$ *that is a Lipschitz function at a neighborhood of* $v = 0$, *such that for every function* $v \in H^s$ *the following estimates are valid:*

$$\|f(v)\|_{H^s} \lesssim \|v\|^p_{H^s} h(\|v\|_{H^s}) \quad \textit{for all} \quad p > 1,$$

$$\|f(v)\|_{L^q} \lesssim \|v\|^{p-2/q}_{L^\infty} \|v\|^{2/q}_{L^2} h(\|v\|_{L^\infty}) \qquad \textit{for all} \quad p \geq 2, \quad 1 \leq q \leq 2, \quad pq \geq 2.$$

The proof of this statement was presented in details by von Wahl in [12].

To estimate different norms which are included in the norm of the solution space $X(t)$ we firstly derive the estimates for the norms containing u_t.

Differentiating the expression (3) with respect to t we obtain

$$\partial_t Nu = v_t + \int\limits_0^t \partial_t\big(K_1(t-\tau, x) *_{(x)} |u_t(\tau, x)|^p\big)\, d\tau,$$

where by v we denote the solution of linear Cauchy problem with the same initial data $u_0,\ u_1$.

Using the estimates (2) with $j = 1,\ \alpha = s$ for v_t and for second expression in the above formula we get

$$\big\|\partial_t(Nu)(t,\cdot)\big\|_{\dot{H}^s} \lesssim (1+t)^{-\frac{n}{4\delta}} \|(u_0, u_1)\|_A + \int\limits_0^t (1+t-\tau)^{-\frac{n+2s}{4\delta}} \big(\||u_t|^p\|_{L^1} + \||u_t|^p\|_{H^s}\big) d\tau. \tag{6}$$

With $s \geq [n/2] + 1 + \left[\dfrac{[n/2]+1}{2}\right]$ we apply Lemma 1 with $q = 1, p > 2$ to have the following useful estimates for the norms of $|u_t|^p$:

$$\big\||u_t|^p\big\|_{H^s} \lesssim \|u_t\|^p_{H^s} h(\|u_t\|_{H^s})$$

and

$$\||u_t|^p\|_{L^1} \lesssim \|u_t\|^{p-2/q}_\infty \|u_t\|^{2/q}_{L^2} h\big(\|u_t\|_\infty\big). \tag{7}$$

By the Sobolev embedding theorem for $L^\infty(\mathbb{R}^n)$ under the given condition for s and from the monotonicity of h, the right-hand side of the (7) is dominated by $\|u_t\|^p_{H^s} h\big(\|u_t\|_{H^s}\big)$ with some suitable multiplicative constant. Adding these last estimates together we get

$$\||u_t|^p\|_{L^1} + \||u_t|^p\|_{H^s} \lesssim \|u_t\|^p_{H^s} h\big(\|u_t\|_{H^s}\big).$$

Substituting the last estimate into the expression in the right-hand side of (6) we obtain

$$\big\|\partial_t(Nu)(t,\cdot)\big\|_{\dot{H}^s} \lesssim (1+t)^{-\frac{n}{4\delta}} \|(u_0, u_1)\|_A + \int\limits_0^t (1+t-\tau)^{-\frac{n+2s}{4\delta}} \|u_t\|^p_{H^s} h\big(\|u_t\|_{H^s}\big)\, d\tau.$$

Putting

$$M(t) = \sup_{0\le\tau\le t} (1+\tau)^{\frac{n}{4\delta}} \|u_t(\tau,\cdot)\|_{H^s}$$

and using the monotonicity of h we may conclude

$$\|\partial_t(Nu)(t,\cdot)\|_{\dot H^s} \lesssim$$

$$\lesssim (1+t)^{-\frac{n}{4\delta}} \|(u_0,u_1)\|_A + h(M(t)) \int\limits_0^t (M(\tau))^p (1+t-\tau)^{-\frac{n+2s}{4\delta}} (1+\tau)^{-p\left(\frac{n}{4\delta}\right)}\, d\tau.$$

Now recalling that for $\max\{\alpha;\beta\} > 1$ the inequality

$$\int\limits_0^t (1+t-\tau)^{-\alpha}(1+\tau)^{-\beta}\, d\tau \lesssim (1+t)^{-\min(\alpha,\beta)}$$

holds, we see that for $p > 1 + \dfrac{2s}{n}$

$$\big\|\partial_t(Nu)(t,\cdot)\big\|_{\dot H^s} \lesssim (1+t)^{-\frac{n}{4\delta}} \|(u_0,u_1)\|_A + (1+t)^{-\frac{n+2s}{4\delta}} (M(t))^p h(M(t)) \tag{8}$$

is valid.

By using the estimates for the corresponding linear Cauchy problem together with the second inequality in Lemma 1 for $q = 2$, we also have

$$\big\|\partial_t(Nu)(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\delta}} \|(u_0,u_1)\|_A + (1+t)^{-\frac{n}{4\delta}} (M(t))^p h(M(t)). \tag{9}$$

The estimates (8) and (9) imply the nonhomogeneous norm estimate

$$\big\|\partial_t(Nu)(t,\cdot)\big\|_{H^s} \lesssim (1+t)^{-\frac{n}{4\delta}} \|(u_0,u_1)\|_A + \big[(1+t)^{-\frac{n+2s}{4\delta}} + (1+t)^{-\frac{n}{4\delta}}\big] (M(t))^p h(M(t)).$$

The last inequality implies that for the local (in time) solution of $Nu = u$ with small data u_0, u_1 suitable chosen in A the inequality

$$M(t) \le c + \big[(1+t)^{-\frac{2s}{4\delta}} + 1\big] (M(t))^p h(M(t))$$

is valid. By a well-known argument, the last estimate implies that $M(t)$ is bounded for small data u_0, u_1 suitable chosen, i.e.,

$$\big\|u_t(t,\cdot)\big\|_{H^s} \lesssim (1+t)^{-\frac{n}{4\delta}}. \tag{10}$$

The last estimate is still weak and is not the one we need to prove. However, we will use it as an intermediate step in the following estimate.

Next, we will strengthen estimate (10) by differentiating directly both sides in the equation (11). For the local (in time) solution of $u = Nu$ the estimate

$$\big\|\partial_t^i \partial_x^\alpha u(t,\cdot)\big\|_{L^2} \lesssim \big\|\partial_t^i \partial_x^\alpha v(t,\cdot)\big\|_{L^2} + \int\limits_0^t \big\|\partial_t^i \partial_x^\alpha \big(K_1(t-\tau,x) *_{(x)} f(u_t)(\tau,x)\big)\big\|_{L^2}\, d\tau$$

is satisfied. The first term in the right-hand side has a growth $(1+t)^{-\frac{n+2\alpha+2\sigma i-2\sigma}{4\delta}}\|(u_0,u_1)\|_A$ from the estimates for the corresponding linear cauchy problem. The second term containing an integral, we estimate as follows:

$$\int\limits_0^t \big\|\partial_t^j\partial_x^\alpha\big(K_1(t-\tau,x)*_{(x)}f(u_t)(\tau,x)\big)\big\|_{L^2}d\tau \lesssim \int\limits_0^t (1+t-\tau)^{-\frac{n+2\alpha+2\sigma j-2\sigma}{4\delta}}\big\||u_t|^p\big\|_{H^{|\alpha|+2(j-1)\delta}}$$

for $j+|\alpha|\leq s$. The term containing $h\big(\|u_t\|_{H^{k-\sigma+2j(\sigma-\delta)}}\big)$ is bounded thanks to the estimate (10) and by the monotonicity of $h(\cdot)$. Therefore it can be dropped out.

We finally arrive at

$$\big\|\partial_t^j u(t,\cdot)\big\|_{H^k}\lesssim (1+t)^{-\frac{n+2k+2\sigma j-2\sigma}{4\delta}}\|(u_0,u_1)\|_A+(1+t)^{-\frac{n+2k+2\sigma j-2\sigma}{4\delta}}\|u_t\|^p_{H^{k-\sigma+j(\sigma-\delta)}}$$

provided that $p>\dfrac{n+2s}{n}$ for the last Matsumura technique to be applied. This estimate implies the first inequality (4).

Now we will present the main steps how to prove the second estimate of (5), that is

$$\|Nu-Nv\|_{X(t)}\leq C\|u-v\|_{X(t}\Big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\Big).$$

For $f(u)=|u_t|^p$ writing

$$\|Nu-Nv\|_{X(t)}=\left\|\int\limits_0^t K_1(t-\tau,x)*_{(x)}\big(f\big(u(\tau,x)\big)-f\big(v(\tau,x)\big)\big)\,d\tau\right\|_{X(t)}$$

by means of the estimate for the corresponding linear Cauchy problem we estimate several norms of the difference $f(u(s,x))-f(v(s,x))$. The estimate of the $L^1\cap L^2$-norm can be obtained by noting that

$$|f(u)-f(v)|\lesssim|u_t-v_t|\big(|u_t|^{p-1}+|v_t|^{p-1}\big)$$

(this fact can be proved, for example, by applying Lagrange's mean value theorem). Hence, by Hölder's inequality, we see that

$$\big\|f(u(\tau))-f(v(\tau))\big\|_{L^1}\lesssim\|u_t(\tau)-v_t(\tau)\|_{L^p}\Big(\|u_t(\tau)\|^{p-1}_{L^p}+\|v_t(\tau)\|^{p-1}_{L^p}\Big),$$

$$\big\|f(u(\tau))-f(v(\tau))\big\|_{L^2}\lesssim\|u_t(\tau)-v_t(\tau)\|_{L^{2p}}\Big(\|u_t(\tau)\|^{p-1}_{L^{2p}}+\|v_t(\tau)\|^{p-1}_{L^{2p}}\Big).$$

The L^p- and L^{2p}-norms of the difference u_t-v_t are estimated by the fractional Gagliardo–Nirenberg inequality and they are dominated by $\|u-v\|_{X(t)}$. More precisely, to estimate the norm $\|w(\tau,\cdot)\|_{L^{kp}}$, $k=1,m$, with $w=u_t-v_t,u_t,v_t$ we apply the fractional Gagliardo–Nirenberg inequality in the form

$$\|w(\tau,\cdot)\|_{L^q}\lesssim\big\||D|^{s-\sigma}w(\tau,\cdot)\big\|^{\theta_{0,s-\sigma}(q,2)}_{L^2}\|w(\tau,\cdot)\|^{1-\theta_{0,s-\sigma}(q,2)}_{L^2}$$

where for $q=p,\ 2p\geq 2$ we need

$$\theta_{0,s-\sigma}(q,2)=\frac{n}{s-\sigma}\left(\frac{1}{2}-\frac{1}{q}\right)\in\big[0,1\big).$$

This leads to the additional condition for p that requires $p < \dfrac{n}{[n-2(s-\sigma)]^+}$ which is always valid if we suppose $s > \dfrac{n}{2} + \sigma$.

The more difficult estimates are those that should be established with the Sobolev norms in the definition of $X(t)$. For example, we will explain how to estimate $\big\||u_t|^p - |v_t|^p\big\|_{H^\gamma}$ for $\gamma > 0$. By the estimates (2), in order to bound the norm $\|u-v\|_{X(t)}$, we pay attention to two special values $\gamma = \alpha + 2(j-1)\delta$ with $(j,\alpha) = (0,s)$ or $(j,\alpha) = (1, s-\sigma)$. These values correspond to $\gamma = s - 2\delta$ and $\gamma = s - \sigma$. Here we apply the following estimate for the nonhomogeneous scales (known as the Kato – Ponce inequality, see [3]). We formulate this well-known inequality below for reference.

Denote by $J^s(f) = (1-\Delta)^{s/2}(f) = \mathcal{F}^{-1}\big((1+4\pi^2|\xi|^2)^{s/2}\mathcal{F}(f)(\xi)\big)$, and $H^{s,p} := \big\{f \in L^p(\mathbb{R}^n) : J^s(f) \in L^p(\mathbb{R}^n\big\}$ for $s > 0$ — special fractional Sobolev spaces on $\mathbb{R}^n$.

Proposition 4 (Kato – Ponce inequality). *For all functions $f \in H^{s,p_2} \cap L^{q_1}$ and $g \in H^{s,q_2} \cap L^{p_1}$ it holds*

$$\|J^s(fg)\|_{L^r(\mathbb{R}^n)} \lesssim \big[\|f\|_{L^{p_1}(\mathbb{R}^n)}\|J^s g\|_{L^{q_1}(\mathbb{R}^n)} + \|J^s f\|_{L^{p_2}\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)}\big],$$

where $s > 0$ and $\dfrac{1}{r} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$ for $1 < r < \infty,\ 1 < p_1,\ q_2 \le \infty,\ 1 < p_2,\ q_1 < \infty$.

The next estimates for the nonlinear part are carried out with the application of Lemma 1 instead of the composition lemma. We write $g(w) = |w|^p$, $G(w) = \dfrac{1}{p} g'(w) = w|w|^{p-2}$. By the fundamental theorem of calculus it follows

$$g(u_t(s)) - g(v_t(s)) = p\int_0^1 (u_t(s) - v_t(s))G(\theta u_t + (1-\theta)v_t)\,d\theta.$$

Therefore,

$$\||u_t|^p - |v_t|^p\|_{H^\gamma} \lesssim \int_0^1 \big\|(u_t(s) - v_t(s))G(\theta u_t + (1-\theta)v_t)\big\|_{H^\gamma}\,d\theta.$$

Applying the Kato – Ponce inequality for the H^γ-norm of the product fg, where $f := G(\theta u_t + (1 - \theta)v_t)$, $g := u_t(s) - v_t(s)$ in the right-hand side of the above estimate with $r = p_2 = q_1 = 2, = p$, $p_1 = q_2 = \infty$ we obtain

$$\big\||u_t|^p - |v_t|^p\big\|_{H^\gamma} \lesssim \int_0^1 \big\|(u_t(s) - v_t(s))\big\|_{H^\gamma}\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{L^\infty}\,d\theta +$$

$$+\big\|(u_t(s) - v_t(s))\big\|_{L^\infty}\int_0^1 \big\|G(\theta u_t + (1-\theta)v_t)\big\|_{H^\gamma}\,d\theta.$$

The norm $\big\|(u_t(s) - v_t(s))\big\|_{H^\gamma}$ is estimated by the Gagliardo – Nirenberg inequality and is dominated from above by the norms that appear in the definition of the solution space $X(t)$. The terms $\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{L^\infty}$ and $\big\|(u_t(s) - v_t(s))\big\|_{L^\infty}$ are estimated from above by the Sobolev embedding. The more interesting is the last term $\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{H^\gamma}$.

In the case of a general nonlinearity $f(u_t)$ that has power growth $f(w) \sim |w|^p$ (see the Remark 1 below), we apply the Lemma 1 once again, since $G(w)$ has the power growth as $|w|^{p-1}$ too. This brings

$$\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{H^\gamma} \lesssim \big\|\theta u_t + (1-\theta)v_t\big\|_{H^\gamma}^{p-1} h\big(\|\theta u_t + (1-\theta)v_t\|_{H^\gamma}\big) \lesssim$$
$$\lesssim \big(\|u_t\|_{H^\gamma}^{p-1} + \|v_t\|_{H^\gamma}^{p-1}\big) h\big(\|u_t\|_{H^\gamma} + \|v_t\|_{H^\gamma}\big).$$

The factor $h\big(\|u_t\|_{H^\gamma} + \|v_t\|_{H^\gamma}\big)$ can be dropped out of the integral by the monotonicity of h and by the boundedness of $\|u_t\|_{H^\gamma}$ and $\|v_t\|_{H^\gamma}$ (we see that $\gamma \le s - \sigma$ in all cases of our interests). The second estimate (5) follows from the above arguments. The condition for p now becomes $p-1 > s-\sigma$ that is true since $\sigma > 1$. We should note that we need also the condition that $s-\sigma \in \mathbb{N}$ for the validity of Lemma 1 applied to G.

However in the case $f(u_t) = |u_t|^p$ the estimate can be obtained directly by means of the Corollary 2. According this composition result we may derive

$$\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{H^\gamma} \lesssim \big\|G(\theta u_t + (1-\theta)v_t)\big\|_{H^{s-\sigma}} \lesssim$$
$$\lesssim \big(\|G(u_t)\|_{H^{s-\sigma}}^{p-1} + \|G(v_t)\|_{H^{s-\sigma}}^{p-1}\big).$$

The last inequality leads to the second estimate (5).

By (4) and (5) the existence of local (in time) solution follows immediately for large data and the existence of global (in time) solutions will be guaranteed for sufficiently small data.

Theorem 1 is proved.

***Remark* 1.** We have proved in [8] another global existence result with the same nonlinearity $|u_t|^p$ but only for the special case $\delta = \dfrac{\sigma}{2}$. In that paper, in order to estimate the integral term including $|u_t|^p$ in the Sobolev scales we had applied the following composition lemma (see [7]).

Proposition 5. *Let $p > 1$ and $v \in H^{s,m}$, where $s > \dfrac{n}{m}$ and $s \in (0,p)$. Then the following estimate holds*:

$$\||v|^p\|_{H^{s,m}} \le C\|v\|_{H^{s,m}}\|v\|_{L^\infty}^{p-1}.$$

The following corollary follows from Proposition 5 immediately.

***Corollary* 1.** *Under the assumptions of Proposition* 5 *it holds*

$$\big\||v|^p\big\|_{\dot H^{s,m}} \le C\|v\|_{\dot H^{s,m}}\|v\|_{L^\infty}^{p-1}.$$

It seems that the conditions $s > \left[\dfrac{n}{2}\right] + 1 + \left[\dfrac{[n/2]+1}{2}\right]$ and s integer are too strong in the statement of Theorem 1. However we present the proof using Lemma 1 since this approach is simple and it does not require more detailed information about the function $h(\cdot)$ except the monotonicity and the Lipschitz property near 0. Additionally, the condition $p > s$ comes simply from the assumption that $f(v) = |v|^p \in C^s$. Moreover, this approach also allows us to consider a broader class of the nonlinearities $f(u_t)$. Actually, we can apply this method to obtain the following general result.

Theorem 3. *Consider the Cauchy problem for the structurally damped model*

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = f(u_t), \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

with $\sigma \geq 1,\ \mu > 0,\ \delta \in \left(\dfrac{\sigma}{2}, \sigma\right)$ *and* $s \geq [n/2] + 1 + \left[\dfrac{[n/2]+1}{2}\right]$ *such that* $s,\ \sigma$ *are integer. The function* $f(v)$ *satisfies the following conditions*:

$$f(v) \in C^s,$$

$$|f(v)| \lesssim |v|^p,$$

$$|f^{(\alpha)}(v)| \lesssim |v|^{p-\alpha} \quad \text{for} \quad 1 \leq \alpha < \min\{s; p\},$$

$$|f^{(\alpha)}(v)| \lesssim 1 \quad \text{if} \quad p \leq \alpha < s.$$

The data (u_0, u_1) *are supposed to belong to the function space* $(L^1 \cap H^s) \times (L^1 \cap H^{s-\sigma})$ *with* $n > 2\sigma$. *Then for all*

$$p \in \left(1 + \frac{2s}{n}, \frac{n}{(n + 2\sigma - 2s)_+}\right)$$

there exists a uniquely determined global (*in time*) *small data solution in* $C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-\sigma})$.

We see that in the above theorem, the condition for p is reduced to $p > 1 + \dfrac{2s}{n}$ when $s > \dfrac{n}{2} + \sigma$ and we do not need the inequality $p > s$ in order to guarantee the global existence with small data result in Theorem 3.

The proof of Theorem 3 coincides to that of Theorem 1 with the exploitation of the following result (Lemma 3 in [4]) which generalizes Lemma 1.

Lemma 2. *Suppose that* $f = f(v) \in C^s$, *where* $s \geq \left[\dfrac{n}{2}\right] + 1 + \left[\dfrac{[n/2]+1}{2}\right]$, *satisfies all assumption in the statement of Theorem* 3. *Then there exists a nondecreasing nonnegative locally Lipschitz at* $v = 0$ *function* $h(\cdot)$, $h(0) = 0$, *which is a Lipschitz function at a neighborhood of* $v = 0$, *such that for every function* $v \in H^s$ *the following estimates are valid*:

$$\|f(v)\|_{H^s} \lesssim \|v\|^p_{H^s} h(\|v\|_{H^s}) \quad \text{for all} \quad p > 1,$$

$$\|f(v)\|_{L^q} \lesssim \|v\|^{p-2/q}_{L^\infty} \|v\|^{2/q}_{L^2} h(\|v\|_{L^\infty}) \quad \text{for all} \quad p \geq 2, \quad 1 \leq q \leq 2, \quad pq \geq 2.$$

***3.2. Proof of Theorem* 2.** The proof of Theorem 2 can be served as an illustration to show how to use the composition lemma effectively to estimate the term containing $|u_t|^p$ in Sobolev scales. In order to prove Theorem 2 we need some additional decay estimates for the mixed derivatives of u which are easily obtained from the representation of solution and can be proved by the same method as for proving Lemma 2 (see [9] for details).

Proposition 6. *The solutions of the linear Cauchy problem for classical damped waves satisfy the following* $(L^1 \cap L^2) - L^2$ *estimates for mixed derivatives*:

$$\big\|v_t(t,\cdot)\big\|_{\dot{H}^k} \lesssim (1+t)^{-\frac{n}{4}-1-k/2} \|v_0\|_{L^1 \cap \dot{H}^{k+1}} + (1+t)^{-\frac{n}{4}-1-k/2} \|v_1\|_{L^1 \cap \dot{H}^k}$$

for all $k \geq 0$, *and the* $L^2 - L^2$ *estimates*

$$\|v_t(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{-k-1} \|v_0\|_{\dot{H}^{k+1}} + (1+t)^{-k} \|v_1\|_{\dot{H}^k}.$$

***Proof of Theorem* 2.** We introduce the data space $A := (L^2 \cap H^s) \times (L^1 \cap H^{s-1})$ and the solution space $X(t) = C([0,t], H^s) \cap C^1([0,t], H^{s-1})$ with the norm

$$\|u\|_{X(t)} := \sup_{0\le\tau\le t} \Big((1+\tau)^{\frac{n}{4}} \|u(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n}{4}+1} \|u_t(\tau,\cdot)\|_{L^2} +$$

$$+ (1+\tau)^{\frac{n}{4}+\frac{s+1}{2}} \||D|^{s-1} u_t(\tau,\cdot)\|_{L^2} + (1+\tau)^{\frac{n}{4}+\frac{s}{2}} \||D|^{s} u(\tau,\cdot)\|_{L^2} \Big).$$

As in the proof of Theorem 1 we define a mapping N in the following way:

$$N : u \in X(t) \to Nu \in X(t) \qquad \text{with}$$

$$Nu = G_0(t,x) *_{(x)} u_0 + G_1(t,x) *_{(x)} u_1 + \int_0^t G_1(t-\tau,x) *_{(x)} |u_t(\tau,\cdot)|^p d\tau. \tag{11}$$

By standard arguments the uniqueness, local and global in time existence will be concluded from the following pair of inequalities:

$$\|Nu\|_{X(t)} \le C\|(u_0,u_1)\|_A + C\|u\|^p_{X(t)}, \tag{12}$$

$$\|Nu - Nv\|_{X(t)} \le C\|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)} \big). \tag{13}$$

We begin by estimating the L^2 norm of Nu itself. To do that we apply the $(L^1 \cap L^2) - L^2$ estimates on the interval $\left[0, \frac{t}{2}\right]$ and $L^2 - L^2$ estimates on the interval $\left[\frac{t}{2}, t\right]$ to conclude

$$\big\|Nu(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}} \|(u_0,u_1)\|_A + \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}} \big\| |u_t(\tau,\cdot)|^p \big\|_{L^1\cap L^2} d\tau +$$

$$+ \int_{\frac{t}{2}}^{t} (1+t-\tau) \big\| |u_t(\tau,\cdot)|^p \big\|_{L^2} d\tau.$$

We get immediately from the definitions of L_p norms that

$$\big\| |u_t(\tau,\cdot)|^p \big\|_{L^1\cap L^2} \lesssim \|u_t(\tau,\cdot)\|^p_{L^p} + \|u_t(\tau,\cdot)\|^p_{L^{2p}}.$$

To estimate the norm $\|u_t(\tau,\cdot)\|_{L^{kp}}$, $k = 1, 2$, we apply the fractional Gagliardo – Nirenberg inequality in the form

$$\|w(\tau,\cdot)\|_{L^q} \lesssim \big\||D|^{s-1} w(\tau,\cdot)\big\|^{\theta_{0,s-1}(q,2)}_{L^2} \|w(\tau,\cdot)\|^{1-\theta_{0,s-1}(q,2)}_{L^2}$$

with $w(\tau,\cdot) = u_t(\tau,\cdot)$, where for $q \ge 2$ we need

$$\theta_{0,s-1}(q,2) = \frac{n}{s-1}\left(\frac{1}{2} - \frac{1}{q}\right) \in [0,1)$$

that is, $2 \le q$ if $\dfrac{n}{2(s-1)} < 1$. Since $\theta_{0,s-1}(p,2) < \theta_{0,s-1}(2p,2)$ we obtain on the interval $\left(0, \dfrac{t}{2}\right)$ the estimate

$$\int\limits_0^{t/2} (1+t-\tau)^{-\frac{n}{4}} \||u_t(\tau,\cdot)|^p\|_{L^1\cap L^2} d\tau \lesssim (1+t)^{-\frac{n}{4}} \|u\|_{X(t)}^p \int\limits_0^{t/2} (1+\tau)^{-p\left(\frac{n}{4}+\frac{s+1}{2(s-1)}n\left(\frac{1}{2}-\frac{1}{p}\right)\right)} d\tau.$$

We see that

$$-p\left(\frac{n}{4} + n\frac{s+1}{2(s-1)}\left(\frac{1}{2}-\frac{1}{p}\right)\right) < -1 \qquad \text{for} \quad p > 1 + \frac{n+2(s-1)}{ns}.$$

On the interval $\left(\dfrac{t}{2}, t\right)$, meanwhile, we proceed as follows:

$$\int\limits_{\frac{t}{2}}^{t} (1+t-\tau)\big\||u_t(\tau,\cdot)|^p\big\|_{L^m} \lesssim (1+t)\|u\|_{X(t)}^p \int\limits_{\frac{t}{2}}^{t} (1+\tau)^{-p\left(\frac{n}{4}+\frac{s+1}{2(s-1)}n\left(\frac{1}{2}-\frac{1}{2p}\right)\right)} d\tau.$$

The inequality

$$-p\left(\frac{n}{4} + \frac{s+1}{2(s-1)}n\left(\frac{1}{2}-\frac{1}{2p}\right)\right) < -\frac{n}{4} - 2$$

holds for

$$p > 1 + \frac{4(s-1)}{ns}.$$

Noting that $1 + \dfrac{4(s-1)}{ns} > 1 + \dfrac{n+2(s-1)}{ns}$ for $s > 1 + \dfrac{n}{2}$ we arrive at the first condition $p > 1 + \dfrac{4(s-1)}{ns}$ for the exponent p. By this approach we have proved the estimate (12) for Nu itself.

Differentiating (11) with respect to t we obtain

$$\partial_t Nu = v_t(t,x) + \int\limits_0^t \partial_t\big(G_1(t-\tau,x) *_{(x)} |u_t(\tau,\cdot)|^p\big)\, d\tau,$$

where we introduce $v := G_0(t,x) *_{(x)} u_0 + G_1(t,x) *_{(x)} u_1$ as the solution of the corresponding linear Cauchy problem with the initial data u_0, u_1.

Using the above techniques for getting the estimate for Nu we arrive at

$$(1+\tau)^{\frac{n}{4}+1}\big\|\partial_t Nu(\tau,\cdot)\big\|_{L^2} \le C\big\|(u_0,u_1)\big\|_A + C\|u\|_{X(t)}^p \qquad \text{for all} \quad \tau \in [0,t]$$

under the same assumption for p.

Now let us turn to estimate $\big\|\partial_t|D|^{s-1}Nu(t,\cdot)\big\|_{L^2}$. We use the following:

$$\partial_t|D|^{s-1}Nu = |D|^{s-1}v_t(t,x) + \int\limits_0^t \partial_t|D|^{s-1}\big(G_1(t-\tau,x) *_{(x)} |u_t(\tau,\cdot)|^p\big) d\tau.$$

Taking account of the estimate in Proposition 6 with $k = s-1$ and using the $(L^1 \cap L^2) - L^2$ estimates on the interval $\left(0, \dfrac{t}{2}\right)$ and $L^2 - L^2$ estimates on the interval $\left(\dfrac{t}{2}, t\right)$ we have

$$\big\|\partial_t |D|^{s-1}(Nu)\big\|_{L^2} \lesssim (1+t)^{-\frac{n+2(s+1)}{4}} \|(u_0,u_1)\|_A +$$

$$+ \int\limits_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n+2(s+1)}{4}} \big(\||u_t(\tau,\cdot)|^p\|_{L^1\cap L^2} + \||u_t(\tau,\cdot)|^p\|_{\dot H^{s-1}}\big)\, d\tau +$$

$$+ \int\limits_{\frac{t}{2}}^{t} (1+t-\tau)^{-s+1} \big(\||u_t(\tau,\cdot)|^p\|_{L^2} + \||u_t(\tau,\cdot)|^p\|_{\dot H^{s-1}}\big)\, d\tau.$$

The integrals with $\big\||u_t(\tau,\cdot)|^p\big\|_{L^1\cap L^2}$ or $\big\||u_t(\tau,\cdot)|^p\big\|_{L^2}$ will be handled as before if we apply the condition $p > 1 + \dfrac{4(s-1)}{ns}$. To estimate the integrals with $\big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-1}}$ we apply the composition result for $p > s$. In this way we may proceed further as follows:

$$\int\limits_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{s+1}{2}} \||u_t(\tau,\cdot)|^p\|_{\dot H^{s-1}} d\tau \lesssim$$

$$\lesssim \int\limits_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{s+1}{2}} \||u_t(\tau,\cdot)|\|_{\dot H^{s-1}} \||u_t(\tau,\cdot)|\|_{L^\infty}^{p-1} d\tau \lesssim$$

$$\lesssim \int\limits_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{s+1}{2}} \||u_t(\tau,\cdot)|\|_{\dot H^{s-1}} \||u_t(\tau,\cdot)|\|_{H^{s_0}}^{p-1} d\tau \lesssim$$

$$\lesssim \int\limits_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{s+1}{2}} \||u_t(\tau,\cdot)|\|_{\dot H^{s-1}} \Big(\||u_t(\tau,\cdot)|\|_{L^2} + \||u_t(\tau,\cdot)|\|_{\dot H^{s-1}}\Big)^{p-1} d\tau$$

with $s - 1 > s_0 > \dfrac{n}{2}$. Using again the estimates from Proposition 3 we get

$$\int\limits_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{s+1}{2}} \big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-1}} d\tau \lesssim$$

$$\lesssim (1+t)^{-\frac{n}{4}-\frac{s+1}{2}} \|u\|_{X(t)}^p \int\limits_0^{t/2} (1+\tau)^{-\frac{n}{4}-1} \Big((1+\tau)^{-\frac{n}{4}-1} + (1+\tau)^{-\frac{n}{4}-\frac{s+1}{2}}\Big)^{p-1} d\tau. \qquad (14)$$

It is obvious that for $p > 2$ and $s > 1$ the integral in (14) is uniformly bounded. The same argument is applied on the interval $\left(\dfrac{t}{2}, t\right)$ to conclude that

$$(1+\tau)^{\frac{n}{4}+\frac{s+1}{2}} \||u_t(\tau,\cdot)|^p\|_{\dot{H}^{s-1}} \lesssim \|(u_0,u_1)\|_A + \|u\|_{X(t)}^p \qquad \text{for all} \quad \tau \in (0,t).$$

An analogous reasoning leads to the other estimates which are required by the definition of $X(t)$-norm.

The second inequality (13) is obtained by an analogous approach that was carried out in the proof of Theorem 1. Namely, the $L^1 \cap L^2$-norm of $f(u(s,x)) - f(v(s,x))$ for $f(u) = |u_t|^p$ is estimated by using

$$\big|f(u) - f(v)\big| \lesssim |u_t - v_t|\big(|u_t|^{p-1} + |v_t|^{p-1}\big),$$

and by Hölder's inequality, which leads to

$$\big\|f(u(\tau)) - f(v(\tau))\big\|_{L^1} \lesssim \|u_t(\tau) - v_t(\tau)\|_{L^p}\Big(\|u_t(\tau)\|_{L^p}^{p-1} + \|v_t(\tau)\|_{L^p}^{p-1}\Big),$$

$$\big\|f(u(\tau)) - f(v(\tau))\big\|_{L^2} \lesssim \|u_t(\tau) - v_t(\tau)\|_{L^{2p}}\Big(\|u_t(\tau)\|_{L^{2p}}^{p-1} + \|v_t(\tau)\|_{L^{2p}}^{p-1}\Big).$$

After that the L^p- and L^{2p}-norms of the difference $u_t - v_t$ are estimated by the fractional Gagliardo–Nirenberg inequality and they are dominated by $\|u - v\|_{X(t)}$.

Now we will use another version of Kato–Ponce inequality which is stated in terms of homogeneous Sobolev spaces.

Proposition 7 (Kato–Ponce inequality for the homogeneous Sobolev spaces, see [3]). *For all functions $f \in \dot{H}^{s,p_2} \cap L^{q_1}$ and $g \in \dot{H}^{s,q_2} \cap L^{p_1}$ it holds*

$$\big\||D|^s(fg)\big\|_{L^r(\mathbb{R}^n)} \lesssim \big[\|f\|_{L^{p_1}(\mathbb{R}^n)}\||D|^s g\|_{L^{q_1}(\mathbb{R}^n)} + \||D|^s f\|_{L^{p_2}\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)}\big]$$

where $s > 0$ and $\dfrac{1}{r} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$ for $1 < r < \infty,\ 1 < p_1,\ q_2 \le \infty,\ 1 < p_2,\ q_1 < \infty$.

By writing

$$\big\||u_t|^p - |v_t|^p\big\|_{\dot{H}^\gamma} \lesssim \int\limits_0^1 \big\|(u_t(s) - v_t(s))G(\theta u_t + (1-\theta)v_t)\big\|_{\dot{H}^\gamma}\, d\theta$$

and by applying the Kato–Ponce inequality for the $\dot{H}^\gamma$-norm for the product fg with $f := G(\theta u_t + (1-\theta)v_t)$ and $g := u_t(s) - v_t(s)$ in the right-hand side of the above estimate with suitable constants $p_1,\ q_1,\ p_1,\ q_2 > 0$ we are able to bound the norm $\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{\dot{H}^{\gamma,q_1}}$ for $G = u|u|^{p-2}$ with some constant $q_1 > 0$ to be chosen later.

The norm in $\dot{H}^{\gamma,q_2}$ for $G\big(\theta u_t + (1-\theta)v_t\big)$ can be estimated if we apply the general composition result which is stated for a broader class of functions. We introduce the class $\operatorname{Lip}\mu$ in the following (see [7]).

Definition 1. *Let $\mu > 0,\ N \in \mathbb{N}_0$ and $0 < \alpha \le 1$ such that $\mu = N + \alpha$. Then we define*

$$\operatorname{Lip}\mu = \left\{ f \in C^{N,loc}(\mathbb{R}) : f^{(j)}(0) = 0,\ j = 0,\ldots,N,\ \textit{and}\ \sup_{t_0 \ne t_1} \frac{\big|f^{(N)}(t_0) - f^{(N)}(t_1)\big|}{|t_0 - t_1|^\alpha} < \infty \right\}.$$

Further we put

$$\|f\|_{\operatorname{Lip}\mu} = \sum_{j=0}^{N-1} \frac{|f^{(j)}(t)|}{|t|^{\mu-j}} + \sup_{t_0 \ne t_1} \frac{\big|f^{(N)}(t_0) - f^{(N)}(t_1)\big|}{|t_0 - t_1|^\alpha}.$$

It is clear that $|t|^\mu \in \mathrm{Lip}\,\mu,\ t|t|^{\mu-2} \in \mathrm{Lip}\,(\mu-1)$ for $\mu > 1$. The following helpful general composition result for the class $\mathrm{Lip}\,\mu$ was obtained in [7]. Let us denote $\sigma_p = n \max\left\{0; \dfrac{1}{p} - 1\right\}$.

Proposition 8 (Theorem 6.3.4 (i) in [7]). *Let $\sigma_p < s < \mu$ and $\mu > 1$.*

Then there exists some constant c such that

$$\|G(f)\|_{F^s_{p,q}} \le c\|G\|_{\mathrm{Lip}\,\mu}\|f\|_{F^s_{p,q}}\|f\|^{\mu-1}_{L^\infty}$$

holds for all $f \in F^s_{p,q} \cap L_\infty$ and all $G \in \mathrm{Lip}\,\mu$.

Proposition 8 together with the Sobolev embedding imply immediately the following consequence in the supercritical case $s > \dfrac{n}{2}$.

***Corollary* 2.** *Let $s \in \left(\dfrac{n}{2}, p\right)$. Denote either $G(u) = |u|^p$ or $G = \pm u|u|^{p-1}$ with $p > 1$. Then for all $u \in H^s$ the following composition estimate holds*:

$$\|G(u)\|_{H^s} \lesssim \|u\|^p_{H^s}.$$

With $G = u|u|^{p-2}$, by choosing $p_i,\ q_i,\ i = 1, 2$, as in the proof of Theorem 1 and by estimating the homogeneous $\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{\dot H^\gamma}$ from above by the nonhomogeneous norm $\big\|G(\theta u_t + (1-\theta)v_t)\big\|_{H^\gamma}$ Corollary 2 allows us now to obtain the second estimate (13). It is possible since $p - 1 > \gamma = s - \sigma$ for $\sigma > 1,\ p > s$ and both of the norms $\|u_t\|_{L^2},\ \|u_t\|_{\dot H^\gamma}$ are included in the norm of $X(t)$.

Theorem 2 is proved.

***Remark* 2.** We have applied two different approaches to deal with the nonlinearity of the form $|u_t|^p$. The first one, which was used in the proof of Theorem 1, follows the Matsumura technique to use some monotone function $h(t)$ that doesn't require the exact expression. Therefore the condition for p could be relaxed very nicely up to $p > 1 + \dfrac{2s}{n}$ for the general nonlinearity function $f(u_t)$ satisfying a certain power growth at infinity. However the regularity s of the solution must be higher than that which is obtained by the second approach in the proof of Theorem 2. We have applied the Runst – Sickel composition there to estimate the term $|u_t|^p$ with the weaker condition on s that is $s > 1 + \dfrac{n}{2}$. On the other hand the condition for the exponent p is more strict by following the second approach. We see that for the solvability of the Cauchy problem in Theorem 2 the admissible values for p must be larger than $\max\left\{2; s; 1 + \dfrac{4(s-1)}{ns}\right\}$.

4. Further studies and concluding remarks. The Cauchy problem that has been treated in Sections 2 and 3 could be generalized to the case of x-dependent coefficient models. We will deal with an operator $B = B(x)$ that behaves in some sense as $b(x,t)(-\Delta)^\sigma$.

Consider the x-dependent functional matrix $b(x,t) = b(x)$, where $b(x) = (b_{ij}(x))_{1\le i,j\le n},\ b_{ij} = b_{ji}$ and $b_{ij} \in C^0$. We assume that the $(n \times n)$-matrix $b = b(x)$ satisfies the following growth condition at infinity:

$$b_0\big(1 + |x|\big)^\beta|\xi|^2 \le b(x)\xi\cdot\xi \le b_1\big(1 + |x|\big)^\beta|\xi|^2$$

for some positive constants b_0, b_1.

In order to define $b(x)(-\Delta)^\sigma$, we restrict ourselves to the following reduction of the divergence-like form. Assume that $\sqrt{B}$ is a self-adjoint operator acting on $L^2(\Omega)$ such that the following identity for the scalar product is satisfied:

$$\langle\sqrt{B}u, \sqrt{B}v\rangle_2 = \Big\langle b(x)(-\Delta)^{\sigma/2}u, (-\Delta)^{\sigma/2}v\Big\rangle_2, \quad u, v \in \mathcal{D}\big(\sqrt{B}\big), \tag{15}$$

where $\mathcal{D}\big(\sqrt{B}\,\big)$ is the completion of C_0^∞ with respect to the norm

$$u \mapsto \Big(\Big\langle b(x)(-\Delta)^{\sigma/2}u, (-\Delta)^{\sigma/2}u\Big\rangle_2 + \|u\|^2\Big)^{1/2}.$$

The existence of such self-adjoint operator is guaranteed by a result on the Friedrichs extension of the nonnegative operators. Actually, the operator $b(x)(-\Delta)^\sigma$ may not be self-adjoint, however by considering the completion of the space C_0^∞ with respect to the above norm, in which all Cauchy sequences are converging, there exists in a unique way such an operator $\sqrt{B}$ that can be denoted by $\big(b(x)(-\bar{\Delta})^\sigma\big)^{1/2}$ in order to distinguish it from the usual notation $\big(b(x)(-\Delta)^\sigma\big)^{1/2}$.

We see that the formula (15) is valid for such functions u, v the fractional Laplacian of which have some behavior at infinity as $|x|^{-\beta/2}$. This means

$$\mathcal{D}\big(\sqrt{B}\big) \supset \big\{u \in H_0^{\delta/2}(\Omega) \colon \big(1 + |x|^{\beta/2}\big)|(-\Delta)^{\delta/2}u| \in L_2(\Omega)\big\}.$$

We define the operator B and its fractional powers as operator functions of $\sqrt{B}$ using the spectral calculus for operators (known also as the identity resolution of the spectrum). Particularly, $B := \big(\sqrt{B}\,\big)^2$, where $\sqrt{B}$ is the above Friedrichs construction based on $\big(b(x)(-\Delta)^\sigma\big)^{1/2}$.

Let Ω be the exterior of a compact set K in $\mathbb{R}^n$ with smooth boundary. Now we consider the following mixed problem in Ω:

$$\begin{aligned} &u_{tt} + Bu + u_t = 0, \quad x \in \Omega, \quad t > 0, \\ &u(0,x) = u_0(x), \\ &u_t(0,x) = u_1(x), \\ &u(t,x) = 0, x \in \partial\Omega, \quad t \geq 0. \end{aligned} \tag{16}$$

The decay estimates, which have a close relation to solutions to the exterior damped model with constant coefficients, will be proved by using a generalization of Gagliardo – Nirenberg's and Hardy's inequalities with arbitrary derivative orders. In order to obtain the decay estimates stated in the Conjecture 1, which will be formulated later, we need some information for the solution to the parabolic mixed problem

$$\begin{aligned} &v_t + Bv = 0, \quad x \in \Omega, \quad t > 0, \\ &v(0,x) = v_0(x), \\ &v(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0. \end{aligned}$$

First of all, using the equivalence of the Nash inequality and the $L^\infty - L^1$ decay of the semigroup $\{e^{-tB}\}$, we will show that the following inequality holds for the operator B:

$$\|f\|_{L^2}^{2+2(2\sigma-\beta)/n} \lesssim \langle Bf, f\rangle \|f\|_{L^1}^{2(2\sigma-\beta)/n} \tag{17}$$

for all function $f \in L^1(\Omega) \cap \mathcal{D}(B)$.

In fact, we have the following application of Hölder's inequality:

$$\|f\|_{L^2}^2 \le \|f\|_{L^1}^{1/p} \|f\|_{L^{(2p-1)/(p-1)}}^{(2p-1)/p}.$$

Choosing now $p = \dfrac{n+2\sigma-\beta}{2(2\sigma-\beta)}$ and raising the last inequality to $1 + \dfrac{2\sigma-\beta}{n}$ we arrive at

$$\|f\|_{L^2}^{2+2(2\sigma-\beta)/n} \le \|f\|_{L^1}^{2(2\sigma-\beta)/n} \|f\|_{L^{2n/(n+\beta-2\sigma)}}^2.$$

In order to estimate the norm $\|f\|_{L^{\frac{2n}{n+\beta-2\sigma}}}$ of f by the scalar product $\langle Bf, f\rangle$ we need the following generalization of the classical Hardy–Littlewood result on fractional derivatives inequality of Sobolev type which was obtained by E. Stein and G. Weiss in [11].

Let us introduce the operator T_λ acting on the functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ as follows:

$$T_\lambda : f \mapsto T_\lambda f = \int\limits_{\mathbb{R}^n} \frac{f(x)}{|x-y|^\lambda}\, dy, \quad 0 < \lambda < n.$$

Proposition 9 (Theorem **B*** in [11]). *Let* $0 < \lambda < n,\ 1 < p < \infty,\ \alpha < n/p',\ \beta < n/q,$ $\alpha + \beta \ge 0,\ 1/q = (1/p) + [(\lambda + \alpha + \beta)/n] - 1$. *If* $p \le q < \infty$, *then*

$$\left(\int\limits_{\mathbb{R}^n} \{|T_\lambda f|\, |x|^{-\beta}\}^q dx \right)^{1/q} \lesssim \left(\int\limits_{\mathbb{R}^n} \{|f(x)|\, |x|^{\alpha}\}^p \right)^{1/p}.$$

Now by the definition of the Riesz potential, it is clear that T_λ is related to the fractional Laplacian with a negative exponent by the following relation:

$$T_\lambda = c_\lambda (-\Delta)^{\frac{\lambda-n}{2}}$$

for $0 < \lambda < n$ with a constant $c_\lambda > 0$.

Proposition 9 can be rewritten in a more convenient form involved the fractional Laplacian with positive exponent as follows:

$$\|u\|_{L^q} \lesssim \big\| |x|^\alpha (-\Delta)^\mu u \big\|_{L^p}$$

for $0 < \mu < \dfrac{n}{2}$, $0 < \alpha < \dfrac{n}{p'}$ with $p,\ q,\ \alpha,\ \mu$ being related by $\dfrac{1}{q} = \dfrac{1}{p} + \dfrac{\alpha - 2\mu}{n}$.

Choosing

$$q = \frac{2n}{n+\beta-2\sigma}, \qquad p = \frac{n+2\sigma-\beta}{2(2\sigma-\beta}, \qquad \alpha = \frac{\beta}{2}$$

in the last inequality we get, for $\beta \in (0,n)$ and $\sigma \in (0,n)$, the following inequality:

$$\|f\|_{L^{\frac{2n}{n+\beta-2\sigma}}} \lesssim \big\| |x|^{\beta/2} \nabla^\sigma f \big\|_{L^2}.$$

Recalling the condition for B we arrive at the desired estimate (17).

It is well known (see [1]) that (17) is equivalent to the following relation:

$$\|e^{-tB}\|_{L^\infty} \lesssim \|f\|_{L^1} \quad \forall f \in L^1(\Omega) \quad \forall t > 0. \tag{18}$$

Usually, the inequalities (17) or (18) are not sufficient to imply the decay rate of $\{e^{-tB}\}$ in other $L^1 - L^q$ scales with arbitrary $q > 1$. The so-called Markov property including the positivity and the contraction for the semigroup $\{e^{-tB}\}$ must be satisfied in order to obtain further energy estimates for the solution u of the mixed problem (16). The positivity, however, does not hold for general σ even for the constant coefficient case.

For example, if we can prove the Markovian property for $\{e^{-tB}\}$, then the above Nash-type estimates allow us to obtain the following result.

Conjecture 1. *Suppose $n > 2\sigma$. Moreover, we assume that the initial data belong to*

$$(u_0, u_1) \in \mathcal{D}\big(\sqrt{B} \cap L^1(\Omega)\big) \times \big(L^2(\Omega) \cap L^1(\Omega)\big),$$

where B is defined as above. Then the solution of (16) *satisfies the following decay estimates for $t \geq 1$:*

$$\|u\|_{L_2} \lesssim t^{-\frac{n}{2(2\sigma-\beta)}} \Big(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_0\|_{L^2} + \|\big(\sqrt{B}+1\big)^{-1} u_1\|_{L^2}\Big),$$

$$\|\sqrt{B}u\|_{L_2} \lesssim t^{-\frac{n}{2(2\sigma-\beta)}-\frac{1}{2}} \Big(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|\sqrt{B}u_0\|_{L^2} + \|u_1\|_{L^2}\Big),$$

$$\|u_t\|_{L_2} \lesssim t^{-\frac{n}{2(2\sigma-\beta)}-1} \Big(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|\sqrt{B}u_0\|_{L^2} + \|u_1\|_{L^2}\Big).$$

After getting these estimates we can follow the approach presented in Section 3 to study the mixed problem

$$\begin{aligned}
&u_{tt} + Bu + u_t = |u_t|^p, \quad x \in \Omega, \quad t > 0, \\
&u(0, x) = u_0(x), \\
&u_t(0, x) = u_1(x), \\
&u(x, t) = 0, \quad x \in \partial\Omega, \quad t \geq 0.
\end{aligned}$$

In the next step of our study, we will try to answer the question proposed above about the Markov property of the semigroup $\{e^{-tB}\}$. The case of t-dependent coefficients brings some interest to our models and must be treated carefully. These results for the nonlinear problem with x-dependent coefficient will appear in a forthcoming paper.

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