UDC 517.9

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CACCIOPPOLI-TYPE ESTIMATES FOR A CLASS OF NONLINEAR DIFFERENTIAL OPERATORS

ОЦІНКИ ТИПУ КАЧЧІОППОЛІ ДЛЯ ОДНОГО КЛАСУ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We establish Caccioppoli-type estimates for a class of nonlinear differential equations with the aid of a differential identity that generalizes the well-known multidimensional Picone's formula. In special cases, these estimates give the Finsler p-Laplacian, the p-Laplacian and the pseudo-p-Laplacian.

Встановлено оцінки типу Каччіопполі для одного класу нелінійних диференціальних рівнянь за допомогою диференціальної тотожності, що узагальнює відому багатовимірну формулу Піконе. В частинних випадках ці оцінки дають p-лапласіан Фінслера, p-лапласіан та p-псевдолапласіан.

1. Introduction. The Caccioppoli inequality is an important tool for the regularity estimate of elliptic partial differential equations (see [1, 2] and references therein). In the basic Caccioppoli inequality, L^2-norm of the gradient of a harmonic (or subharmonic) function v is estimated in terms of the L^2-norm of v itself. Such an estimate may not hold for arbitrary functions v, but we can typically use them for solutions of elliptic equations or systems in which integrals of higher derivatives can be bounded in terms of integrals of lower derivatives, usually over a slightly larger set. One of the alternative ways to establish the basic Caccioppoli inequality is to derive it from Picone's identity [3, 4]:

$$\left\|\nabla u\right\|_2^2 - \left\langle \nabla\left(\frac{u^2}{v}\right), \nabla v\right\rangle = \left\|\nabla u - \frac{u}{v}\nabla v\right\|_2^2, \tag{1.1}$$

where u and v are differentiable functions in a domain $\Omega \subset R^n$, $v(x) \neq 0$ in Ω, $\|\cdot\|_2$, ∇ and $\langle , \rangle$ denote the Euclidean norm, the usual gradient and the inner product in R^n, respectively.

Let $v > 0$ be a weak (continuous) solution of

$$\triangle v = 0 \quad \text{in} \quad \Omega$$

and $\eta \in C_0^\infty(\Omega)$ be a nonnegative test function.

If we integrate (1.1) by substituting $u = \eta v$, and use Young's inequality and Cauchy – Schwartz inequality, respectively, then we can obtain the desired estimate

$$\int\limits_\Omega \left\|\eta\nabla v\right\|_2^2 dx \leq 4\int\limits_\Omega \left\|v\nabla\eta\right\|_2^2 dx.$$

More generally, if $p > 1$ is fixed and $v > 0$ is a weak continuous solution (or subsolution) of the p-harmonic equation

$$\operatorname{div}\left(\|\nabla v\|_2^{p-2}\nabla v\right) = 0 \quad \text{in} \quad \Omega,$$

then the L^p-version of the Caccioppoli estimate

$$\int\limits_{\Omega} \|\eta\nabla v\|_2^p dx \le p^p \int\limits_{\Omega} \|v\nabla\eta\|_2^p dx$$

([5], see also inequality (5.27) in [6] and Corollary A6 in [7]) can be easily obtained from the p-Laplacian generalization of Picone's identity

$$\|\nabla u\|_2^p - \left\langle \nabla\left(\frac{|u|^p}{v^{p-1}}\right), \|\nabla v\|_2^{p-2}\nabla v \right\rangle = \Phi_p(u,v),$$

where

$$\Phi_p(u,v) := \|\nabla u\|_2^p + (p-1)\frac{\|u\|^p}{v^p}\|\nabla v\|_2^p - p\frac{|u|^{p-1}}{v^{p-1}}\left\langle \nabla u, \|\nabla v\|_2^{p-2}\nabla v \right\rangle \ge 0$$

by setting $u = \eta v$ and making use of Young's and Hölder inequality [8, 9].

Recently, Jaroš [10] extended the Caccioppoli inequality to a class of differential operators of the form

$$\Delta_{H,p} v := \operatorname{div}\left(H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\right), \tag{1.2}$$

where $p > 1,\ H : R^n \to [0,\infty),\ n \ge 2$ is a convex function (see, for example, [11]) of the class $C^1(R^n\backslash\{0\})$ which is positively homogeneous of degree 1 and ∇ and ∇_ξ stand for usual gradient operators with respect to the variables x and ξ, respectively.

We refer to the operator $\Delta_{H,p}$ as the Finsler p-Laplacian (or the anisotropic p-Laplacian). A prototype of H satisfying the above conditions is the l_r-norm

$$H(\xi) = \|\xi\|_r = \left(\sum_{i=1}^n |\xi_i|^r\right)^{1/r}, \quad r > 1,$$

for which the operator defined by (1.2) has the form

$$\Delta_{r,p} v := \operatorname{div}\left(\|\nabla v\|_r^{p-r}\nabla^r v\right), \tag{1.3}$$

where

$$\nabla^r v := \left(\left|\frac{\partial v}{\partial x_1}\right|^{r-2}\frac{\partial v}{\partial x_1}, \ldots, \left|\frac{\partial v}{\partial x_n}\right|^{r-2}\frac{\partial v}{\partial x_n}\right). \tag{1.4}$$

The class of operators of the form (1.3) includes the usual p-Laplacian and the so-called pseudo-p-Laplace operator as the special cases corresponding to $r = 2$ and $p \in (1,\infty)$ and $r = p > 1$, respectively. Clearly, if $p = r = 2$, then (1.3) reduces to the standard Laplacian Δ.

Anisotropic elliptic problems involving this kind of operator have recently been studied in several papers including [6, 12–22].

Recently, Jaroš obtained the following interesting result related to Caccioppoli inequality:

Theorem 1.1 (Caccioppoli-type inequality) [10]. *Let $v > 0$ be a weak subsolution of*

$$-\Delta_{H,p} v = g(x)|v|^{p-2} v \quad in \quad \Omega.$$

Then, for any fixed $q > p - 1$ and $w = v^{q/p}$, the inequality

$$\int\limits_{\Omega} H(\eta \nabla w)^p dx \leq \left(\frac{q}{q-p+1} \right)^p \int\limits_{\Omega} H(w \nabla \eta)^p dx + \frac{q^p p^{1-p}}{q-p+1} \int\limits_{\Omega} g(x) w^p \eta^p dx \tag{1.5}$$

holds for all $0 \leq \eta \in C_0^\infty(\Omega)$, where $0 \leq g \in L^\infty_{\mathrm{Loc}}(\Omega)$.

The purpose of this paper is to extend the Caccioppoli inequality to a class of A-harmonic operators of the form

$$\Delta_A v := \operatorname{div} \big(\nabla_\xi A(x, \nabla v) \big), \tag{1.6}$$

where $A \colon \Omega \times R^n \to R_+$ is measurable function such that $\xi \to A(x, \xi)$ is continuously differentiable function, convex and homogeneous of degree $p > 1$ with $A(x, \xi) \leq A_1(x) A_2(\xi)$. A typical example of functions A satisfying the above hypothesis is

$$A(x, \xi) = \frac{1}{p} \|\xi\|_r^p, \qquad p > 1, \quad r > 1, \quad \xi \in R^n,$$

or more generally

$$A(x, \xi) = \frac{1}{p} H(\xi)^p, \quad p > 1.$$

In these two important particular cases, (1.6) reduces to (1.3) and (1.2), respectively.

2. Preliminaries. In this section we give some properties of the A-harmonic operator and general norms in R^n which will be used in the sequel. The proofs can be obtained similarly as in [15] or [16].

Let $\langle , \rangle$ denote the usual inner product in R^n and $A \colon \Omega \times R^n \to R_+$ be a measurable function such that $\xi \to A(x, \xi)$ is convex and homogeneous of degree $p > 1$ so that

$$A(x, t\xi) = |t|^p A(x, \xi) \tag{2.1}$$

for all $t \in R$ and $(x, \xi) \in \Omega \times R^n$. If we assume that $A \in C^1(\Omega \times R^n \backslash \{0\})$, then from (2.1) it follows that

$$\langle \xi, \nabla_\xi A(x, t\xi) \rangle = p A(x, \xi)$$

for all $(x, \xi) \in \Omega \times R^n$ [23].

Let A_2 be an arbitrary norm in R^n. If we define the dual norm A_0 of A_2 by

$$A_0(u) = p^{-\frac{1}{p}} \sup \frac{\langle u, \xi \rangle}{A_2(\xi)^{\frac{1}{p}}} \quad \text{for} \quad u \in R^n$$

and if we assume $A_2 \in C^1\big(R^n\backslash\{0\}\big)$, then from (2.1) we obtain

$$\nabla_\xi A_2(t\xi) = \operatorname{sgn} t\,|t|^p \nabla_\xi A_2(\xi) \quad \text{for all} \quad \xi \neq 0 \quad \text{and} \quad t \neq 0 \tag{2.2}$$

and

$$\langle \xi, \nabla_\xi A_2(\xi)\rangle = pA_2(\xi) \quad \text{for all} \quad \xi \in R^n,$$

where the left-hand side is defined to be 0 if $\xi = 0$.

Moreover,

$$A_0\left(\nabla_\xi A_2(\xi)^{\frac{1}{p}}\right) = p^{-\frac{1}{p}} \quad \text{for all} \quad \xi \in R^n\backslash\{0\}.$$

Similarly, if A_0 is of class C^1 for $u \neq 0$, then

$$A_2\Big(\nabla A_0(u)^{\frac{1}{p}}\Big) = p^{-\frac{1}{p}} \quad \text{for all} \quad u \in R^n\backslash\{0\}.$$

From (2.2), we obtain the Hölder-type inequality

$$\langle u, \xi\rangle \leq p^{\frac{1}{p}} A_0(u) A_2(\xi)^{\frac{1}{p}} \quad \text{for all} \quad u, \xi \in R^n.$$

We will also need the following lemmas.

Lemma 2.1 [23]. *Let the function $\xi \to A(x,\xi)$ be continuously differentiable convex and homogeneous of degree $p > 1$, that is, $A(x,t\xi) = |t|^p A(x,\xi)$ for all $t \in R$ and $(x,\xi) \in \Omega \times R^n$. Then*

$$A(x,\eta) + (p-1)A(x,\xi) \geq \langle \nabla_\xi A(x,\xi), \eta\rangle \tag{2.3}$$

for all $\xi, \eta \in R^n$. If in addition, $\xi \to A(x,\xi)$ is a strictly convex function, then the equality in (2.3) *holds if and only if $\xi = \eta$.*

Lemma 2.2 (A generalization of Picone's identity) [23]. *Let the function $\xi \to A(x,\xi)$ be continuously differentiable convex and homogeneous of degree $p > 1$. Assume that u and v are differentiable in a given domain $\Omega \subset R^n$ with $v(x) \neq 0$ in Ω and denote*

$$\Phi_A(u,v) := A(x,\nabla u) + (p-1)A\left(x, \frac{u}{v}\nabla v\right) - \Big\langle \nabla_\xi A\Big(x, \frac{u}{v}\nabla v\Big), \nabla u\Big\rangle.$$

Then

$$A(x,\nabla u) - \frac{1}{p}\left\langle \nabla_\xi A(x,\nabla v), \nabla\left(\frac{|u|^p}{|v|^{p-2}v}\right)\right\rangle = \Phi_A(u,v) \tag{2.4}$$

and $\Phi_A(u,v) \geq 0$ a.e. in Ω. If, in addition, the function $\xi \to A(x,\xi)$ is strictly convex in R^n, then $\Phi_A(u,v) = 0$ a.e. in Ω if and only if $u\nabla v = v\nabla u$ in Ω.

***Remark* 2.1.** In the special case, where $A(x,\xi) = \dfrac{1}{p}H(\xi)^p$, $p > 1$, Lemmas 2.1 and 2.2 reduce to Lemmas 2.1 and 2.2 in [10], respectively. In [10], this special formula was used to establish the Caccioppoli-type inequality (1.5) for the subsolution (resp. the supersolution) of the nonlinear equation involving a Finsler p-Laplace operator $\triangle_{H,p}$. Similarly in the special case, where $A(x,\xi) = \dfrac{1}{p}\|\xi\|_r^p$, where $p, r > 1$, $\xi \in R^n$, the identity (2.4) reduces to

$$\|\nabla u\|_r^p - \left\langle \nabla \left(\frac{|u|^p}{|v|^{p-2}v} \right), \|\nabla v\|_r^{p-r} \nabla^r v \right\rangle = \Phi_{p,r}(u,v),$$

where $\nabla^r v$ is defined in (1.4) and

$$\Phi_{p,r}(u,v) := \|\nabla u\|_r^p + (p-1)\frac{|u|^p}{|v|^p}\|\nabla v\|_r^p - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle \nabla u, \|\nabla v\|_r^{p-2}\nabla^r v\rangle \geq 0$$

which is the special case of (2.4).

3. Caccioppoli-type estimates. In this section we establish Caccioppoli-type estimates for positive sub- and supersolutions of nonlinear equations involving anisotropic elliptic operators $\triangle_A$ where $A(x,\xi) \leq A_1(x)A_2(\xi)$.

Let Ω be a domain in R^n and $A\colon \Omega \times R^n \to [0,\infty)$ be a special Caratheodory function, i.e., $A_1 \in C(\Omega)$ with $A_1(x) > 0,$ A_2 is measurable nonnegative function for all $\xi \in R^n$. Moreover we assume that $A_2(\xi)$ is differentiable and satisfies the properties given in Section 2.

Consider the equation

$$-\frac{1}{p}\triangle_A v = g(x)|v|^{p-2}v + f(x), \tag{3.1}$$

where $p > 1,$ $0 \leq g \in L^\infty(\Omega),$ $0 \leq f \in L^{p'}(\Omega)$ and $p' = \dfrac{p-1}{p}$.

As usual we will say that a continuous function $v \in W^{1,p}(\Omega)$ is a (weak) solution of Eq. (3.1) in a domain $\Omega \subset R^n$ if it satisfies

$$\frac{1}{p}\int_\Omega \langle \nabla_\xi A(x,\nabla u), \nabla\eta\rangle dx - \int_\Omega g(x)|v|^{p-2}v\eta dx = \int_\Omega f(x)\eta dx \tag{3.2}$$

for all $\eta \in W_0^{1,p}(\Omega)$.

(Weak) subsolution and supersolution (3.1) are defined analogously using the nonnegative test functions $\eta \in W_0^{1,p}(\Omega)$ by replacing “ $=$ ” in (3.2) with “ $\leq$ ” and “ $\geq$ ”, respectively.

We define the functional J_A as

$$J_A(u;\Omega) := \int_\Omega A(x,\nabla u)dx - \int_\Omega g(x)|u|^p dx, \quad u \in W_0^{1,p}(\Omega)$$

associated with (3.1).

Let $v > 0$ be a (continuous) weak subsolution of (3.1) in Ω and $u \in W_0^{1,p}(\Omega)$. Then we can choose

$$\eta = \frac{|u|^p}{v^{p-1}}$$

as a test function in (3.1) and conclude by (2.4), that

$$J_A(u;\Omega) \leq \int_\Omega \left\{ A(x,\nabla u) - \frac{1}{p}\left\langle \nabla_\xi A(x,\nabla u), \nabla\left(\frac{|u|^p}{v^{p-1}}\right)\right\rangle\right\} dx + \int_\Omega f(x)\frac{|u|^p}{v^{p-1}}dx =$$

$$= \int_\Omega \Phi_A(u,v)dx + \int_\Omega f(x)\frac{|u|^p}{v^{p-1}}dx. \tag{3.3}$$

Clearly, for positive supersolution v of (3.1) and any $u \in W_0^{1,p}(\Omega)$, the reversed inequality

$$J_A(u;\Omega) \geq \int_\Omega \Phi_A(u,v)dx + \int_\Omega f(x)\frac{|u|^p}{v^{p-1}}dx.$$

holds true. If, in particular, $v \in W_0^{1,p}(\Omega)$ is a positive solution of (3.1), then (3.3) becomes an equality.

Then we have the following theorem.

Theorem 3.1 (Caccioppoli-type inequality)**.** *Let* $v > 0$ *be weak subsolution of* (3.1) *in* Ω*. Then for any fixed* $q > p-1$ *and* $w = v^{\frac{q}{p}}$*, the inequality*

$$\frac{qp}{q-p+1}\int\limits_\Omega A(x,\eta\nabla w)dx - \frac{(p-1)(q+1)}{q-p+1}\int\limits_\Omega A_1(x)A_2(\eta\nabla w)dx \leq$$

$$\leq \left(\frac{q}{q-p+1}\right)^p \int\limits_\Omega A_1(x)A_2(w\nabla\eta)dx + \frac{q^p p^{1-p}}{q-p+1}\int\limits_\Omega g(x)w^p\eta^p dx + \frac{q^p p^{1-p}}{q-p+1}\int\limits_\Omega f(x)\eta dx \tag{3.4}$$

holds for all $0 \leq \eta \in C_0^\infty(\Omega)$.

Proof. Let v be a positive subsolution of (3.1) in Ω. Fix a nonnegative function $\eta \in C_0^\infty(\Omega)$. Then $u := v^{\frac{q}{p}}\eta$ belongs to $W_0^{1,p}(\Omega)$ and we can use it as a test function in (3.3) to get

$$J_A(v^{\frac{q}{p}}\eta;\Omega) \leq \int\limits_\Omega A(x,\nabla v^{\frac{q}{p}}\eta)dx + (p-1)\int\limits_\Omega A\left(x, v^{\frac{q-p}{p}}\eta\nabla v\right)dx -$$

$$-\int\limits_\Omega \left\langle \nabla_\xi A\big(x, v^{\frac{q-p}{p}}\eta\nabla v\big), \nabla\big(v^{\frac{q}{p}}\eta\big)\right\rangle dx + \int\limits_\Omega f(x)\eta dx. \tag{3.5}$$

By using

$$\nabla\big(v^{\frac{q}{p}}\eta\big) = \frac{q}{p}v^{\frac{q}{p}-1}\eta\nabla v + v^{\frac{q}{p}}\nabla\eta$$

and

$$A(x,\xi) \leq A_1(x)A_2(\xi)$$

in the above inequality (3.5), we have

$$J_A\big(v^{\frac{q}{p}}\eta;\Omega\big) \leq \int\limits_\Omega A\big(x,\nabla v^{\frac{q}{p}}\eta\big)dx + (p-1)\int\limits_\Omega A_1(x)A_2(v^{\frac{q-p}{p}}\eta\nabla v)dx -$$

$$-\int\limits_\Omega \left\langle \nabla_\xi A\big(x, v^{\frac{q-p}{p}}\eta\nabla v\big), \frac{q}{p}v^{\frac{q-p}{p}}\eta\nabla v\right\rangle dx +$$

$$+\int\limits_\Omega \left\langle \nabla_\xi A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big), v^{\frac{q}{p}}\nabla\eta\right\rangle dx + \int\limits_\Omega f(x)\eta dx. \tag{3.6}$$

On the other hand by using properties of A_0 and A_2, we can easily obtain

$$\left\langle \nabla_\xi A_1(x) A_2(v^{\frac{q-p}{p}}\eta\nabla v), v^{\frac{q}{p}}\nabla\eta \right\rangle = A_1(x)\left\langle \nabla_\xi \left[A_2(v^{\frac{q-p}{p}}\eta\nabla v)^{\frac{1}{p}}\right]^p, v^{\frac{q}{p}}\nabla\eta \right\rangle \leq$$

$$\leq pA_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)^{\frac{p-1}{p}} A_2\big(v^{\frac{q}{p}}\nabla\eta\big)^{\frac{1}{p}}.$$

By using this inequality in (3.6), we get

$$J_A(v^{\frac{q}{p}}\eta;\Omega) \leq \int\limits_\Omega A\big(x, \nabla v^{\frac{q}{p}}\eta\big)dx-$$

$$-(1-p)\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)dx - q\int\limits_\Omega A\big(x, v^{\frac{q-p}{p}}\eta\nabla v\big)dx+$$

$$+p\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)^{\frac{p-1}{p}} A_2(v^{\frac{q}{p}}\nabla\eta)^{\frac{1}{p}}dx + \int\limits_\Omega f(x)\eta dx.$$

Applying Young's inequality in the form

$$a^{\frac{1}{p}}b^{p-1} \leq \frac{1}{p}\frac{1}{\tau^{p-1}}a + \frac{p-1}{p}\tau b^p, \qquad a,b\geq 0, \quad \tau>0,$$

we have

$$J_A(v^{\frac{q}{p}}\eta;\Omega) \leq \int\limits_\Omega A\big(x, \nabla v^{\frac{q}{p}}\eta\big)dx-$$

$$-\big[1-p+(1-p)\tau\big]\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)dx - q\int\limits_\Omega A\big(x, v^{\frac{q-p}{p}}\eta\nabla v\big)dx+$$

$$+\frac{1}{\tau^{p-1}}\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q}{p}}\nabla\eta\big)dx + \int\limits_\Omega f(x)\eta dx.$$

Now making use of the definition of J_A we obtain

$$q\int\limits_\Omega A\big(x, v^{\frac{q-p}{p}}\eta\nabla v\big)dx \leq (p-1)(1+\tau)\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)dx+$$

$$+\frac{1}{\tau^{p-1}}\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q}{p}}\nabla\eta\big)dx\int\limits_\Omega g(x)v^q\eta^p dx + \int\limits_\Omega f(x)\eta dx,$$

which after choosing the constant $\tau := \dfrac{q-p+1}{p}$ as in [10], leads to

$$\frac{qp}{q-p+1}\int\limits_\Omega A\big(x, v^{\frac{q-p}{p}}\eta\nabla v\big)dx - \frac{(p-1)(q+1)}{q-p+1}\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)dx \leq$$

$$\leq \left(\frac{p}{q-p+1}\right)^p \int\limits_\Omega A_1(x)A_2\big(v^{\frac{q}{p}}\nabla\eta\big)dx +$$

$$+\frac{p}{q-p+1}\int\limits_\Omega g(x)v^q\eta^p dx + \frac{p}{q-p+1}\int\limits_\Omega f(x)\eta dx.$$

Finally, the substitution of $w = v^{\frac{q}{p}}$ yields (3.4) as claimed.

Theorem 3.1 is proved.

An analogous result as Theorem 3.1 above holds for positive supersolution of (3.1) and a range of q smaller than $p-1$. The proof is similar to that of Theorem 3.1, hence omitted.

Theorem 3.2. *Let $v > 0$ be a weak supersolution of* (3.1) *in Ω. Then, for any fixed $q < p-1$, the inequality*

$$\frac{(p-1)(q+1)}{p-q-1}\int\limits_\Omega A_1(x)A_2\big(v^{\frac{q-p}{p}}\eta\nabla v\big)dx - \frac{qp}{p-q-1}\int\limits_\Omega A(x, v^{\frac{q-p}{p}}\eta\nabla v)dx \leq$$

$$\leq \left(\frac{p}{p-q-1}\right)^p \int\limits_\Omega A_1(x)A_2(v^{\frac{q}{p}}\nabla\eta)dx -$$

$$-\frac{p}{p-q-1}\int\limits_\Omega g(x)v^q\eta^p dx - \frac{p}{p-q-1}\int\limits_\Omega f(x)\eta dx \tag{3.7}$$

holds for all $\eta \in C_0^\infty(\Omega)$ with $\eta \geq 0$.

Now we consider the equality case, that is, $A(x,\xi) = A_1(x)A_2(\xi)$. From Theorems 3.1 and 3.2 we have the following results.

Theorem 3.3 (Caccioppoli-type inequality)**.** *Let $v > 0$ be weak subsolution of* (3.1) *in Ω. Then, for any fixed $q > p-1$ and $w = v^{\frac{q}{p}}$, the inequality*

$$\int\limits_\Omega A_1(x)A_2(\eta\nabla w)dx \leq \left(\frac{q}{q-p+1}\right)^p \int\limits_\Omega A_1(x)A_2(w\nabla\eta)dx +$$

$$+\frac{q^p p^{1-p}}{q-p+1}\int\limits_\Omega g(x)w^p\eta^p dx + \frac{q^p p^{1-p}}{q-p+1}\int\limits_\Omega f(x)\eta dx$$

holds for all $0 \leq \eta \in C_0^\infty(\Omega)$.

***Corollary* 3.1.** *Let $q = p$, $g(x) \equiv 0$, and $f(x) \equiv 0$ in Ω. If $v > 0$ is a weak subsolution of* (3.1) *in Ω, then*

$$\int\limits_\Omega A_1(x)A_2(\eta\nabla v)dx \leq p^p \int\limits_\Omega A_1(x)A_2(v\nabla\eta)dx$$

for any nonnegative $\eta \in C_0^\infty(\Omega)$.

Theorem 3.4. *Let* $r > 0$ *be a weak supersolution of* (3.1) *in* Ω. *Then, for any fixed* $q < p-1$, *the inequality*

$$\int\limits_{\Omega} A_1(x)A_2\big(v^{\frac{q}{p}-1}\eta\nabla v\big)dx \le \left(\frac{p}{p-q-1}\right)^p \int\limits_{\Omega} A_1(x)A_2\big(v^{\frac{q}{p}}\nabla\eta\big)dx-$$

$$-\frac{p}{p-q-1}\int\limits_{\Omega} g(x)v^q\eta^p dx - \frac{p}{p-q-1}\int\limits_{\Omega} f(x)\eta dx$$

holds for all $\eta \in C_0^\infty(\Omega)$ *with* $\eta \ge 0$.

The particular case of the above theorem when $q = 0$ is interesting in the sense that the right-hand side of (3.7) does not contain v. The result specializes as follows.

***Corollary* 3.2** (Logarithmic Caccioppoli-type inequality)**.** *Let* $v > 0$ *be weak subsolution of* (3.1) *in* Ω. *Then*

$$\int\limits_{\Omega} A_1(x)A_2(\eta\nabla log v)dx \le \left(\frac{p}{p-1}\right)^p \int\limits_{\Omega} A_1(x)A_2(\nabla\eta)dx-$$

$$-\frac{p}{p-1}\int\limits_{\Omega} g(x)\eta^p dx - \frac{p}{p-1}\int\limits_{\Omega} f(x)\eta dx \tag{3.8}$$

whenever $0 \le \eta \in C_0^\infty(\Omega)$.

If $A(\xi) = \dfrac{1}{p}\|\xi\|_2^p$, $p > 1$, $\xi \in R^n$, and $g(x) \equiv 0$ and $f(x) \equiv 0$ in Ω, then (3.8) reduces to the well-known logarithmic Caccioppoli inequality for the positive p-superharmonic functions [24].

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Received 16.03.16