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A STUDY OF MODULES OVER RINGS AND THEIR EXTENSIONS

ДОСЛІДЖЕННЯ МОДУЛІВ НАД КІЛЬЦЯМИ ТА ЇХ РОЗШИРЕННЯ

We study the transfer of properties of some types of modules under identity preserving ring homomorphisms. These studies seem to be overlooked in literature. We have picked a few types of modules and provided proofs for these properties that are transferable and suitable counterexamples for the properties that are not transferable.

Вивчається проблема передачі властивостей деяких типів модулів під дією гомоморфізмів, що зберігають одиницю. Подібні дослідження, здається, відсутні в літературі. Вибрано кілька типів модулів та наведено доведення для тих властивостей, що передаються, та придатні контрприклади для тих властивостей, що не передаються.

Introduction. In this note we consider the ordinary ring extension under identity preserving ring homomorphisms between two rings with identity and use it to study the transfer of properties of different types of modules and their submodules over these rings. These studies seem to be overlooked in literature and in several cases may not be trivial to prove. Moreover, several properties do not transfer under such extensions. In this paper we have picked some very commonly studied classes of modules and planned to cover more in future. Almost all statements and counter statements are defended and followed by examples and counterexamples. As a summary, the lists of types of modules that we have studied here are given at the end.

1. Preliminaries. Throughout this note, unless otherwise specified, the term ring means an associative ring with identity $1 \neq 0$, homomorphisms are identity preserving, and all modules are unital. For a ring R, we shall denote the Jacobson radical of R by $J(R)$. We will denote the annihilator of a module M by $\operatorname{ann} M$. The symbols $\mathbb{N}^*$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ ($\mathbb{Z}_n$, $n \in \mathbb{N}^*$) will denote the set of all positive integers, integers, rational numbers, real numbers and integers modulo n, respectively.

Suppose that R and S are rings and $\Phi\colon R \to S$ is a ring homomorphism, then S is called an extension of R by Φ. Let V_S be a right S-module. Then V_S can be considered as a right R-module by the action

$$x \cdot r = x\Phi(r) \quad \forall x \in V_S \quad \forall r \in R \tag{1.1}$$

and is denoted by $V_{|R}$. As usual, we will write $x \cdot r = xr$.

Note that, if $\Phi(1) = 1$, and V_S is a unital S-module, then $V_{|R}$ is a unital R-module too. However, in general this statement is not true. For instance, suppose that the ring R contains a nontrivial idempotent e and consider the ring homomorphism $\Phi\colon \mathbb{Z} \longrightarrow R$, which is defined by $\Phi(n) = ne$ for all $n \in \mathbb{Z}$. Then R is a unital R-module, but not a unital $\mathbb{Z}$-module.

For any right R-module M, a pair (E, i) is an injective envelope of M in case E is an injective right R-module and $i\colon M \longrightarrow E$ is an essential monomorphism. A projective cover of M means an epimorphism $\theta\colon P \longrightarrow M$, where P_R is a projective module and $\ker\theta$ is a small (= superfluous) submodule of P. A projective module M is called semiperfect if every factor module of M has a projective cover.

The following propositions list some useful and known results.

Proposition 1.1 ([2, p. 439], Theorem 3.7). (a) *The following conditions on a nonzero ring R are equivalent*:

(i) *R is semisimple*;

(ii) *every right R-module is projective*;

(iii) *every right R-module is injective*;

(iv) *every right R-module is semisimple.*

(b) *Let M be a semisimple right R-module and $N \subseteq M$ be a submodule. Then N is simple iff N is indecomposable.*

Proposition 1.2. (a) *Let I be an ideal in a ring R and $\overline{R} = R/I$. Let M be a right $\overline{R}$-module, which is, therefore a right R- module. If M_R has a projective cover over R, then $M_{\overline{R}}$ also has a projective cover over $\overline{R}$.*

(b) *Every module over an Artinian ring has a projective cover; hence, every projective module over an Artinian ring is semiperfect.*

Proof. See [6] (Lemma 24.15) and [7] (Corollary 5.4).

2. Main results. Some elementary observations are entered in the following theorem.

Theorem 2.1. *Suppose that R and S are rings and $\Phi \colon R \to S$ is a ring homomorphism. Then the following statements hold. The converse of each statement also holds provided Φ is surjective.*

(a) *If $V_{|R}$ is generated by X, then V_S is generated by X.*

(b) *If $V_{|R}$ is a cyclic R-module, then V_S is a cyclic S-module.*

(c) *If V' is an S-submodule of V_S, then V' is an R-submodule of $V_{|R}$.*

(d) *If $V_{|R}$ is a simple R-module, then V_S is a simple S-module.*

(e) *If $V_{|R}$ is an indecomposable R-module, then V_S is an indecomposable S-module.*

(f) *If $V_{|R}$ is Artinian (Noetherian), then V_S is Artinian (Noetherian).*

Proof. (a) Since $V_{|R}$ is generated by X, by (1.1),

$$v = \sum_{i=1}^{n} x_i r_i = \sum_{i=1}^{n} x_i \Phi(r_i),$$

where $v \in V_S$, $n \in \mathbb{N}^*$, $x_i \in X$, $r_i \in R$. Thus, V_S is also generated by X.

Conversely, suppose that V_S is generated by X and Φ is surjective, then $s_i = \Phi(r_i)$, $s_i \in S$, $r_i \in R$. Hence, by (1.1),

$$v = \sum_{i=1}^{n} x_i s_i = \sum_{i=1}^{n} x_i \Phi(r_i) = \sum_{i=1}^{n} x_i r_i,$$

where $v \in V_S$, $n \in \mathbb{N}^*$, $x_i \in X$. Thus, $V_{|R}$ is also generated by X.

(b) This is a special case of (a).

(c) This holds by following the definition (1.1).

(d) Follows by applying (c).

(e) By (c), V_S is a decomposable S-module iff $V_S = M_S \oplus N_S$, where M_S and N_S are S-submodules of V_S iff $V_{|R} = M_R \oplus N_R$, where M_R and N_R are R-submodules of $V_{|R}$ iff $V_{|R}$ is a decomposable R-module.

(f) Again by (c), any descending (ascending) chain of S-submodules of V_S is a descending (ascending) chain of R-submodules of $V_{|R}$ and vice versa.

Theorem 2.1 is proved.

***Corollary* 2.1.** *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. Then $V_{|R}$ is a semisimple R-module iff V_S is a semisimple S-module.*

Proof. By Theorem 2.1(c) and Theorem 2.1(d), $V_{|R}$ is a semisimple R-module iff $V_{|R} = \sum_{i\in I} \oplus V_i$, where V'_{iS} are simple R-submodules of $V_{|R}$ iff $V_S = \sum_{i\in I} \oplus V_i$, where V'_{iS} are simple S-submodules of V_S iff V_S is a semisimple S-module.

Some examples and counterexamples relevant to the above observations are as under.

***Example* 2.1.** Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$. Define $\Phi\colon R \to \mathbb{Z}_2$ by $\Phi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = a$. Then Φ is a surjective ring homomorphism and $\mathbb{Z}_2$ is a cyclic $\mathbb{Z}_2$-module generated by 1. Consider $\mathbb{Z}_2$ as an R-module by the action

$$x \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = x\Phi\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = xa, \quad x, a, b \in \mathbb{Z}_2.$$

Since $(1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ and $(1)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$, $\mathbb{Z}_2$ is also a cyclic R-module generated by 1.

***Example* 2.2.** Moreover, in the case of simple modules, obviously in Example 2.1, $\mathbb{Z}_2$ is a simple $\mathbb{Z}_2$-module as well as an R-module.

***Example* 2.3.** Let $R = UTM_2(\mathbb{Z}_6)$ be the ring of all 2×2 upper triangular matrices over $\mathbb{Z}_6$. Define $\Phi\colon R \to \mathbb{Z}_6$ by $\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = a$. Then Φ is a surjective ring homomorphism. Since $3\mathbb{Z}_6$ and $4\mathbb{Z}_6$ are simple $\mathbb{Z}_6$-submodules of $\mathbb{Z}_6$ and $\mathbb{Z}_6 = 3\mathbb{Z}_6 \oplus 4\mathbb{Z}_6$, by Theorem 2.1(d), $3\mathbb{Z}_{6|R}$ and $4\mathbb{Z}_{6|R}$ are simple R-modules. Therefore, $\mathbb{Z}_6$ is a semisimple R-module.

The following is a counterexample of Corollary 2.1 for Φ not to be surjective.

***Example* 2.4.** Let $R = UTM_2(\mathbb{Z}_6)$ and $\Phi\colon \mathbb{Z}_6 \longrightarrow R$ be defined by $\Phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then Φ is a ring homomorphism. Since $J(R) = \begin{pmatrix} 0 & \mathbb{Z}_6 \\ 0 & 0 \end{pmatrix}$, R is not a semisimple R-module. Consider R as a $\mathbb{Z}_6$-module by the action

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Phi(x) = \begin{pmatrix} ax & bx \\ 0 & cx \end{pmatrix}, \quad x, a, b, c \in \mathbb{Z}_6.$$

Since $\mathbb{Z}_6$ is semisimple, by Proposition 1.1(a), R is a semisimple $\mathbb{Z}_6$-module.

The following is a counterexample of Theorem 2.1(a, c, d, f), and Corollary 2.1 for Φ not to be surjective.

***Example* 2.5.** Let $\Phi\colon \mathbb{Z} \longrightarrow \mathbb{Q}$ be defined by $\Phi(a) = a$. Then Φ is a ring homomorphism. Consider $\mathbb{Q}$ as a $\mathbb{Q}$-module. Since it is a field, it is a simple $\mathbb{Q}$-module. Therefore, it is a semisimple and a cyclic $\mathbb{Q}$-module. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then it is not a cyclic $\mathbb{Z}$-module. Since it contains $\mathbb{Z}$ as a $\mathbb{Z}$-submodule, which is neither semisimple nor Artinian, it is not a semisimple or an Artinian $\mathbb{Z}$-module.

***Example* 2.6.** Let $R = UTM_2(\mathbb{Z}_6)$. Define $\Phi\colon R \longrightarrow \mathbb{Z}_6$ by $\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = a$. Since $3\mathbb{Z}_6$ is a simple $\mathbb{Z}_6$-module and R-module, it is an indecomposable $\mathbb{Z}_6$-module and R-module.

The following is a counterexample of Theorem 2.1(e, f) for Φ not to be surjective.

***Example* 2.7.** Let $\Phi : \mathbb{Q} \longrightarrow \mathbb{R}$ be defined by $\Phi(a) = a$. Then Φ is a ring homomorphism. Since $\mathbb{R}_{\mathbb{R}}$ is simple, it is indecomposable. Consider $\mathbb{R}$ as a $\mathbb{Q}$-module. Since $\mathbb{Q}$ is semisimple, by Proposition 1.1 (a), $\mathbb{R}$ is a semisimple $\mathbb{Q}$-module, which is not simple, since it contains $\mathbb{Q}$ as a $\mathbb{Q}$-submodule. Therefore, by Proposition 1.1(b), it is decomposable. Let $s_i = \left\{\sqrt{p_i} : p_i \text{ is a prime number}\right\}$, $i \in \mathbb{N}^*$. Then the chain of $\mathbb{Q}$-submodules of $\mathbb{R}$, $\langle s_1 \rangle \subseteq \left\langle \overset{2}{\underset{i=1}{\cup}} s_i \right\rangle \subseteq \left\langle \overset{3}{\underset{i=1}{\cup}} s_i \right\rangle \subseteq \ldots$ does not terminate. Thus, $\mathbb{R}$ is not a Noetherian $\mathbb{Q}$-module, but since it is a field, it is a Noetherian $\mathbb{R}$-module.

Theorem 2.2. *Suppose that R and S are rings and $\Phi : R \to S$ is a surjective ring homomorphism, then*:

(a) V'_R *is an essential R-submodule of* $V_{|R}$ *iff* V'_S *is an essential S-submodule of* V_S.

(b) $\mathrm{Soc}(V_{|R}) = \mathrm{Soc}(V_S)$, *where* $\mathrm{Soc}(V_{|R})$ *and* $\mathrm{Soc}(V_S)$ *denote, respectively, the socle of* $V_{|R}$ *and* V_S.

Proof. (a) Suppose that V'_R is an essential R-submodule of $V_{|R}$. Since Φ is surjective, by Theorem 2.1(c), V'_S is an S-submodule of V_S. Suppose that M_S is an S-submodule of V_S such that $V'_S \cap M_S = 0$. By Theorem 2.1(c), M_R is an R-submodule of $V_{|R}$. Therefore, $V'_R \cap M_R = 0$. Since V'_R is an essential R-submodule of $V_{|R}$, $M_R = 0$. Therefore, $M_S = 0$. Thus, V'_S is an essential S-submodule of V_S. Similarly, we can verify the only if part.

(b) Since $\mathrm{Soc}(V_{|R}) = \cap\left\{L \leq V_{|R} : L \text{ is essential in} V_{|R}\right\}$, the result is satisfied immediately from (a).

Theorem 2.2 is proved.

***Example* 2.8.** Let $R = UTM_2(\mathbb{Z}_8)$ and $\Phi : R \to \mathbb{Z}_8$ be defined by $\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = a$. Then Φ is a surjective ring homomorphism. Consider $\mathbb{Z}_8$ as an R-module by the action

$$x\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = x\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = xa, \quad x, a, b, c \in \mathbb{Z}_8.$$

Since $\mathbb{Z}_8$ and R are Artinian, where $J(\mathbb{Z}_8) = 2\mathbb{Z}_8$ and $J(R) = \begin{pmatrix} 2\mathbb{Z}_8 & \mathbb{Z}_8 \\ 0 & 2\mathbb{Z}_8 \end{pmatrix}$, by [1] (Corollary 15.21),

$$\mathrm{Soc}(\mathbb{Z}_{8_{\mathbb{Z}_8}}) = \mathrm{Soc}(\mathbb{Z}_{8_{|R}}) = \{x \in \mathbb{Z}_8 : xa = 0,\ a \in 2\mathbb{Z}_8\} = 4\mathbb{Z}_8.$$

Theorem 2.3. *Suppose that R and S are rings and $\Phi : R \to S$ is a surjective ring homomorphism, then*:

(a) V'_R *is a maximal R-submodule of* $V_{|R}$ *iff* V'_S *is a maximal S-submodule of* V_S;

(b) V'_R *is a small R-submodule of* $V_{|R}$ *iff* V'_S *is a small S-submodule of* V_S;

(c) $J(V_{|R}) = J(V_S)$, *where* $J(V_{|R})$ *and* $J(V_S)$ *denote, respectively, the Jacobson radicals of* $V_{|R}$ *and* V_S.

Proof. (a) Suppose that V'_R is a maximal submodule of $V_{|R}$. Then $V'_R \neq V_{|R}$. Hence, $V'_S \neq V_S$. Since Φ is surjective, by Theorem 2.1(c), V'_S is a submodule of V_S. Now suppose that there exists a submodule M_S of V_S such that $V'_S \subseteq M_S \subseteq V_S$. By Theorem 2.1(c), M_R is a submodule of $V_{|R}$. Since, $V'_R \subseteq M_R \subseteq V_{|R}$ and V'_R is a maximal submodule of $V_{|R}$, $M_R = V'_R$ or $M_R = V_{|R}$. Thus, $M_S = V'_S$ or $M_S = V_S$. Therefore, V'_S is is a maximal submodule of V_S. Similarly, we can verify the only if part.

(b) Suppose that V'_R is a small submodule of $V_{|R}$. Since Φ is surjective, by Theorem 2.1(c), V'_S is a submodule of V_S. Suppose that M_S is a submodule of V_S such that $V'_S + M_S = V_S$. By Theorem 2.1(c), M_R is a submodule of $V_{|R}$. Therefore, $V'_R + M_R = V_{|R}$. Since V'_R is a small submodule of $V_{|R}$, $M_R = V_{|R}$. Therefore, $M_S = V_S$. Thus, V'_S is a small submodule of V_S. Similarly, we can verify the only if part.

(c) By definition of the Jacobson radical of a module and (a) or (b), we have $J(V_{|R}) = J(V_S)$.

Theorem 2.3 is proved.

***Example* 2.9.** Let $R = \begin{pmatrix} \mathbb{Z}_6 & 0 \\ 0 & \mathbb{Z}_6 \end{pmatrix}$ and $\Phi\colon R \longrightarrow \mathbb{Z}_6$ be defined by $\Phi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = a$. Then Φ is a surjective ring homomorphism. Since $J(\mathbb{Z}_6) = 0$, $J(R) = 0$. Consider $\mathbb{Z}_6$ as an R-module. Since R is semisimple, by Proposition 1.1(a), $\mathbb{Z}_6$ is a projective R-module. Thus, $J(\mathbb{Z}_{6_{|R}}) = \mathbb{Z}_6 J(R) = 0 = J(\mathbb{Z}_6)$ [6] (Theorem 24.7).

The following example shows that if Φ is not surjective, then

$$\operatorname{Soc}(V_{|R}) \neq \operatorname{Soc}(V_S) \qquad \text{and} \qquad J(V_{|R}) \neq J(V_S).$$

***Example* 2.10.** Let $\Phi : \mathbb{Z} \longrightarrow \mathbb{Q}$ be defined by $\Phi(a) = a$. Then $\operatorname{Soc}(\mathbb{Q}_{\mathbb{Q}}) = \mathbb{Q}$ and $J(\mathbb{Q}_{\mathbb{Q}}) = 0$. Since $\mathbb{Q}_{\mathbb{Z}}$ has no maximal and no minimal $\mathbb{Z}$-submodules, $\operatorname{Soc}(\mathbb{Q}_{\mathbb{Z}}) = 0$ and $J(\mathbb{Q}_{\mathbb{Z}}) = \mathbb{Q}$.

Theorem 2.4. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is a faithful R-module, then V_S is a faithful S-module.*

Proof. Suppose $s \in S$ such that $vs = 0$ for all $v \in V_S$. Since Φ is surjective, there exists $r \in R$ such that $s = \Phi(r)$. Hence, $v\Phi(r) = 0$, which implies by (1.1), $vr = 0$ for all $v \in V_{|R}$. Since $V_{|R}$ is faithful, $r = 0$. Therefore, $s = \Phi(0) = 0$. Thus, V_S is a faithful S-module.

Theorem 2.4 is proved.

***Example* 2.11.** Let $R = UTM_2(\mathbb{Z}_6)$ and $\Phi\colon R \longrightarrow \mathbb{Z}_6$ be defined by $\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = a$. Consider $3\mathbb{Z}_6 = \{0, 3\}$ as a $\mathbb{Z}_6$-module. Then $ann\, 3\mathbb{Z}_6 = \{0, 2, 4\}$. Thus, $3\mathbb{Z}_6$ is not a faithful $\mathbb{Z}_6$-module. Now consider $3\mathbb{Z}_6$ as an R-module by the action

$$x \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = x\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = xa, \qquad x \in 3\mathbb{Z}_6, \quad a, b, c \in \mathbb{Z}_6.$$

Then

$$ann 3\mathbb{Z}_{6|R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R\colon a \in \{0, 2, 4\} \right\}.$$

Thus, $3\mathbb{Z}_{6|R}$ is not a faithful R-module.

The following example shows that the converse of Theorem 2.4 is not true in general.

***Example* 2.12.** Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix}\colon a, b \in \mathbb{R} \right\}$. Then R is a subring of $M_2(\mathbb{R})$. Define $\Phi\colon R \longrightarrow \mathbb{R}$ by $\Phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) = a - b$. Then Φ is a surjective ring homomorphism. Let $A = \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix}$. Then A is a left $\mathbb{R}$-module by the action $r \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix}$, $r \in \mathbb{R}$. If $0 \neq \begin{pmatrix} a \\ b \end{pmatrix} \in A$ and $r \begin{pmatrix} a \\ b \end{pmatrix} = 0$, then $ra = 0$ and $rb = 0$. Since $\mathbb{R}$ is a field, $r = 0$. Thus, A is a faithful $\mathbb{R}$-module. Consider A as a left R-module by the action

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \Phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right)\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} (a-b)a' \\ (a-b)b' \end{pmatrix}.$$

If $0 \neq \begin{pmatrix} a' \\ b' \end{pmatrix} \in A$ and $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = 0$, then $(a-b)a' = 0$ and $(a-b)b' = 0$. Since $\mathbb{R}$ is a field, $a = b$. Hence, $ann\, A = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R}\right\}$. Thus, A is not a faithful R-module.

The following is a counterexample for Φ not to be surjective.

***Example* 2.13.** Let $R = UTM_2(\mathbb{Z}_6)$ and $\Phi\colon 3\mathbb{Z}_6 \longrightarrow R$ be defined by $\Phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \in 3\mathbb{Z}_6 = \{0, 3\}$. Consider $A = \begin{pmatrix} 3\mathbb{Z}_6 \\ 3\mathbb{Z}_6 \end{pmatrix}$. Then A is a left R-module by the action

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} aa' + bb' \\ cb' \end{pmatrix} \in A, \quad a, b, c \in \mathbb{Z}_6, \quad a', b' \in 3\mathbb{Z}_6.$$

Also A is a left $3\mathbb{Z}_6$-module by the action

$$a\begin{pmatrix} b \\ c \end{pmatrix} = \Phi(a)\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} ab \\ ac \end{pmatrix}, \quad a, b, c \in 3\mathbb{Z}_6.$$

If $0 \neq \begin{pmatrix} b \\ c \end{pmatrix}$ and $a\begin{pmatrix} b \\ c \end{pmatrix} = 0$, then $ab = 0$ and $ac = 0$. Since $a, b \in 3\mathbb{Z}_6$, $a = 0$. Thus, A is a faithful $3\mathbb{Z}_6$-module. But A is not a faithful R-module, since for all $\begin{pmatrix} a \\ b \end{pmatrix} \in A$, $\begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$.

Lemma 2.1. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $s \in S$ is not a zero divisor, then there exists $r \in R$, which is not a zero divisor such that $s = \Phi(r)$.*

Proof. Suppose $s \in S$ is not a zero divisor. Then for all $0 \neq s' \in S$, we have $ss' \neq 0$ and $s's \neq 0$. Since Φ is surjective, there exist r, $r' \in R$ such that $s = \Phi(r)$, $s' = \Phi(r')$. Since $\Phi(0) = 0$ and $s' \neq 0$, $r' \neq 0$. Therefore, $ss' \neq 0$ implies $\Phi(rr') \neq 0$ and so $rr' \neq 0$. Similarly, if $s's \neq 0$, then $r'r \neq 0$. Hence, r is not a zero divisor.

Lemma 2.1 is proved.

Theorem 2.5. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is a divisible R-module, then V_S is a divisible S-module.*

Proof. Suppose $s \in S$ is not a zero divisor and $v \in V_S$. Then by Lemma 2.1, there exists $r \in R$, which is not a zero divisor such that $s = \Phi(r)$. Since $V_{|R}$ is divisible, there exists $v' \in V_{|R}$ such that $v = v'r$, so by (1.1), $v = v'r = v'\Phi(r) = v's$. Thus, V_S is divisible.

Theorem 2.5 is proved.

***Example* 2.14.** Let $R = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$ and $\Phi: R \to \mathbb{Z}$ be defined by $\Phi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = a$. Then Φ is a surjective ring homomorphism. Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\mathbb{Z}$ is not divisible, since there does not exist $x \in \mathbb{Z}$ such that $3 = 2x$. Now consider $\mathbb{Z}$ as an R-module by the action

$$z\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = z\Phi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = za, \quad z, a, b \in \mathbb{Z}.$$

Then $\mathbb{Z}$ is not a divisible R-module, since for $r = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in R$, there does not exist $x \in \mathbb{Z}$ such that $5 = x \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

The converse of Theorem 2.5 is not true in general and the following example illustrates this.

***Example* 2.15.** Let $\Phi\colon \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\Phi(a) = \overline{a}$. Then Φ is a surjective ring homomorphism. For $\overline{1}$, $\overline{2} \in \mathbb{Z}_3$, we have $\overline{x} = \overline{1}\overline{(x)}$ for all $\overline{x} \in \mathbb{Z}_3$, $\overline{0} = \overline{2}\overline{(0)}$, $\overline{1} = \overline{2}\overline{(2)}$ and $\overline{2} = \overline{2}\overline{(1)}$. Thus, $\mathbb{Z}_3$ is a divisible $\mathbb{Z}_3$-module. Consider $\mathbb{Z}_3$ as a $\mathbb{Z}$-module by the action $\overline{a}z = \overline{a}\Phi(z) = \overline{az}$. Then $\mathbb{Z}_3$ is not a divisible $\mathbb{Z}$-module, since there does not exist $\overline{a} \in \mathbb{Z}_3$ such that $\overline{2} = 3\overline{a}$.

***Remark* 2.1.** Since every Abelian group is a unital $\mathbb{Z}$-module in a unique way [1, p. 27], the action of $\mathbb{Z}$ on $\mathbb{Z}_3$ in the previous example is the usual action by simple calculations.

The following is a counterexample for Φ not to be surjective.

***Example* 2.16.** Let $\Phi\colon \mathbb{R} \to \mathbb{R}[x]$ be defined by $\Phi(a) = a$. Then Φ is a ring homomorphism. Consider the ideal $V = \langle x^2 \rangle$ of $\mathbb{R}[x]$. Let $f(x) = x \in \mathbb{R}[x]$. Then there does not exist $g(x) \in V$ such that $x^2 = f(x)g(x)$, since $\deg(g(x)) \geq 2$. Thus, V is not a divisible $\mathbb{R}[x]$-module. Consider V as a left $\mathbb{R}$-module by the action $af(x) = \Phi(a)f(x) \in V$, $a \in \mathbb{R}$, $f(x) \in V$. Then for all $0 \neq r \in \mathbb{R}$ and $f(x) \in V$, there exists $r^{-1}f(x) \in V$ such that $f(x) = r[r^{-1}f(x)]$. Thus, V is a divisible $\mathbb{R}$-module.

Theorem 2.6. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism.*

(a) *If V_S is a torsion S-module, then $V_{|R}$ is a torsion R-module.*

(b) *If $V_{|R}$ is a torsion free R-module, then V_S is a torsion free S-module.*

Proof. (a) Suppose $v \in V_S$. Since V_S is a torsion S-module, there exists $s \in S$, which is not a zero divisor such that $vs = 0$. Therefore, by Lemma 2.1, there exists $r \in R$, which is not a zero divisor such that $s = \Phi(r)$. Hence, by (1.1), $0 = vs = v\Phi(r) = vr$. Thus, $V_{|R}$ is a torsion R-module.

(b) Suppose $s \in S$, which is not a zero divisor such that $vs = 0$. Then by Lemma 2.1, there exists $r \in R$, which is not a zero divisor such that $s = \Phi(r)$. Therefore, by (1.1), $0 = vs = v\Phi(r) = vr$. Since $V_{|R}$ is torsion free, $v = 0$. Thus, V_S is a torsion free S-module.

Theorem 2.6 is proved.

The converse of Theorem 2.6(a) or Theorem 2.6(b) is not true in general and the following example illustrates this.

***Example* 2.17.** Let $\Phi\colon \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\Phi(a) = \overline{a}$. Since $\mathbb{Z}_3$ is a field, it is a torsion free $\mathbb{Z}_3$-module, whence it is not a torsion $\mathbb{Z}_3$-module. But $\mathbb{Z}_3$ is clearly a torsion $\mathbb{Z}$-module, whence, it is not a torsion free $\mathbb{Z}$-module.

The following is a counterexample of both cases of Theorem 2.6 for Φ not to be surjective.

***Example* 2.18.** Let $\Phi\colon \mathbb{R} \to \mathbb{R}[x]$ be defined by $\Phi(a) = a$. Consider the $\mathbb{R}[x]$-module $\mathbb{R}^2$. Then $\mathbb{R}^2$ is a torsion $\mathbb{R}[x]$-module [3, p. 185]. Therefore, it is not a torsion free $\mathbb{R}[x]$-module. Consider $\mathbb{R}^2$ as an $\mathbb{R}$-module by the action $r(a, b) = \Phi(r)(a, b) = (ra, rb)$. If $0 \neq r \in \mathbb{R}$ and $r(a, b) = 0$, then $ra = 0$ and $rb = 0$ implies $a = 0$ and $b = 0$. Hence, $\mathbb{R}^2$ is a torsion free $\mathbb{R}$-module. Therefore, it is not a torsion $\mathbb{R}$-module.

Lemma 2.2. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a ring homomorphism, then*:

(a) *If M_S and V_S are S-modules and $\theta\colon M_S \to V_S$ is an S-homomorphism, then $\theta\colon M_{|R} \to V_{|R}$ is an R-homomorphism.*

(b) *If* Φ *is surjective and* $\theta\colon M_{|R} \to V_{|R}$ *is an* R*-homomorphism, then* $\theta\colon M_S \to V_S$ *is an* S*-homomorphism.*

(c) $\mathrm{End}_S(V_S) \subseteq \mathrm{End}_R(V_{|R})$. *Equality holds if* Φ *is surjective.*

Proof. (a) Suppose $m \in M_{|R}$, $r \in R$. Then by (1.1),

$$\theta(mr) = \theta(m\Phi(r)) = \theta(m)\Phi(r) = \theta(m)r.$$

Therefore, θ is an R-homomorphism.

(b) Suppose $m \in M_S; s \in S$. Since Φ is surjective, there exists $r \in R$ such that $s = \Phi(r)$. Hence, by (1.1),

$$\theta(ms) = \theta(m\Phi(r)) = \theta(mr) = \theta(m)r = \theta(m)\Phi(r) = \theta(m)s.$$

Therefore, θ is an S-homomorphism.

(c) Put $M_S = V_S$ and $M_{|R} = V_{|R}$, then by (a) $\mathrm{End}_S(V_S) \subseteq \mathrm{End}_R(V_{|R})$ and if Φ is surjective, then by (b) $\mathrm{End}_S(V_S) = \mathrm{End}_R(V_{|R})$.

Lemma 2.2 is proved.

The following is a counterexample for Φ not to be surjective.

***Example* 2.19.** Let $R = UTM_2(\mathbb{Z}_6)$. Define $\Phi\colon \mathbb{Z}_6 \to R$ by $\Phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Consider $\mathbb{Z}_6$ as an R-module by the action

$$x \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = xc \in \mathbb{Z}_6. \tag{2.1}$$

Then $\mathbb{Z}_{6|\mathbb{Z}_6}$ is a $\mathbb{Z}_6$-module by the action

$$xz = x\Phi(z) = x \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \quad x, z \in \mathbb{Z}_6. \tag{2.2}$$

Consider $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ as an R-module by the action

$$(x, y) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (xa, xb + yc) \in \mathbb{Z}_6 \oplus \mathbb{Z}_6. \tag{2.3}$$

Then $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ is a $\mathbb{Z}_6$-module by the action

$$(x, y)z = (x, y)\Phi(z) = (xz, yz), \quad x, y, z \in \mathbb{Z}_6. \tag{2.4}$$

Now define $\theta\colon \mathbb{Z}_{6|\mathbb{Z}_6} \longrightarrow \mathbb{Z}_6 \oplus \mathbb{Z}_{6|\mathbb{Z}_6}$ by $\theta(x) = (x, 0)$. It is clear that θ is a group homomorphism. By (2.4), $\theta(xz) = (xz, 0) = (x, 0)z = \theta(x)z$. Hence, θ is a $\mathbb{Z}_6$-homomorphism. But θ is not an R-homomorphism, since by (2.1) and (2.3),

$$\theta\left(x \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \theta(xc) = (xc, 0),$$

while

$$\theta(x) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (x, 0) \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (xa, xb), \qquad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R, \quad x, z \in \mathbb{Z}_6.$$

Theorem 2.7. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism, which is an R-homomorphism also. If $\theta \in \mathrm{Hom}_R(V_{|R}, R) = V^*_{|R}$, then $\Phi\circ\theta \in \mathrm{Hom}_S(V_S, S) = V^*_S$.*

Proof. Since S is an R-module by the action $sr = s\Phi(r)$, $s \in S$; $r \in R$, we can consider Φ as an R-homomorphism. Suppose $v \in V_S$. Since Φ is surjective, $s = \Phi(r)$ for $s \in S$, $r \in R$. Hence, by (1.1),

$$(\Phi\circ\theta)(vs) = \Phi(\theta(vs)) = \Phi(\theta(v\Phi(r)) = \Phi(\theta(vr)) = \Phi[(\theta(v)r] =$$

$$= \Phi[(\theta(v)]r = [(\Phi\circ\theta)(v)]r = [(\Phi\circ\theta)(v)]\Phi(r) = [(\Phi\circ\theta)(v)]s.$$

Thus, $\Phi\circ\theta \in \mathrm{Hom}_S(V_S, S) = V^*_S$.

Theorem 2.7 is proved.

Theorem 2.8. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is an injective R-module, then V_S is an injective S-module.*

Proof. Let L_S and M_S be S-modules, $f: L_S \longrightarrow M_S$ be an S-monomorphism and $g: L_S \longrightarrow V_S$ be an S-homomorphism. Then by Lemma 2.2, $f: L_{|R} \longrightarrow M_{|R}$ is an R-monomorphism and $g: L_{|R} \longrightarrow V_{|R}$ is an R-homomorphism. Since $V_{|R}$ is injective, there exists an R-homomorphism $h: M_{|R} \longrightarrow V_{|R}$ such that $h\circ f = g$. Since Φ is surjective, by Lemma 2.2, h is an S-homomorphism. Thus, V_S is an injective S-module.

Theorem 2.8 is proved.

Remark **2.2.** (a) The converse of Theorem 2.8 is not true in general.

(b) The injective envelopes of $V_{|R}$ and V_S, which are denoted, respectively, by $E(V_{|R})$ and $E(V_S)$, may not be the same. These facts are illustrated in the following example.

Example **2.20.** Let $\Phi: \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\Phi(a) = \overline{a}$. Then $\mathbb{Z}_3$ is an injective $\mathbb{Z}_3$-module and $E(\mathbb{Z}_3) = \mathbb{Z}_3$. Consider $\mathbb{Z}_3$ as a $\mathbb{Z}$-module. By Example 2.15, $\mathbb{Z}_3$ is not a divisible $\mathbb{Z}$-module. Thus, $\mathbb{Z}_3$ is not an injective $\mathbb{Z}$-module and $E(\mathbb{Z}_3) = \mathbb{Z}_{3^\infty}$ (the Prüfer 3-group) [5] (Proposition 3.19, Example 3.36).

The following is a counterexample for Φ not to be surjective.

Example **2.21.** Let $\Phi: \mathbb{R} \to \mathbb{R}[x]$ be defined by $\Phi(a) = a$ and $V = \langle x^2\rangle$. By Example 2.16, V is not a divisible $\mathbb{R}[x]$-module. Therefore, V is not an injective $\mathbb{R}[x]$-module [3] (Proposition 5.2.11). Since $\mathbb{R}$ is semisimple, by Proposition 1.1(a), V is an injective $\mathbb{R}$-module.

Lemma 2.3. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a ring homomorphism. If M is a right S-module, N is a left S-module and B is an Abelian group, then*:

(a) *If $\tau: M_S \times_S N \to B$ is an S-balanced map, then $\tau: M_{|R} \times_{R|} N \to B$ is an R-balanced map.*

(b) *If Φ is surjective and $\tau: M_{|R} \times\, _{R|}N \to B$ is an R-balanced map, then $\tau: M_S \times_S N \to B$ is an S-balanced map.*

Proof. (a) Since $\tau: M_S \times_S N \to B$ is an S-balanced map,

$\tau(m + m', n) = \tau(m, n) + \tau(m', n)$,

$\tau(m, n + n') = \tau(m, n) + \tau(m, n')$,

$\tau(ms, n) = \tau(m, sn)\ \forall m, m' \in M\ \forall n, n' \in N\ \forall s \in S$.

So, to make τ an R-balanced map, we need only to verify the third condition. Suppose $r \in R$. Then by (1.1),

$$\tau(mr, n) = \tau(m\Phi(r), n) = \tau(m, \Phi(r)n) = \tau(m, rn).$$

Thus, τ is an R-balanced map.

(b) Since τ is an R-balanced map and Φ is surjective, by (1.1), we have

$$\tau(ms, n) = \tau(m\Phi(r), n) = \tau(m, \Phi(r)n) = \tau(m, sn), \qquad s \in S, \quad r \in R, \quad \text{with} \quad s = \Phi(r).$$

Hence, τ is an S-balanced map.

Lemma 2.3 is proved.

Lemma 2.4. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $M = M_S = M_{|R}$ and $N = {}_SN = {}_{R|}N$, then*

$$M \otimes_R N \cong M \otimes_S N \quad \text{as groups.}$$

Proof. The pair $(\tau,\ M \otimes_R N)$ is a tensor product of M and N over R, where $\tau\colon M \times N \to \to M \otimes_R N$ is an R-balanced map. Since Φ is surjective, by Lemma 2.3, τ is an S-balanced map. Now suppose that A is an Abelian group and $\beta\colon M \times N \to A$ is an S-balanced map. Then, by Lemma 2.3, β is an R-balanced map. By definition of a tensor product, there exists a unique $\mathbb{Z}$-homomorphism $f\colon M \otimes_R N \longrightarrow A$ such that the diagram

$$\begin{array}{ccc} & M \times N & \\ \tau \swarrow & & \searrow \beta \\ M \otimes_R N & \xrightarrow{f} & A \end{array}$$

commutes. Since τ and β are S-balanced maps, the pair $(\tau,\ M \otimes_R N)$ is a tensor product of M and N over S. Since the tensor product is unique to within isomorphism, $M \otimes_R N \cong M \otimes_S N$.

Lemma 2.4 is proved.

***Example* 2.22.** Let R be a ring and $\Phi\colon R \longrightarrow R/I$ be the natural epimorphism, where I is an ideal of R. Let $B = U/I$, where U is an ideal of R such that $I \subseteq U$. Then B is an ideal of R/I. Since $IB = \{I\}$, $R/I \otimes_R B \cong B/IB \cong B/\{I\} \cong B$ [2, p. 217] (Execise 9). Also, $R/I \otimes_{R/I} B \cong B$.

The following is a counterexample for Φ not to be surjective.

***Example* 2.23.** Let $\Phi : \mathbb{R} \longrightarrow \mathbb{C}$ be defined by $\Phi(a) = a$. Then Φ is a ring homomorphism. By considering $\mathbb{C}$ as a vector space over $\mathbb{C}$, we obviously have

$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}.$$

and as vector spaces over $\mathbb{R}, \mathbb{C} \cong \mathbb{R}^2$. Hence,

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2 \cong \mathbb{C}^2.$$

Theorem 2.9. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is a flat R-module, then V_S is a flat S-module.*

Proof. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of left S-modules. Then by Lemma 2.2,

$$0 \longrightarrow_{R|} A \longrightarrow_{R|} B \longrightarrow_{R|} C \longrightarrow 0$$

is a short exact sequence of left R-modules. Since $V_{|R}$ is a flat R-module,

$$0 \longrightarrow V_{|R} \otimes_R A \longrightarrow V_{|R} \otimes_R B \longrightarrow V_{|R} \otimes_R C \longrightarrow 0$$

is a short exact sequence of Abelian groups. Hence, by Lemma 2.4,

$$0 \longrightarrow V_S \otimes_S A \longrightarrow V_S \otimes_S B \longrightarrow V_S \otimes_S C \longrightarrow 0$$

is a short exact sequence of Abelian groups. Thus, V_S is a flat S-module.

Theorem 2.9 is proved.

***Example* 2.24.** Let $R = \begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Define $\Phi\colon R \to \mathbb{Z}_2$ by $\Phi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = a$. Then Φ is a surjective ring homomorphism. Since R and $\mathbb{Z}_2$ are semisimple, by Proposition 1.1(a) and [1] (Proposition 19.16), $\mathbb{Z}_2$ is a flat $\mathbb{Z}_2$-module and R-module.

The following example shows that the converse of Theorem 2.9 is not true in general.

***Example* 2.25.** Let $\Phi\colon \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\Phi(a) = \overline{a}$. Then $\mathbb{Z}_3$ is a flat $\mathbb{Z}_3$-module. Since $\mathbb{Z}_3$ is not a torsion free $\mathbb{Z}$-module, it is not a flat $\mathbb{Z}$-module [8] (Corollary 3.51).

Theorem 2.10. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is a free R-module, then V_S is a free S-module.*

Proof. Suppose that $X = \{x_i\}_{i \in I}$ is an R-basis for $V_{|R}$. Then by Theorem 2.1(a), X generates V_S, so it remains to prove that X is linearly independent over S. Suppose $s_i \in S$ and $\sum_{i \in I} x_i s_i = 0$, which implies $\sum_{i \in I} x_i \Phi(r_i) = 0$, $r_i \in R$, since Φ is surjective. By (1.1), we get $\sum_{i \in I} x_i r_i = 0$. Since X is linearly independent over R, $r_i = 0$ $\forall i \in I$. Therefore, $s_i = \Phi(r_i) = \Phi(0) = 0$ $\forall i \in I$. Thus, V_S is a free S-module.

Theorem 2.10 is proved.

***Example* 2.26.** Let $\Phi\colon \mathbb{Z}[x] \to \mathbb{Z}$ be defined by $\Phi(f(x)) = a$, where a is the constant term of $f(x)$. Then Φ is a surjective ring homomorphism. Consider $\mathbb{Z}[x]$ as a $\mathbb{Z}$-module. Then $\mathbb{Z}[x]$ is a free $\mathbb{Z}$-module with basis $\{x^n : n \in \mathbb{Z}^+\}$. Consider $\mathbb{Z}[x]$ as a $\mathbb{Z}[x]$-module by the action $f(x)g(x) = f(x)\Phi(g(x)) = f(x)a$, $f(x), g(x) \in \mathbb{Z}[x]$, a is the constant term of $g(x)$. Then $\mathbb{Z}[x]$ is a free $\mathbb{Z}[x]$-module with the same basis.

The following is a counterexample for Φ not to be surjective.

***Example* 2.27.** Let $R = UTM_2(\mathbb{Z}_2)$. Define $\Phi\colon \mathbb{Z}_2 \to R$ by $\Phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Consider $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as an R-module by the action

$$(x, y)\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (xa, xb + yc) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Since $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and R^n, for any $n \in \mathbb{N}^*$, are of different orders, they can not be isomorphic. Hence, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not a free R-module [2, p. 181] (Theorem 2.1). On the other hand, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a free $\mathbb{Z}_2$-module.

Theorem 2.11. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is a projective R-module, then V_S is a projective S-module.*

Proof. Suppose that $V_{|R}$ is a projective R-module and let

$$0 \longrightarrow L_S \xrightarrow{f} M_S \xrightarrow{g} V_S \longrightarrow 0$$

be a short exact sequence of S-modules. Since f and g are S-homomorphism, by Lemma 2.2, f and g are R-homomorphism. Thus,

$$0 \longrightarrow L_{|R} \xrightarrow{f} M_{|R} \xrightarrow{g} V_{|R} \longrightarrow 0$$

is a short exact sequence of R-modules. Since $V_{|R}$ is a projective R-module, this sequence splits. Therefore, there exists an R-homomorphism $g'\colon V_{|R} \longrightarrow M_{|R}$ such that $gg' = 1_{V_{|R}}$. Since Φ is surjective, by Lemma 2.2, g' is an S-homomorphism and $1_{V_{|R}} = 1_{V_S}$. Thus, V_S is a projective S-module [2, p. 192] (Theorem 3.4).

***Example* 2.28.** Let $\Phi : \mathbb{Z}[x] \to \mathbb{Z}$ be defined by $\Phi(f(x)) = a$, where a is the constant term of $f(x)$. By Example 2.26, $\mathbb{Z}[x]$ is a free $\mathbb{Z}$-module and $\mathbb{Z}[x]$-module. Therefore, $\mathbb{Z}[x]$ is a projective $\mathbb{Z}$-module and $\mathbb{Z}[x]$-module [2, p. 191] (Theorem 3.2).

The following is a counterexample of Theorems 2.9 and 2.11 for Φ not to be surjective.

***Example* 2.29.** Let $\Phi\colon \mathbb{R} \to \mathbb{R}[x]$ be defined by $\Phi(a) = a$. Since $\mathbb{R}$ is semisimple, by Proposition 1.1(a), $\mathbb{R}^2$ is a projective $\mathbb{R}$-module, therefore, it is a flat $\mathbb{R}$-module [1] (Proposition 9.16). Since $\mathbb{R}^2$ is not a torsion free $\mathbb{R}[x]$-module by Example 2.18, it is not a flat $\mathbb{R}[x]$-module [8] (Corollary 3.51), so it is not a projective $\mathbb{R}[x]$-module [1] (Proposition 19.16).

The converse of Theorem 2.10 and the converse of Theorem 2.11 are not true in general. The following example illustrates these facts.

***Example* 2.30.** Let $R = \left\{\begin{pmatrix} a & b \\ 0 & a\end{pmatrix} : a \in \mathbb{Z}_2\right\}$. Define $\Phi\colon R \to \mathbb{Z}_2$ by $\Phi\left(\begin{pmatrix} a & b \\ 0 & a\end{pmatrix}\right) = a$. Then $\mathbb{Z}_2$ is a free (projective) $\mathbb{Z}_2$-module. Consider $\mathbb{Z}_2$ as an R-module. Since $\mathbb{Z}_2$ and R^n, for any $n \in \mathbb{N}^*$, are of different orders, they can not be isomorphic. Hence, $\mathbb{Z}_2$ is not a free R-module [2, p. 181] (Theorem 2.1). Since $J(R) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right\}$, R is a local ring [1] (Proposition 15.15). Therefore, $\mathbb{Z}_2$ is not a projective R-module [4] (Theorem 2).

Theorem 2.12. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ has a projective cover over R, then V_S has a projective cover over S also.*

Proof. Since Φ is surjective, by the first isomorphism theorem

$$R/\ker\Phi \cong S.$$

Therefore, by Proposition 1.2(a), we get the result.

Theorem 2.12 is proved.

***Example* 2.31.** Let $R = \begin{pmatrix} \mathbb{Z}_3 & 0 \\ 0 & \mathbb{Z}_3\end{pmatrix}$ and $\Phi\colon R \to \mathbb{Z}_3$ be defined by $\Phi\left(\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\right) = a$. Since R and $\mathbb{Z}_3$ are semisimple rings, by Proposition 1.1(a), $\mathbb{Z}_3$ is a projective $\mathbb{Z}_3$-module and R-module. Therefore, it has a projective cover over $\mathbb{Z}_3$ and R [6] (Remark 24.11(5)).

The converse of Theorem 2.12 is not true in general and the following example illustrates this.

***Example* 2.32.** Let $\Phi\colon \mathbb{Z} \to \mathbb{Z}_3$ be defined by $\Phi(a) = \overline{a}$. Since $\mathbb{Z}_3$ is a projective $\mathbb{Z}_3$-module and J-semisimple, it has a projective cover over $\mathbb{Z}_3$. On the other hand, $\mathbb{Z}_3$ is not a projective $\mathbb{Z}$-module, it also does not have a projective cover over $\mathbb{Z}$ [6] (Remark 24.11(5)).

Theorem 2.13. *Suppose that R and S are rings and $\Phi\colon R \to S$ is a surjective ring homomorphism. If $V_{|R}$ is a semiperfect R-module, then V_S is a semiperfect S-module.*

Proof. Since $V_{|R}$ is projective, by Theorem 2.11, V_S is projective. Let $\overline{V} = V_S \diagup M$ be a factor module of V_S, where M is an S-submodule of V_S. By Theorem 2.1(c), M is an R-submodule of $V_{|R}$. Hence, $V_{|R}\diagup M$ is an R-module by the action $(v + M)r = vr + M = v\Phi(r) + M = (v + M)\Phi(r)$. Therefore, $V_{|R}\diagup M = (\overline{V})_{|R}$. Since $V_{|R}$ is semiperfect, $(\overline{V})_{|R}$ has a projective cover over R. Thus, by Theorem 2.12, $\overline{V}$ has a projective cover over S. Hence, V_S is a semiperfect S-module.

Theorem 2.13 is proved.

***Example* 2.33.** Let $R = \begin{pmatrix} \mathbb{Z}_3 & 0 \\ 0 & \mathbb{Z}_3 \end{pmatrix}$. Define $\Phi\colon R \to \mathbb{Z}_3$ by $\Phi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = a$. Since R and $\mathbb{Z}_3$ are semisimple, by Proposition 1.1(a), $\mathbb{Z}_3$ is a projective $\mathbb{Z}_3$-module and R-module. Thus, by Proposition 1.2(b), it is a semiperfect $\mathbb{Z}_3$-module and R-module.

The following is a counterexample for Φ not to be surjective.

***Example* 2.34.** Let $\Phi\colon \mathbb{R} \to \mathbb{R}[x]$ be defined by $\Phi(a) = a$. By Example 2.29, $\mathbb{R}^2$ is not a projective $\mathbb{R}[x]$-module. Therefore, it is not a semiperfect $\mathbb{R}[x]$-module. Since $\mathbb{R}$ is Artinian and $\mathbb{R}^2$ is a projective $\mathbb{R}$-module, by Proposition 1.2(b) it is a semiperfect $\mathbb{R}$-module.

***Remark* 2.3.** The converse of Theorem 2.13 is not true in general, since if V_S is a projective S-module, $V_{|R}$ may not be a projective R-module (see Example 2.30).

3. Conclusion. A summary of above studies is entered in the following tables.

3.1. First column on left-hand side of the following table represents the type of the module $V_{|R}$, and the remaining two columns illustrate whether V_S is of the same type or not.

Type of module	General case	Surjective case
1-generated	yes	yes
2-simple	yes	yes
3-semisimple	no (Example 2.4)	yes
4-indecomposable	yes	yes
5-Artinian	yes	yes
6-Noetherian	yes	yes
7-faithful	no (Example 2.13)	yes
8-divisible	no (Example 2.16)	yes
9-torsion	no	no (Example 2.17)
10-torsion free	no (Example 2.18)	yes
11-injective	no (Example 2.21)	yes
12-flat	no (Example 2.29)	yes
13-free	no (Example 2.27)	yes
14-projective	no (Example 2.29)	yes
15-semiperfect	no (Example 2.34)	yes

3.2. First column on left-hand side of the following table represents the type of the module V_S, and the remaining two columns illustrate whether $V_{|R}$ is of the same type or not.

Type of module	General case	Surjective case
1-generated	no (Example 2.5)	yes
2-simple	no (Example 2.5)	yes
3-semisimple	no (Example 2.5)	yes
4-indecomposable	no (Example 2.7)	yes
5-Artinian	no (Example 2.5)	yes
6-Noetherian	no (Example 2.7)	yes
7-faithful	no	no (Example 2.12)
8-divisible	no	no (Example 2.15)
9-torsion	no (Example 2.18)	yes
10-torsion free	no	no (Example 2.17)
11-injective	no	no (Example 2.20)
12-flat	no	no (Example 2.25)
13-free	no	no (Example 2.30)
14-projective	no	no (Example 2.30)
15-semiperfect	no	no (Example 2.30)

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