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THE S-JERIBI ESSENTIAL SPECTRUM

СУТТЄВИЙ S-СПЕКТР ДЖЕРІБІ

We study some properties and results on the S-Jeribi essential spectrum of linear bounded operators on a Banach space. In particular, we give some criteria for coincidence of this spectrum for two linear operators and the relation of this type of spectrum with the well-known S-Schechter essential spectrum.

Вивчено властивості суттєвого S-спектра Джерібі для лінійних операторів у просторі Банаха та отримано деякі відповідні результати. Зокрема, сформульовано критерії збігу цих спектрів для двох лінійних операторів і встановлено зв'язок між цим типом спектра та відомим суттєвим S-спектром Шехтера.

1. Introduction. The spectral theory of operator pencils $\lambda S - T,\ \lambda \in \mathbb{C}$ (operator-valued functions of a complex argument) play a crucial role in several branches of mathematical physics (see, for example, [1, 5, 4, 11]), this notion has recently attracted the attention of many mathematicians. In particular, F. Abdmouleh, A. Ammar and A. Jeribi investigate, in [1], the S-Browder essential spectrum of bounded linear operators on a Banach space X and extended many results to various types of S-essential spectra. Moreover, in [6], we have introduced the notion of the S-Jeribi essential spectrum and we gave some localization of this S-essential spectrum using the notion of measure of noncompactness. An obvious question to ask, when introducing this notion, is whether the different results of the S-essential spectra continue to hold for the S-Jeribi essential spectrum?

This work continues research begun in [6], we will give many properties for the S-Jeribi essential spectrum. In particular, a characterization of the S-Jeribi essential spectrum is given, in Theorem 3.1, when the operator S is Fredholm with index zero. In Theorem 3.2, we show the existence of a weakly compact operator W for which the S-Jeribi essential spectrum of T coincides with the S-Schechter essential spectrum of the operator $(T+W)$. Furthermore, we state a condition for which the S-Jeribi essential spectrum coincides with the S-Schechter essential spectrum in Theorem 3.5.

The paper is organized as follows. Section 2 contains an overview of the necessary background. The main results of the paper are found in Section 3.

2. Preliminaries. The purpose of this section is to recall some results on Fredholm operators and S-essential spectra. Let X be a Banach space. Denote by $\mathcal{L}(X)$ (resp., $\mathcal{K}(X)$) the set of all bounded linear (resp., compact) operators on X. The nullity, $\alpha(T)$, of an operator T is defined as the dimension of $\ker(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of the range $R(T)$ in X.

The sets of upper and lower semi-Fredholm operators in $\mathcal{L}(X)$ are defined, respectively, by

$$\Phi_+(X) = \{T \in \mathcal{L}(X) \colon \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\},$$

$$\Phi_-(X) = \{T \in \mathcal{L}(X) \colon \text{ such that } \beta(T) < \infty\}.$$

The sets of Fredholm and semi-Fredholm operators are defined, respectively, by

$$\Phi(X) = \Phi_+(X) \cap \Phi_-(X) \quad \text{and} \quad \Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X).$$

The index of T, denoted $\operatorname{ind}(T)$, is defined as $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. An operator F is called a Fredholm perturbation if $T+F \in \Phi(X)$ whenever $T \in \Phi(X)$. Denote by $\mathcal{F}(X)$ the set of Fredholm perturbations. Note that $\mathcal{F}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ (see [7, 17]). For more details about the theory of Fredholm operators we refer to [2, 3, 14, 17].

Now, we recall the definitions of strictly singular operators and weakly compact operators. The concept of strictly singular operators was introduced in the pioneering paper by Kato [12] as a generalization of the notion of compact operators.

Definition 2.1. *Let $T \in \mathcal{L}(X)$. T is said to be strictly singular if the restriction of T to any infinite-dimensional subspace of X is not an homeomorphism.*

The set of strictly singular operators on X is denoted by $\mathcal{S}(X)$. Note that $\mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [8, 12]). If X is a Hilbert space, then $\mathcal{K}(X) = \mathcal{S}(X)$. In general, we have

$$\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}(X).$$

Definition 2.2. *An operator $T \in \mathcal{L}(X)$ is said to be weakly compact if for every bounded subset $B \subset X$, $T(B)$ is relatively weakly compact.*

The family of weakly compact operators on X is denoted by $\mathcal{W}(X)$. Note that $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [8, 12]). Let us notice that, according to [15] (Theorem 1), we have $\mathcal{W}(L_1(\Omega, d\mu)) = \mathcal{S}(L_1(\Omega, d\mu))$, where $(\Omega, \Sigma, d\mu)$ be an arbitrary positive measure space. If $1 < p < \infty$, $L_p(\Omega, d\mu)$ is reflexive and then $\mathcal{L}(L_p(\Omega, d\mu)) = \mathcal{W}(L_p(\Omega, d\mu))$. Moreover, from [7] (Theorem 5.2) we deduce that

$$\mathcal{L}(L_p(\Omega, d\mu)) \subsetneq \mathcal{S}(L_p(\Omega, d\mu)) \subsetneq \mathcal{W}(L_p(\Omega, d\mu)) \text{ with } p \neq 2.$$

For $p = 2$, we obtain

$$\mathcal{L}(L_2(\Omega, d\mu)) = \mathcal{S}(L_2(\Omega, d\mu)) = \mathcal{W}(L_2(\Omega, d\mu)).$$

Let T, S be two bounded linear operators on $\mathcal{L}(X)$ such that S is non-zero. We define the S-spectrum of T as $\sigma_S(T) = \mathbb{C} \setminus \rho_S(T)$, where $\rho_S(T)$ is the S-resolvent set defined by

$$\rho_S(T) = \{\lambda \in \mathbb{C} \colon \lambda S - T \text{ has a bounded inverse}\}.$$

There are several definitions of the S-essential spectra of an operator T defined on a Banach space (see, for example, [1, 4, 5]). In this paper, we are concerned with the S-Jerbi essential spectrum introduced in [6] and the S-Schechter essential spectrum.

Definition 2.3. *Let S and T be two bounded linear operators on a Banach space X. The S-Jeribi essential spectrum of an operator $T \in \mathcal{L}(X)$ is defined by*

$$\sigma_{j,S}(T) := \bigcap_{K \in \mathcal{W}^*(X)} \sigma_S(T + K),$$

where $\mathcal{W}^*(X)$ stands for each one of the sets $\mathcal{W}(X)$ and $\mathcal{S}(X)$. In other words, $\mathcal{W}^*(X)$ is either the set $\mathcal{W}(X)$ or the set $\mathcal{S}(X)$.

The S-Schechter essential spectrum of an operator T, denoted $\sigma_{e_{1,S}}(T)$, is defined as follow:

$$\sigma_{e_{1,S}}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_S(T + K).$$

Since $\mathcal{K}(X) \subset \mathcal{W}^*(X)$, then $\sigma_{j,S}(T) \subset \sigma_{e_{1,S}}(T)$. Note that if $S = I$ (the identity operator), we recover the definition of the Jeribi essential spectrum (see [9, 10, 13]) and the Schechter essential spectrum, respectively,

$$\sigma_j(T) = \bigcap_{K \in \mathcal{W}^*(X)} \sigma(T+K) \qquad \text{and} \qquad \sigma_{e_1}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K).$$

We conclude this section with a useful characterization of the S-Schechter essential spectrum by means of Fredholm operators established in [5].

Proposition 2.1. *Let T, S be in $\mathcal{L}(X)$. Then*

$$\lambda \notin \sigma_{e_{1,S}}(T) \quad \textit{if and only if} \quad (\lambda S - T) \in \Phi(X) \qquad \textit{and} \qquad \operatorname{ind}(\lambda S - T) = 0.$$

3. Main results. The purpose of this section is to present our main results on the S-Jeribi essential spectrum of linear bounded operators on a Banach space X. In this paper, we choose the definition of the S-Jeribi essential spectrum of T, $\sigma_{j,S}(T)$, when K belongs to $\mathcal{W}(X)$

$$\sigma_{j,S}(T) := \bigcap_{K \in \mathcal{W}(X)} \sigma_S(T+K).$$

From this definition, it is obviously that the S-Jeribi essential spectrum is invariant under weakly compact perturbations. Now, from the stability of the S-Jeribi essential spectrum under weakly compact perturbations and Proposition 2.1, we can easily obtain a characterization of the S-Jeribi essential spectrum by means of Fredholm operators.

Lemma 3.1. *Let T, $S \in \mathcal{L}(X)$. Then the S-Jeribi essential spectrum is given by*

$$\sigma_{j,S}(T) = \mathbb{C} \setminus \{\lambda \in \mathbb{C} \colon \exists W \in \mathcal{W}(X), (\lambda S - T - W) \in \Phi(X) \quad \textit{with} \quad \operatorname{ind}(\lambda S - T - W) = 0\}.$$

The following theorems give a characteristic of the S-Jeribi essential spectrum, in the particular case, when S is a Fredholm operator with zero index. Let us consider $S \in \Phi(X)$, then by Atkinson theorem (see [2], Theorem 4.46) there exist compact operators $K_1, K_2 \in \mathcal{K}(X)$ and an operator $B \in \mathcal{L}(X)$ such that

$$SB = I + K_1 \qquad \text{and} \qquad BS = I + K_2.$$

The statement (iii) of [1] (Proposition 2.2) remains valid for the S-Jeribi essential spectrum. The following theorem holds.

Theorem 3.1. *Let $T \in \mathcal{L}(X)$, $S \in \Phi(X)$ such that $\operatorname{ind}(S) = 0$. Then*

$$\sigma_{j,S}(T) = \sigma_j(TB),$$

where B is introduced above.

Proof. Let $\lambda \notin \sigma_j(TB)$, then there exists a weakly compact operator W on X such that $(\lambda I - TB - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda I - TB - W) = 0$. Since $S \in \Phi(X)$, the use of [14] (Theorem 5) implies that $(\lambda I - TB - W)S \in \Phi(X)$ with $\operatorname{ind}((\lambda I - TB - W)S) = 0$, i.e., $(\lambda S - T - TK_2 - WS) \in$ $\in \Phi(X)$ with $\operatorname{ind}(\lambda S - T - TK_2 - WS) = 0$. Since the operator TK_2 is compact, it follows from the stability of Fredholm operators under compact perturbations that $(\lambda S - T - WS) \in \Phi(X)$ and $\operatorname{ind}(\lambda S - T - WS) = 0$. Hence, there exists a weakly compact operator $W' = WS$ such that $(\lambda S - T - W') \in \Phi(X)$ and $\operatorname{ind}(\lambda S - T - W') = 0$. So, $\lambda \notin \sigma_{j,S}(T)$. This shows the inclusion $\sigma_{j,S}(T) \subset \sigma_j(TB)$.

By the same argument we get the inverse inclusion.

Theorem 3.1 is proved.

The next main theorem provides conditions for the existence of a weakly compact operator W' on X such that the S-Jeribi essential spectrum of an operator T coincides with the S-Schechter essential spectrum of the operator $(T + W')$.

Theorem 3.2. *Let* $T, S \in \mathcal{L}(X)$, $S \in \Phi(X)$ *with* $\operatorname{ind}(S) = 0$. *Then there exists a weakly compact operator* W' *on* X *such that* $\sigma_{e_{1,S}}(T + W') = \sigma_{j,S}(T)$.

Proof. Let $\lambda \notin \sigma_{j,S}(T)$. From Theorem 3.1, it follows that $\lambda \notin \sigma_j(TB)$. So, there exists a weakly compact operator W on X such that $(\lambda I - TB - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda I - TB - W) = 0$. Since $S \in \Phi(X)$, the use of [14] (Theorem 5) shows that $(\lambda I - TB - W)S \in \Phi(X)$ with $\operatorname{ind}((\lambda I - TB - W)S) = 0$, i.e., $(\lambda S - T - TK_2 - WS) \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T - TK_2 - WS) = 0$. By using the fact that $TK_2 \in \mathcal{K}(X)$ and the stability of Fredholm operators under compact perturbations, we obtain $(\lambda S - T - WS) \in \Phi(X)$ and $\operatorname{ind}(\lambda S - T - WS) = 0$. Consequently, there exists a weakly compact operator $W' = WS$ such that $\lambda \notin \sigma_{e_{1,S}}(T + W')$. Hence, $\sigma_{e_{1,S}}(T + W') \subset \sigma_j(TB) = \sigma_{j,S}(T)$. The reverse inclusion is obvious since $\sigma_{j,S}(T) = \sigma_{j,S}(T + W) \subset \sigma_{e_{1,S}}(T + W)$ for all $W \in \mathcal{W}(X)$.

Theorem 3.2 is proved.

The other main result is the following theorem.

Theorem 3.3. *Let* T, $S \in \mathcal{L}(X)$ *such that* $0 \in \rho(T) \cap \rho(S)$. *Then, for all* $\lambda \neq 0$, *we have*

$$\lambda \in \sigma_{j,S}(T) \quad \textit{if and only if} \quad \lambda^{-1} \in \sigma_{j,S^{-1}}(T^{-1}).$$

Proof. Assume that $\lambda^{-1} \notin \sigma_{j,S^{-1}}(T^{-1})$. Then there exists a weakly compact operator W on X such that $(\lambda^{-1}S^{-1} - T^{-1} - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda^{-1}S^{-1} - T^{-1} - W) = 0$. We obtain (using the fact that T and S are Fredholm operators) $S(\lambda^{-1}S^{-1} - T^{-1} - W)T \in \Phi(X)$. Consequently, $-\lambda(\lambda^{-1}T - S - SWT) \in \Phi(X)$ with $\operatorname{ind}(-\lambda(\lambda^{-1}T - S - SWT)) = 0$. So, $(\lambda S - T - W'') \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T - W'') = 0$, where $W'' = -\lambda SWT \in \mathcal{W}(X)$. By using Lemma 3.1, we can see that $\lambda \notin \sigma_{j,S}(T)$. This proves that

$$\lambda \in \sigma_{j,S}(T) \Rightarrow \lambda^{-1} \in \sigma_{j,S^{-1}}(T^{-1}).$$

We now prove that $\lambda^{-1} \in \sigma_{j,S^{-1}}(T^{-1})$ implies $\lambda \in \sigma_{j,S}(T)$. To see this, let $\lambda \notin \sigma_{j,S}(T)$. Then there exists $W \in \mathcal{W}(X)$ such that $(\lambda S - T - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T - W) = 0$. The operator $(\lambda S - T - W)$ can be written in the form

$$(\lambda S - T - W) = -\lambda S \left(\lambda^{-1}S^{-1} - T^{-1} + \lambda^{-1}S^{-1}WT^{-1}\right) T. \tag{3.1}$$

We see that the second member of equation (3.1) is in $\Phi(X)$ with zero index. Applying the index theorem [2] (Theorem 4.43), we get

$$\left(\lambda^{-1}S^{-1} - T^{-1} + \lambda^{-1}S^{-1}WT^{-1}\right) \in \Phi(X)$$

with $\operatorname{ind}\left(\lambda^{-1}S^{-1} - T^{-1} + \lambda^{-1}S^{-1}WT^{-1}\right) = 0$. We put $W' = \lambda^{-1}S^{-1}WT^{-1}$, then $W' \in \mathcal{W}(X)$. Therefore, $\lambda^{-1} \notin \sigma_{j,S^{-1}}(T^{-1})$.

Theorem 3.3 is proved.

From Theorem 3.3 with $S = I$ we obtain the following corollary.

***Corollary* 3.1.** *Let* $T \in \mathcal{L}(X)$ *such that* $0 \in \rho(T)$. *Then, for all* $\lambda \neq 0$, *we have*

$$\lambda \in \sigma_j(T) \quad \textit{if and only if} \quad \lambda^{-1} \in \sigma_j(T^{-1}).$$

In next theorem, we will give a relation between the S-Jeribi essential spectrum and the S-Schechter essential spectrum.

Theorem 3.4. *Let* $T,\ S \in \mathcal{L}(X),\ W \in \mathcal{W}(X)$ *and* $\xi \in \rho_S(T+W)$. *We put* $T_\xi = (\xi S - T - W)^{-1}$. *Then, for* $\lambda \neq \xi$, *we have*

$$\lambda \in \sigma_{j,S}(T) \quad \textit{if and only if} \quad (-\lambda+\xi)^{-1} \in \sigma_{e_1}(T_\xi S).$$

Proof. Let $W \in \mathcal{W}(X)$. The operator $(\lambda S - T - W)$ can be written in the form

$$(\lambda S - T - W) = (-\lambda+\xi)(\xi S - T - W)((-\lambda+\xi)^{-1} - T_\xi S). \tag{3.2}$$

Suppose that $(-\lambda+\xi)^{-1} \notin \sigma_{e_1}(T_\xi S)$. Then we get $((-\lambda+\xi)^{-1}I - T_\xi S) \in \Phi(X)$ and $\operatorname{ind}((-\lambda+\xi)^{-1}I - T_\xi S) = 0$. Since $(\xi S - T - W) \in \Phi(X)$ with $\operatorname{ind}(\xi S - T - W) = 0$, then, by using [14] (Theorem 5), we see that the second member of equation (3.2) is a Fredholm operator with zero index. So, we have $(\lambda S - T - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T - W) = 0$, which means that $\lambda \notin \sigma_{e_{1,S}}(T+W)$, i.e., $\lambda \notin \sigma_{j,S}(T+W)$. By using the stability of S-Jeribi essential spectrum under weakly compact perturbations, we conclude that $\lambda \notin \sigma_{j,S}(T)$.

Conversely, assume that $\lambda \notin \sigma_{j,S}(T)$, then there exists $W \in \mathcal{W}(X)$ which satisfies $(\lambda S - T - W) \in \Phi(X)$ and ind $(\lambda S - T - W) = 0$. By equation (3.2), we have $(\xi S - T - W)((-\lambda+\xi)^{-1} - T_\xi S) \in \Phi(X)$. Since $(\xi S - T - W) \in \Phi(X)$, then by [2] (Theorem 4.43) we get $((-\lambda+\xi)^{-1} - T_\xi S) \in \Phi(X)$ and ind $((-\lambda+\xi)^{-1} - T_\xi S) = 0$. Hence, $(-\lambda+\xi)^{-1} \notin \sigma_{e_1}(T_\xi S)$.

Theorem 3.4 is proved.

In the next theorem, we state a condition under which the S-Jeribi essential spectrum coincides with the S-Schechter essential spectrum.

Theorem 3.5. *Let* X *be a Banach space,* $T,\ S \in \mathcal{L}(X)$ *and* $W \in \mathcal{W}(X)$. *If there exists* $\lambda \in \rho_S(T+W)$ *such that* $W(\lambda S - T - W)^{-1} \in \mathcal{F}(X)$, *then* $\sigma_{e_{1,S}}(T) = \sigma_{j,S}(T)$.

Proof. It suffices to show the following inclusion: $\sigma_{e_{1,S}}(T) \subset \sigma_{j,S}(T)$. Writing the operator $(\lambda S - T)$ as follows:

$$(\lambda S - T) = \left(I + W(\lambda S - T - W)^{-1}\right)(\lambda S - T - W) \quad \text{for any} \quad W \in \mathcal{W}(X). \tag{3.3}$$

Let $\lambda \in \mathbb{C}$ such that $\lambda \notin \sigma_{j,S}(T)$. Then there exists a weakly compact operator W on X satisfies $(\lambda S - T - W) \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T - W) = 0$. Using the hypothesis, Theorem 5 cited in [14] and together with equation (3.3), we get $(\lambda S - T) \in \Phi(X)$ with $\operatorname{ind}(\lambda S - T) = 0$. Then $\lambda \notin \sigma_{e_{1,S}}(T)$.

Theorem 3.5 is proved.

We next show that Theorems 4.7 and 4.8 of [16] are also valid for the S-Jeribi essential spectrum.

Theorem 3.6. *Let* $T, S, L \in \mathcal{L}(X)$ *and* $W \in \mathcal{W}(X)$. *If for some* $\xi \in \rho_S(T+W) \cap \rho_S(L+W)$ *the operator* $(\xi S - T - W)^{-1} - (\xi S - L - W)^{-1}$ *is compact on* X, *then*

$$\sigma_{j,S}(T) = \sigma_{j,S}(L).$$

Proof. Let $\lambda \in \sigma_{j,S}(T)$. Then, according to Theorem 3.4, we have $(-\lambda+\xi)^{-1} \in \sigma_{e_1}(T_\xi S)$ for all $\lambda \neq \xi$. We see that this is equivalent to $(-\lambda+\xi)^{-1} \in \sigma_{e_1}((\xi S - L - W)^{-1}S)$ since $((\xi S - T - W)^{-1} - (\xi S - L - W)^{-1})S \in \mathcal{K}(X)$ and the Schechter essential spectrum is invariant under compact perturbations. This means that $(-\lambda+\xi)^{-1} \in \sigma_{e_1}(L_\xi S)$ where $L_\xi = (\xi S - L - W)^{-1}$. Hence, by Theorem 3.4 again, we have $\lambda \in \sigma_{j,S}(L)$. This proves the claim.

Theorem 3.6 is proved.

Theorem 3.7. *Let T, L operators in $\Phi(X)$ such that $\mathrm{ind}(T) = \mathrm{ind}(L) = 0$ and $S \in \mathcal{L}(X)$. If the operator $(A - B)$ is compact, then*

$$\sigma_{j,S}(T) = \sigma_{j,S}(L).$$

Operators A, B are defined in $\mathcal{L}(X)$ such that $TA = I - K_1$ and $LB = I - K_2$, K_1, $K_2 \in \mathcal{K}(X)$.

Proof. For any scalar $\lambda \in \mathbb{C}$ and $W \in \mathcal{W}(X)$, we can write

$$(\lambda S - T - W)A - (\lambda S - L - W)B = K_1 - K_2 + (\lambda S - W)(A - B). \tag{3.4}$$

Let $\lambda \notin \sigma_{j,S}(T)$. Then there exists a weakly compact operator W on X such that $(\lambda S - T - W) \in \in \Phi(X)$ with $\mathrm{ind}(\lambda S - T - W) = 0$. So, $(\lambda S - T - W)A \in \Phi(X)$ since $A \in \Phi(X)$. The second member of equation (3.4) is compact, then, by the stability of Fredholm operators under compact perturbations, we see that $(\lambda S - L - W)B \in \Phi(X)$. This implies, by the use of index theorem, that $(\lambda S - L - W) \in \Phi(X)$ and $\mathrm{ind}(\lambda S - L - W) = 0$. We deduce from Lemma 3.1 that $\lambda \notin \sigma_{j,S}(L)$. So,

$$\sigma_{j,S}(L) \subset \sigma_{j,S}(T).$$

Similarly we prove the opposite inclusion.

Theorem 3.7 is proved.

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