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MAPPING PROPERTIES FOR CONVOLUTION INVOLVING HYPERGEOMETRIC SERIES

ВЛАСТИВОСТІ ВІДОБРАЖЕННЯ ДЛЯ ЗГОРТКИ, ЩО ВКЛЮЧАЄ ГІПЕРГЕОМЕТРИЧНІ РЯДИ

We introduce sufficient conditions of (Gaussian) hypergeometric functions to be in a subclass of analytic functions. In addition, we investigate several mapping properties for convolution and integral convolution involving hypergeometric functions.

Введено достатні умови того, що (гауссові) гіпергеометричні функції є підкласом аналітичних функцій. Крім того, розглянуто деякі властивості відображення для згортки та інтегральної згортки, що включають гіпергеометричні функції.

1. Introduction. Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad p \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We note that $\mathcal{A}(1) = \mathcal{A}$. Also, for $g(z) \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} g_{p+n} z^{p+n},$$

the Hadamard product (or convolution) of two power series $f(z)$ and $g(z)$ is given by (see [4])

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} g_{p+n} z^{p+n} = (g * f)(z)$$

and the integral convolution is defined by (see [4])

$$(f \circledast g)(z) = z^p + \sum_{n=1}^{\infty} \frac{a_{p+n} g_{p+n}}{p+n} z^{p+n} = (g \circledast f)(z).$$

We recall some definitions which will be used in our paper.

Definition 1.1. *For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence* (*see* [7]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2 [10]. *For $0 \le \alpha < p,\ f(z) \in \mathcal{A}(p)$ is said to be in the class of p-valent starlike of order α, denoted by $\mathcal{S}_p^*(\alpha)$, if*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{U}, \tag{1.2}$$

and in the class of p-valent convex of order α, denoted by $\mathcal{K}_p(\alpha)$, if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathbb{U}. \tag{1.3}$$

From (1.2) *and* (1.3) *we can see that*

$$f(z) \in \mathcal{K}_p(\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha).$$

We denote by $\mathcal{S}^* = \mathcal{S}_1^*(0)$ and $\mathcal{K} = \mathcal{K}_1(0)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of starlike and convex functions, respectively (*see Robertson* [11]).

We also define for $\sigma > 0$, the classes

$$\mathcal{S}^*(p,\sigma) = \left\{ f(z) \in \mathcal{A}(p) \colon \left|\frac{zf'(z)}{f(z)} - p\right| < \sigma,\ z \in \mathbb{U} \right\}$$

and

$$\mathcal{K}(p,\sigma) = \left\{ f(z) \in \mathcal{A}(p) \colon \left|1 + \frac{zf''(z)}{f'(z)} - p\right| < \sigma,\ z \in \mathbb{U} \right\}.$$

It is a known fact that a sufficient condition for $f(z) \in$ *A(p)* to be in the class $\mathcal{S}_p^*(\alpha)$ is that $\sum_{n=p+1}^{\infty}(n-\alpha)|a_n| \le p-\alpha$. A simple extension of this result is (see [8])

$$\sum_{n=p+1}^{\infty}(n-p+\sigma)|a_n| \le \sigma \Rightarrow f(z) \in \mathcal{S}^*(p,\sigma).$$

Since $f(z) \in \mathcal{K}(p,\sigma) \Longleftrightarrow \dfrac{zf'(z)}{p} \in \mathcal{S}^*(p,\sigma)$, we have a corresponding result for $\mathcal{K}(p,\sigma)$,

$$\sum_{n=p+1}^{\infty}\frac{n}{p}(n-p+\sigma)|a_n| \le \sigma \Rightarrow f(z) \in \mathcal{K}(p,\sigma).$$

Definition 1.3 [1]. *For $-1 \le A < B \le 1,\ |\lambda| < \dfrac{\pi}{2}$ and $0 \le \alpha < p$, we define $\mathcal{R}^\lambda(A,B,p,\alpha)$ which consists of functions $f(z)$ of the form* (1.1) *and satisfying the analytic criterion*

$$e^{i\lambda}\frac{f'(z)}{z^{p-1}} \prec \cos\lambda\left[(p-\alpha)\frac{1+Az}{1+Bz} + \alpha\right] + ip\sin\lambda, \quad z \in \mathbb{U}.$$

According to the principle of subordination, $f(z) \in \mathcal{R}^\lambda(A,B,p,\alpha)$ if and only if there exists function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{U}$, such that

$$e^{i\lambda}\frac{f'(z)}{z^{p-1}} = \cos\lambda\left[(p-\alpha)\frac{1+Aw(z)}{1+Bw(z)} + \alpha\right] + ip\sin\lambda, \quad z \in \mathbb{U},$$

or, equivalently,

$$\left|\frac{e^{i\lambda}\left(\dfrac{f'(z)}{z^{p-1}}-p\right)}{Be^{i\lambda}\dfrac{f'(z)}{z^{p-1}}-[pBe^{i\lambda}+(A-B)(p-\alpha)\cos\lambda]}\right|<1, \quad z\in\mathbb{U}.$$

For suitable choices of A, B and α, we obtain the following subclass: $\mathcal{R}^0(-\beta,\beta,p,\alpha) = \mathcal{R}(\beta,p,\alpha)$ (see [9]).

Also, we note that:

(i) $\mathcal{R}^\lambda(-\beta,\beta,p,\alpha)=\mathcal{R}^\lambda(\beta,p,\alpha)=$

$$=\left\{f(z)\in\mathcal{A}(p): \left|\frac{\dfrac{f'(z)}{z^{p-1}}-p}{\dfrac{f'(z)}{z^{p-1}}-[p-2(p-\alpha)e^{-i\lambda}\cos\lambda]}\right|<\beta,\ 0<\beta\le 1;\ z\in\mathbb{U}\right\},$$

(ii) $\mathcal{R}^\lambda(-1,1,p,\alpha)=\mathcal{R}^\lambda(p,\alpha)=$

$$=\left\{f(z)\in\mathcal{A}(p): \operatorname{Re}\left\{e^{i\lambda}\left(\frac{f'(z)}{z^{p-1}}-\alpha\right)\right\}>0,\ |\lambda|<\frac{\pi}{2};\ 0\le\alpha<p;\ z\in\mathbb{U}\right\}.$$

Let ${}_2F_1(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,$$

where $c\neq 0,-1,-2,\ldots,$ and

$$(\gamma)_n=\begin{cases}1, & \text{if} \quad n=0,\\ \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1), & \text{if} \quad n\in\mathbb{N}.\end{cases}$$

We note that ${}_2F_1(a,b;c;1)$ converges for $\operatorname{Re}(c-a-b)>0$ and is related to Gamma function by

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Also, we define the functions

$$g_p(a,b;c;z)=z^p{}_2F_1(a,b;c;z)=z^p+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^{p+n} \tag{1.4}$$

and

$$h_{p,\mu}(a,b;c;z)=(1-\mu)\left(z^p{}_2F_1(a,b;c;z)\right)+\mu\frac{z}{p}\left(z^p{}_2F_1(a,b;c;z)\right)'=$$

$$=z^p+\sum_{n=1}^{\infty}\left(1+\mu\frac{n}{p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}z^{p+n}, \quad \mu\ge 0. \tag{1.5}$$

Corresponding to ${}_2F_1(a,b;c;z)$, we define $I^p_{a,b,c}\colon\mathcal{A}(p)\to\mathcal{A}(p)$ by

$$\left[I^p_{a,b,c}(f)\right](z)=g_p(a,b;c;z)*f(z)=$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} dt * \frac{z^p}{(1-z)^a} =$$

$$= z^p + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} a_{p+n} z^{p+n}$$

and $L_{p,\mu} : \mathcal{A}(p) \to \mathcal{A}(p)$ by

$$\left[L_{p,\mu}(f)\right](z) = h_{p,\mu}(a,b;c;z) * f(z) =$$

$$= z^p + \sum_{n=1}^{\infty} \left(1 + \frac{\mu n}{p}\right) \frac{(a)_n (b)_n}{(c)_n (1)_n} a_{p+n} z^{p+n}.$$

For $p = 1$, the operators $I_{a,b,c}$ and L_μ were introduced by Hohlov (see [5]) and Kim and Shon (see [6]), respectively.

Further, we define $\mathcal{M}^p_{a,b,c} : \mathcal{A}(p) \to \mathcal{A}(p)$ by

$$\left[\mathcal{M}^p_{a,b,c}(f)\right](z) = g_p(a,b;c;z) \circledast f(z) =$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} dt \circledast \frac{z^p}{(1-z)^a} =$$

$$= z^p + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \frac{a_{p+n} z^{p+n}}{(p+n)},$$

and $\mathcal{N}_{p,\mu} : \mathcal{A}(p) \to \mathcal{A}(p)$ by

$$\left[\mathcal{N}_{p,\mu}(f)\right](z) = h_{p,\mu}(a,b;c;z) \circledast f(z) =$$

$$= z^p + \sum_{n=1}^{\infty} \left(1 + \frac{\mu n}{p}\right) \frac{(a)_n (b)_n}{(c)_n (1)_n} \frac{a_{p+n} z^{p+n}}{(p+n)}.$$

For $p = 1$, the operators $\mathcal{M}_{a,b,c}$ and $\mathcal{N}_\mu$ were introduced and studied by Aouf et al. (see [2]).

2. Main results. Unless otherwise mentioned, we assume throughout this paper that $-1 \le A < B \le 1$, $|\lambda| < \dfrac{\pi}{2}$, $0 \le \alpha < p$, $p \in \mathbb{N}$ and $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. To establish our results, we need the following lemmas.

Lemma 2.1 ([1], Theorem 4). *A sufficient condition for* $f(z)$ *defined by* (1.1) *to be in the class* $\mathcal{R}^\lambda(A,B,p,\alpha)$ *is*

$$\sum_{n=1}^{\infty} (1 + |B|)(p+n)|a_{p+n}| \le (B-A)(p-\alpha)\cos\lambda.$$

Lemma 2.2 ([1], Theorem 1). *A function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{R}^\lambda(A,B,p,\alpha)$ *if*

$$|a_{p+n}| \le \frac{(B-A)(p-\alpha)\cos\lambda}{(p+n)}, \quad n \ge 1.$$

The estimate is sharp.

By using Lemmas 2.1 and 2.2, we get the following results.

Theorem 2.1. *Let* $a, b \in \mathbb{C}^*$ *and* $c > |a| + |b| + 1$. *Then the sufficient condition for* $g_p(a, b; c; z)$ *to be in the class* $\mathcal{R}^\lambda(A, B, p, \alpha)$ *is that*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[p+\frac{|ab|}{(c-|a|-|b|-1)}\right] \le p + \frac{(B-A)(p-\alpha)\cos\lambda}{1+|B|}. \tag{2.1}$$

Proof. According to Lemma 2.1 and (1.4), we need only to show that

$$\sum_{n=1}^{\infty}(1+|B|)\,(p+n)\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}\right| \le (B-A)(p-\alpha)\cos\lambda. \tag{2.2}$$

Since

$$|(d)_n| \le (|d|)_n\,, \tag{2.3}$$

then the left-hand side of (2.2) is less than or equal to

$$\sum_{n=1}^{\infty}(1+|B|)\,(p+n)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} = T_0.$$

Now

$$T_0 = p\,(1+|B|)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} + (1+|B|)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}} =$$

$$= p\,(1+|B|)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right] +$$

$$+ (1+|B|)\,\frac{|ab|}{c}\,\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} =$$

$$= (1+|B|)\,\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[p+\frac{|ab|}{(c-|a|-|b|-1)}\right] - p\,(1+|B|)\,.$$

But this last expression is bounded above by $(B-A)(p-\alpha)\cos\lambda$ if (2.1) holds.

Theorem 2.1 is proved.

Remark **2.1.** Putting $p = 1$ in Theorem 2.1, we obtain the result of Aouf et al. [3] (Theorem 2.1).

Theorem 2.2. *Let* $a, b \in \mathbb{C}^*$ *and* $c > |a|+|b|+2$. *Then the sufficient condition for* $h_{p,\mu}(a, b; c; z)$ *to be in the class* $\mathcal{R}^\lambda(A, B, p, \alpha)$ *is that*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[p+\left(1+\mu+\frac{\mu}{p}\right)\frac{|ab|}{(c-|a|-|b|-1)} + \frac{\mu}{p}\frac{(|a|)_2\,(|b|)_2}{(c-|a|-|b|-2)_2}\right] \le$$

$$\le p + \frac{(B-A)(p-\alpha)\cos\lambda}{1+|B|}. \tag{2.4}$$

Proof. According to Lemma 2.1 and (1.5), we need only to show that

$$\sum_{n=1}^{\infty}(1+|B|)\,(p+n)\left|\left(1+\mu\frac{n}{p}\right)\frac{(a)_n(b)_n}{(c)_n(1)_n}\right|\leq (B-A)(p-\alpha)\cos\lambda. \tag{2.5}$$

The left-hand side of (2.5), by (2.3), is less than or equal to

$$\sum_{n=1}^{\infty}(1+|B|)\,(p+n)\left(1+\mu\frac{n}{p}\right)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}=T_1$$

and so

$$T_1=p\,(1+|B|)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}+\left(1+\mu+\frac{\mu}{p}\right)(1+|B|)\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}+$$

$$+\frac{\mu}{p}\,(1+|B|)\sum_{n=2}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-2}}=$$

$$=p\,(1+|B|)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]+$$

$$+(1+|B|)\left(1+\mu+\frac{\mu}{p}\right)\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+$$

$$+\frac{\mu\,(1+|B|)}{p}\frac{(|a|)_2\,(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}=$$

$$=(1+|B|)\,\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[p+\left(1+\mu+\frac{\mu}{p}\right)\frac{|ab|}{(c-|a|-|b|-1)}+\right.$$

$$\left.+\frac{\mu}{p}\frac{(|a|)_2\,(|b|)_2}{(c-|a|-|b|-2)_2}\right]-p\,(1+|B|)\,,$$

the proof now follows by (2.5).

Theorem 2.2 is proved.

***Remark* 2.2.** Putting $p=1$ in Theorem 2.2, we obtain the result of Aouf et al. [3] (Theorem 2.2).

Theorem 2.3. *Let* $a,b\in\mathbb{C}^*$ *and* $c>|a|+|b|$. *If the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\leq 1+\frac{1}{(1+|B|)}, \tag{2.6}$$

is satisfied, then $[I^p_{a,b,c}(f)](z)$ *maps the class* $\mathcal{R}^\lambda(A,B,p,\alpha)$ *into itself.*

Proof. We need to show that

$$T_2=\sum_{n=1}^{\infty}(1+|B|)\,(p+n)\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}a_{p+n}\right|\leq (B-A)(p-\alpha)\cos\lambda. \tag{2.7}$$

By (2.3) and Lemma 2.2, we have

$$T_2 \leq \sum_{n=1}^{\infty} (1+|B|)\,(p+n) \frac{(B-A)(p-\alpha)\cos\lambda}{(p+n)} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \sum_{n=1}^{\infty} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right],$$

the last expression is bounded above by $(B-A)(p-\alpha)\cos\lambda$ if (2.6) holds.

Theorem 2.3 is proved.

Theorem 2.4. *Let* $a, b \in \mathbb{C}^*$ *and* $c > |a| + |b| + 1$. *If the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[1 + \frac{\mu}{p} \frac{|ab|}{(c-|a|-|b|-1)}\right] \leq 1 + \frac{1}{(1+|B|)}, \tag{2.8}$$

holds, then $[L_{p,\mu}(f)]\,(z)$ *maps the class* $\mathcal{R}^{\lambda}(A,B,p,\alpha)$ *into itself.*

Proof. We need to prove that

$$T_3 = \sum_{n=1}^{\infty} (1+|B|)\,(p+n) \left|\left(1 + \frac{\mu n}{p}\right) \frac{(a)_n(b)_n}{(c)_n(1)_n} a_{p+n}\right| \leq (B-A)(p-\alpha)\cos\lambda. \tag{2.9}$$

By (2.3) and Lemma 2.2, we get

$$T_3 \leq \sum_{n=1}^{\infty} (1+|B|)\,(p+n) \left(1 + \frac{\mu n}{p}\right) \frac{(B-A)(p-\alpha)\cos\lambda}{(p+n)} \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \sum_{n=1}^{\infty} \left(1 + \frac{\mu n}{p}\right) \frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} + \right.$$

$$\left. + \frac{\mu}{p} \frac{|ab|}{c} \frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right] =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left(1 + \frac{\mu}{p} \frac{|ab|}{(c-|a|-|b|-1)}\right).$$

It is easy to see that the last expression is bounded above by $(B-A)(p-\alpha)\cos\lambda$ if (2.8) holds.

Theorem 2.4 is proved.

Theorem 2.5. *Assume that* $a, b \in \mathbb{C}^*,\ |a| \neq 1,\ |b| \neq 1$ *and* $c > \max\{0, |a|+|b|-1\}$. *If the inequality*

$$\frac{1}{(|a|-1)(|b|-1)} \frac{\Gamma(c)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \leq 1 + \frac{(c-1)}{(|a|-1)(|b|-1)} + \frac{1}{(1+|B|)} \tag{2.10}$$

is true, then $[\mathcal{M}^{p}_{a,b,c}(f)](z)$ *maps the class* $\mathcal{R}^{\lambda}(A,B,p,\alpha)$ *into itself.*

Proof. It is enough to show that

$$T_4 = \sum_{n=1}^{\infty} (1+|B|)\,(p+n) \left| \frac{(a)_n (b)_n}{(c)_n (1)_n} \frac{a_{p+n}}{(p+n)} \right| \leq (B-A)(p-\alpha)\cos\lambda. \tag{2.11}$$

By using (2.3) and Lemma 2.2, we get

$$T_4 \leq (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \sum_{n=1}^{\infty} \frac{(|a|)_n (|b|)_n}{(c)_n (1)_{n+1}} =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \frac{(c-1)}{(|a|-1)(|b|-1)} \times$$

$$\times \left[\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1 - \frac{(|a|-1)(|b|-1)}{(c-1)} \right].$$

Theorem 2.5 is proved.

Theorem 2.6. *Assume that* $a, b \in \mathbb{C}^*,\ |a| \neq 1,\ |b| \neq 1$ *and* $c > \max\{0, |a|+|b|-1\}$. *If the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[\frac{\mu}{p} + \left(1-\frac{\mu}{p}\right) \frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)} \right] \leq$$

$$\leq 1 + \frac{1}{(1+|B|)} + \left(1-\frac{\mu}{p}\right) \frac{(c-1)}{(|a|-1)(|b|-1)}$$

is satisfied, then $[\mathcal{N}_{p,\mu}(f)]\,(z)$ *maps the class* $\mathcal{R}^{\lambda}(A,B,p,\alpha)$ *into itself.*

Proof. It suffices to show that

$$\sum_{n=1}^{\infty} (1+|B|)\,(p+n) \left| \left(1+\frac{\mu n}{p}\right) \frac{(a)_n (b)_n}{(c)_n (1)_n} \frac{a_{p+n}}{(p+n)} \right| \leq (B-A)(p-\alpha)\cos\lambda. \tag{2.12}$$

The left-hand side of (2.12), is less than or equal to

$$\sum_{n=1}^{\infty} (1+|B|) \left(1+\frac{\mu n}{p}\right) \frac{(B-A)(p-\alpha)\cos\lambda}{(n+1)} \frac{(|a|)_n (|b|)_n}{(c)_n (1)_n} = T_5,$$

where

$$T_5 = (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \sum_{n=1}^{\infty} \left(1+\frac{\mu n}{p}\right) \frac{(|a|)_n (|b|)_n}{(c)_n (1)_{n+1}} =$$

$$= (1+|B|)\,(B-A)(p-\alpha)\cos\lambda \left[\left(1-\frac{\mu}{p}\right) \sum_{n=1}^{\infty} \frac{(|a|)_n (|b|)_n}{(c)_n (1)_{n+1}} + \frac{\mu}{p} \sum_{n=1}^{\infty} \frac{(|a|)_n (|b|)_n}{(c)_n (1)_n} \right] =$$

$$= (1+|B|)\,(B-A)(p-\alpha) \left(1-\frac{\mu}{p}\right) \frac{(c-1)}{(|a|-1)(|b|-1)} \cos\lambda \times$$

$$\times \left[\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1 - \frac{(|a|-1)(|b|-1)}{(c-1)} \right] +$$

$$+\frac{\mu}{p}\left(1+|B|\right)(B-A)(p-\alpha)\cos\lambda\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]=$$

$$=\left(1+|B|\right)(B-A)(p-\alpha)\cos\lambda\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\times$$

$$\times\left[\left(1-\frac{\mu}{p}\right)\frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)}+\frac{\mu}{p}\right]-$$

$$-\left(1+|B|\right)(B-A)(p-\alpha)\cos\lambda\left[1+\left(1-\frac{\mu}{p}\right)\frac{(c-1)}{(|a|-1)(|b|-1)}\right].$$

By a simplification, we see that the last expression is bounded above by $(B-A)(p-\alpha)\cos\lambda$ if (2.12) holds.

Theorem 2.6 is proved.

Theorem 2.7. *Suppose that $a,b\in\mathbb{C}^*$ and c be a real number. If the inequality*

$$\frac{p\,|ab|}{c(p+1)}\;{}_3F_2\left(|a|+1,|b|+1,p+1;c+1,p+2;1\right)+$$

$$+\sigma{}_3F_2\left(|a|,|b|,p;c,p+1;1\right)\le\sigma\left(1+\frac{p}{(B-A)(p-\alpha)\cos\lambda}\right) \tag{2.13}$$

holds, then $[I^p_{a,b,c}(f)](z)$ maps the class $\mathcal{R}^\lambda(A,B,p,\alpha)$ to $\mathcal{S}^*\left(p,\sigma\right)$.

Proof. It is enough to prove that

$$\sum_{n=1}^{\infty}(n+\sigma)\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}a_{p+n}\right|\le\sigma. \tag{2.14}$$

The left-hand side of (2.14), is less than or equal to

$$T_6=\sum_{n=1}^{\infty}(n+\sigma)\frac{(B-A)(p-\alpha)\cos\lambda}{(p+n)}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}=$$

$$=(B-A)(p-\alpha)\cos\lambda\sum_{n=1}^{\infty}\left(\frac{n+\sigma}{p+n}\right)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}=$$

$$=\frac{(B-A)(p-\alpha)\cos\lambda}{p}\sum_{n=1}^{\infty}(n+\sigma)\frac{(|a|)_n(|b|)_n(p)_n}{(c)_n(p+1)_n(1)_n}=$$

$$=\frac{(B-A)(p-\alpha)\,|ab|\cos\lambda}{c(p+1)}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n(p+1)_n}{(c+1)_n(p+2)_n(1)_n}+$$

$$+\frac{(B-A)(p-\alpha)\sigma\cos\lambda}{p}\left[\sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n(p)_n}{(c)_n(p+1)_n(1)_n}-1\right]=$$

$$=\frac{(B-A)(p-\alpha)\,|ab|\cos\lambda}{c(p+1)}\;{}_3F_2\left(|a|+1,|b|+1,p+1;c+1,p+2;1\right)+$$

$$+\frac{(B-A)(p-\alpha)\sigma\cos\lambda}{p}\Big[\,{}_3F_2\left(|a|,|b|,p;c,p+1;1\right)-1\Big],$$

the last expression is bounded above by σ if (2.13) holds.

Theorem 2.7 is proved.

Theorem 2.8. *Suppose that* $a, b \in \mathbb{C}^*$ *and* $c > |a| + |b| + 1$. *If the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{|ab|}{(c-|a|-|b|-1)}+\sigma\right]\leq\sigma\left(1+\frac{p}{(B-A)(p-\alpha)\cos\lambda}\right) \tag{2.15}$$

is true, then $[I^p_{a,b,c}(f)](z)$ *maps the class* $\mathcal{R}^\lambda(A,B,p,\alpha)$ *to* $\mathcal{K}(p,\sigma)$.

Proof. It suffices to prove that

$$\sum_{n=1}^{\infty}(n+p)(n+\sigma)\left|\frac{(a)_n(b)_n}{(c)_n(1)_n}a_{p+n}\right|\leq p\sigma. \tag{2.16}$$

The left-hand side of (2.16), is less than or equal to

$$T_7=(B-A)(p-\alpha)\cos\lambda\sum_{n=1}^{\infty}(n+\sigma)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}=$$

$$=(B-A)(p-\alpha)\cos\lambda\left[\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}+\sigma\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]=$$

$$=(B-A)(p-\alpha)\cos\lambda\left[\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+\sigma\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-\sigma\right]=$$

$$=(B-A)(p-\alpha)\cos\lambda\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{|ab|}{(c-|a|-|b|-1)}+\sigma\right]-\sigma(B-A)(p-\alpha)\cos\lambda.$$

We note that the last expression is bounded above by $p\sigma$ if (2.15) holds.

Theorem 2.8 is proved.

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