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SUBCLASS OF k-UNIFORMLY STARLIKE FUNCTIONS DEFINED BY SYMMETRIC q-DERIVATIVE OPERATOR

ПІДКЛАС k-РІВНОМІРНО ЗІРКОПОДІБНИХ ФУНКЦІЙ, ЩО ВИЗНАЧЕНІ ЗА ДОПОМОГОЮ СИМЕТРИЧНОГО ОПЕРАТОРА q-ПОХІДНОЇ

The theory of q-analogs is frequently encountered in numerous areas, including fractals and dynamical systems. The q-derivatives and q-integrals play an important role in the study of q-deformed quantum-mechanical simple harmonic oscillators. We define a symmetric q-derivative operator and study a new family of univalent functions defined by using this operator. We establish some new relations between the functions satisfying analytic conditions related to conical sections.

Теорія q-аналогів часто зустрічається в багатьох галузях, включаючи фрактали та динамічні системи. Важливу роль у вивченні q-деформованих квантово-механічних простих гармонічних осциляторів відіграють q-похідні та q-інтеграли. Наведено визначення симетричного оператора q-похідної та вивчено нову сім'ю однолистих функцій, що визначені за допомогою цього оператора. Встановлено також деякі нові співвідношення між функціями, що задовольняють аналітичні умови відносно конічних перерізів.

1. Introduction, definitions and notations. The intrinsic properties of q-analogs, including the applications in the study of quantum groups and q-deformed superalgebras, study of fractals and multifractal measures, and in chaotic dynamical systems are known in the literature. Some integral transforms in the classical analysis have their q-analogues in the theory of q-calculus. This has led various researchers in the field of q-theory for extending all the important results involving the classical analysis to their q-analogs.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. Throughout this paper, we will assume that q satisfies the condition $0 < q < 1$. We shall follow the notation and terminology of [**?**]. We first recall the definitions of fractional q-calculus operators of complex valued function f.

Definition 1.1 [**?**]. *Let $q \in (0,1)$ and $\lambda \in \mathbb{C}$. The q-number, denoted $[\lambda]_q$, we define as*

$$[\lambda]_q = \frac{1-q^\lambda}{1-q}.$$

In the case when $\lambda = n \in \mathbb{N}$ we obtain $[\lambda]_q = 1 + q + q^2 + \ldots + q^{n-1}$, and when $q \to 1^-$ then $[n]_q = n$. The symmetric q-number, denoted $\widetilde{[n]}_q$ is defined as a number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

that reduces to n, in the case when $q \to 1^-$.

We note that the symmetric q-number do not reduce to the defined above q-number, and frequently occurs in the study of *q-deformed quantum mechanical simple harmonic oscillator* (see [**?**]).

Applying the above q-numbers we define q-derivative and symmetric q-derivative, below.

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Definition 1.2 [?]. *The q-derivative of a function f, defined on a subset of $\mathbb{C}$, is given by*

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \textit{for} \quad z \neq 0, \\ f'(0) & \textit{for} \quad z = 0. \end{cases}$$

We note that $\lim_{q \to 1^-} (D_q f)(z) = f'(z)$ if f is differentiable at z. Additionally, if $f(z) = z + a_2 z^2 + \ldots$, then

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Definition 1.3 [?]. *The symmetric q-derivative $\widetilde{D}_q f$ of a function f is defined as follows*:

$$(\widetilde{D}_q f)(z) = \begin{cases} \dfrac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} & \textit{for} \quad z \neq 0, \\ f'(0) & \textit{for} \quad z = 0. \end{cases} \tag{1.1}$$

From (1.1), we deduce that $\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}$, and a power series of $\widetilde{D}_q f$, when $f(z) = z + a_2 z^2 + \ldots$, is

$$(\widetilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}.$$

It is easy to check that the following properties hold:

$$\widetilde{D}_q(f(z) + g(z)) = (\widetilde{D}_q f)(z) + (\widetilde{D}_q g)(z),$$

$$\widetilde{D}_q\big(f(z)g(z)\big) = g(q^{-1}z)(\widetilde{D}_q f)(z) + f(qz)(\widetilde{D}_q g)(z) = g(qz)(\widetilde{D}_q f)(z) + f(q^{-1}z)(\widetilde{D}_q g)(z),$$

$$\widetilde{D}_q f(z) = D_{q^2} f(q^{-1}z).$$

The defined above fractional q-calculus are the important tools used in a study of various families of analytic functions, and in the context of univalent functions was first used in a book chapter by Srivastava [23]. In contrast to the Leibnitz notation, being a ratio of two infinitisemals, the notions of q-derivatives are plain ratios. Therefore, it appeared soon a generalization of q-calculus in many subjects, such as hypergeometric series, complex analysis, and particle physics. It is also widely applied in an approximation theory, especially on various operators, which includes convergence of operators to functions in real and complex domain. In the last twenty years q-calculus served as a bridge between mathematics and physics. The field has expanded explosively, due to the fact that applications of basic hypergeometric series to the diverse subjects of combinatorics, quantum theory, number theory, statistical mechanics, are constantly being uncovered. Specially, the theory of univalent functions can be newly described by using the theory of the q-calculus. In recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, [?, ?, ?, ?]). In the present paper we study the symmetric q-operator, and related problems involving univalent functions.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Also, let $\mathcal{S}$, $\mathbf{T}$ be the subclasses of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{D}$, and with negative coefficients, respectively. We denote by $\mathcal{ST}(\alpha)$ $(0 \le \alpha < 1)$ a subset of $\mathcal{S}$ consisting of all functions starlike of order α, i.e., such that $\Re(zf'(z)/f(z)) > \alpha$, $z \in \mathbb{D}$. When $\alpha = 0$ the class $\mathcal{ST}(\alpha)$ becomes the class $\mathcal{ST}$ of functions f that maps $\mathbb{D}$ onto a starlike domain with respect to the origin. By k-$\mathcal{ST}(\alpha)$ we denote the class of k-starlike functions of order α, $0 \le \alpha < 1$, that is a class of function f, which satisfy a condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha, \quad k \ge 0 \tag{1.3}$$

(for details see [**?**] and [**?**]).

We remark here that the class of k-starlike functions of order α is an extension of the relatively more familiar class of k-starlike functions investigated earlier by Kanas et al. [**?** – **?**, **?**] (see also [**?**, **?**]). For the case $k = 1$ that class was studied by Rønning [**?**], and called there "a parabolic class". We mention here that the name k-uniformly starlike was incorrectly attributed to the class of k-starlike functions defined by (1.3) (for $\alpha = 0$), and related to the class of k-uniformly convex functions by the well known Alexander relation. A class of uniformly starlike functions is due to Goodman [**?**] and was defined by the condition

$$\Re\left(\frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)}\right) > 0, \quad z, \zeta \in \mathbb{D},$$

and is completely different that the class k-stalike functions.

Definition 1.4. *Let $0 \le k < \infty$ and $0 \le \alpha < 1$. By k-$\widetilde{\mathcal{ST}}_q(\alpha)$ we denote the class of functions $f \in \mathcal{A}$ satisfying the condition*

$$\Re\left(\frac{z(\widetilde{D}_q f)(z)}{f(z)}\right) > k\left|\frac{z(\widetilde{D}_q f)(z)}{f(z)} - 1\right| + \alpha, \quad z \in \mathbb{D}. \tag{1.4}$$

We also set k-$\widetilde{\mathcal{ST}}_q^-(\alpha) = k$-$\widetilde{\mathcal{ST}}_q(\alpha) \cap \mathbf{T}$. *We note that* $\lim_{q \to 1^-} k$-$\widetilde{\mathcal{ST}}_q(\alpha) = k$-$\mathcal{ST}(\alpha)$.

Let $\mathcal{P}$ be the Carathèodory class of functions with positive real part consisting of all functions p analytic in $\mathbb{D}$ satisfying $p(0) = 1$, and $\Re(p(z)) > 0$. Making use of a properties of the Carathèodory functions we may rewrite a definition of k-$\widetilde{\mathcal{ST}}_q^-(\alpha)$. Setting $p(z) = \dfrac{z(\widetilde{D}_q f)(z)}{f(z)}$ we may rewrite a condition (1.4) in a form $\Re p(z) > k|p(z) - 1| + \alpha$ $(z \in \mathbb{D})$, or $p \prec p_{k,\alpha}$, where $p_{k,\alpha}$ is a function with a positive real part, that maps the unit disk onto a domain $\Omega_{k,\alpha}$, described by the inequality $\Re w > k|w - 1| + \alpha$ (here $\prec$ denotes a symbol of a subordination of the analytic functions). We note that $\Omega_{k,\alpha}$ is a domain bounded by a conic section, symmetric about real axis and contained in a right half plane. It is also known that $p_{k,\alpha}$ has the real and positive coefficients (see [**?**, 13]). We will use the notation $p_{k,\alpha} = 1 + P_1 z + P_2 z^2 + \ldots$.

It is known, that if $p \in \mathcal{P}$ has a Taylor series expansion $p(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, then $|B_n| \le 2$ for $n \in \mathbb{N}$ [**?**].

More refinement result was obtained by Grenander and Szegö [6].

Lemma 1.1 [6]. *If the function* $p \in \mathcal{P}$, *then*

$$2B_2 = B_1^2 + x(4 - B_1^2),$$

$$4B_3 = B_1^3 + 2(4 - B_1^2)B_1 x - B_1(4 - B_1^2)x^2 + 2(4 - B_1^2)(1 - |x|^2)z$$

for some x, z *with* $|x| \leq 1$ *and* $|z| \leq 1$.

2. Fundamental properties. Several new subclasses of the families of k-starlike and k-uniformly convex functions making use of linear operators and fractional calculus were studied (see, for example, [**?**, **?**]), and various interesting properties were obtained. In light of this, it is of interest to consider the behaviour of the classes k-$\widetilde{\mathcal{ST}}_q(\alpha)$ and k-$\widetilde{\mathcal{ST}}_q^-(\alpha)$ defined by symmetric q-derivative operator. We provide necessary and sufficient coefficient conditions, distortion bounds, and extreme points. In the first theorems we provide a necessary and a necessary and sufficient conditions to be a member of k-$\widetilde{\mathcal{ST}}_q(\alpha)$ and k-$\widetilde{\mathcal{ST}}_q^-(\alpha)$, respectively.

Theorem 2.1. *Let* $0 < q < 1$, *and* $f \in \mathcal{S}$ *be given by* (1.2). *If the inequality*

$$\sum_{n=2}^{\infty} \left[\widetilde{[n]}_q (k+1) - (k+\alpha)\right] |a_n| \leq 1 - \alpha \tag{2.1}$$

holds true for some k, $0 \leq k < \infty$ *and* α, $0 \leq \alpha < 1$, *then* $f \in k$-$\widetilde{\mathcal{ST}}_q(\alpha)$.

Proof. By a Definition 1.4, it suffices to prove that

$$k\left|\frac{z(\widetilde{D}_q f)(z)}{f(z)} - 1\right| - \Re\left(\frac{z(\widetilde{D}_q f)(z)}{f(z)} - 1\right) < 1 - \alpha.$$

Observe that

$$k\left|\frac{z(\widetilde{D}_q f)(z)}{f(z)} - 1\right| - \Re\left(\frac{z(\widetilde{D}_q f)(z)}{f(z)} - 1\right) \leq (k+1)\left|\frac{z(\widetilde{D}_q f)(z)}{f(z)} - 1\right| =$$

$$= (k+1)\left|\frac{\sum_{n=2}^{\infty}\left(\widetilde{[n]}_q - 1\right) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}\right| \leq$$

$$\leq (k+1)\frac{\sum_{n=2}^{\infty}\left(\widetilde{[n]}_q - 1\right) |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}.$$

The last expression is bounded by $1 - \alpha$, if the inequality (2.1) holds.

Theorem 2.1 is proved.

The inequality (2.1) gives a tool to obtain some special members k-$\widetilde{\mathcal{ST}}_q(\alpha)$. For example, we have the following corollary.

***Corollary* 2.1.** *Let* $0 \leq k < \infty$, $0 < q < 1$ *and* $0 \leq \alpha < 1$. *If*, *for* $f(z) = z + a_n z^n$, *the inequality*

$$|a_n| \leq \frac{1-\alpha}{\widetilde{[n]}_q (k+1) - (k+\alpha)}, \quad n \geq 2,$$

holds, *then* $f \in k$-$\widetilde{\mathcal{ST}}_q(\alpha)$. *Specially* $f(z) = z + \dfrac{(1-\alpha)q}{(q^2+1)(k+1) - q(k+\alpha)} z^2 \in k$-$\widetilde{\mathcal{ST}}_q(\alpha)$.

Theorem 2.2. *Let* $0 \le k < \infty,\ 0 < q < 1,$ *and* $0 \le \alpha < 1$. *A necessary and sufficient condition for* f *of the form* $f(z) = z - a_2 z^2 - \ldots,\ a_n \ge 0,$ *to be in the class* $k\text{-}\widetilde{\mathcal{ST}}_q^-(\alpha)$ *is that*

$$\sum_{n=2}^{\infty}\left[\widetilde{[n]}_q(k+1) - (k+\alpha)\right]a_n \le 1-\alpha. \tag{2.2}$$

The result is sharp, equality holds for the function f *given by*

$$f(z) = z - \frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)}z^n.$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f \in k\text{-}\widetilde{\mathcal{ST}}_q^-(\alpha)$, then by $|\Re(z)| \le |z|$ for any z we get

$$\left|\frac{1-\sum_{n=2}^{\infty}\widetilde{[n]}_q a_n z^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}} - \alpha\right| \ge k\left|\frac{\sum_{n=2}^{\infty}\left(\widetilde{[n]}_q - 1\right)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}\right|. \tag{2.3}$$

Choose values of z on the real axis so that $\widetilde{D}_q f(z)$ is real. Upon clearing the dominator of (2.3) and letting $z \to 1^-$ through the real values, we obtain (2.2).

Theorem 2.2 is proved.

Theorem 2.3. *Let* $0 \le k < \infty,\ 0 < q < 1$ *and* $0 \le \alpha < 1$. *Let the function* f *defined by* $f(z) = z - a_2 z^2 - \ldots,\ a_n \ge 0,$ *be in the class* $k\text{-}\widetilde{\mathcal{ST}}_q^-(\alpha)$. *Then for* $|z| = r < 1$ *it holds*

$$r - \frac{q\big(1-\alpha\big)}{\big(q^2+1\big)(k+1)-q(k+\alpha)}r^2 \le \big|f(z)\big| \le r + \frac{q(1-\alpha)}{\big(q^2+1\big)(k+1)-q(k+\alpha)}r^2. \tag{2.4}$$

Equality in (2.4) *holds true for the function* f *given by*

$$f(z) = z + \frac{q(1-\alpha)}{(q^2+1)(k+1)-q(k+\alpha)}z^2. \tag{2.5}$$

Proof. Since $f \in k\text{-}\widetilde{\mathcal{ST}}_q^-(\alpha)$, then in view of Theorem 2.2, we have

$$\left[\widetilde{[2]}_q(k+1)-(k+\alpha)\right]\sum_{n=2}^{\infty}a_n \le \sum_{n=2}^{\infty}\left[\widetilde{[n]}_q(k+1)-(k+\alpha)\right]|a_n| \le 1-\alpha,$$

which gives

$$\sum_{n=2}^{\infty}a_n \le \frac{1-\alpha}{\widetilde{[2]}_q(k+1)-(k+\alpha)}. \tag{2.6}$$

Therefore

$$|f(z)| \le |z| + \sum_{n=2}^{\infty}a_n\,|z|^n \le r + \frac{q\,(1-\alpha)}{(q^2+1)\,(k+1)-q(k+\alpha)}r^2$$

and

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty}a_n\,|z|^n \ge r - \frac{q\,(1-\alpha)}{(q^2+1)\,(k+1)-q(k+\alpha)}r^2.$$

The results follows by letting $r \to 1^-$.

Theorem 2.3 is proved.

Theorem 2.4. *Let* $0 \leq k < \infty,\ 0 < q < 1$ *and* $0 \leq \alpha < 1$. *Let the function* f *with the Taylor series* $f(z) = z - a_2 z^2 - \ldots,\ a_n \geq 0$, *be a member of the class* $k\text{-}\widetilde{\mathcal{ST}}_q^{-}(\alpha)$. *Then for* $|z| = r < 1$

$$1 - \frac{2q\,(1-\alpha)}{(q^2+1)\,(k+1) - q(k+\alpha)} r \leq \left|f'(z)\right| \leq 1 + \frac{2q\,(1-\alpha)}{(q^2+1)\,(k+1) - q(k+\alpha)} r. \tag{2.7}$$

Proof. Differentiating f and using triangle inequality for the modulus, we obtain

$$\left|f'(z)\right| \leq 1 + \sum_{n=2}^{\infty} n a_n\,|z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n a_n \tag{2.8}$$

and

$$\left|f'(z)\right| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n a_n. \tag{2.9}$$

The assertion (2.7) now follows from (2.8) and (2.9) by means of a rather simple consequence of (2.6) given by

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1-\alpha)}{\widetilde{[2]}_q (k+1) - (k+\alpha)}.$$

Theorem 2.4 is proved.

Theorem 2.5. *Let* $0 \leq k < \infty,\ 0 < q < 1,\ 0 \leq \alpha < 1$, *and set*

$$f_1(z) = z, \qquad f_n(z) = z - \frac{1-\alpha}{\widetilde{[n]}_q (k+1) - (k+\alpha)} z^n, \quad n = 2, 3, \ldots.$$

Then $f \in k\text{-}\widetilde{\mathcal{ST}}_q^{-}(\alpha)$ *if and only if* f *can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad \lambda_n > 0, \qquad \sum_{n=1}^{\infty} \lambda_n = 1.$$

Proof. Suppose that

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) =$$

$$= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n \left[z - \frac{1-\alpha}{\widetilde{[n]}_q (k+1) - (k+\alpha)} z^n \right] =$$

$$= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q (k+1) - (k+\alpha)} z^n =$$

$$= \left(\sum_{n=1}^{\infty} \lambda_n \right) z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q (k+1) - (k+\alpha)} z^n =$$

$$= z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q (k+1) - (k+\alpha)} z^n.$$

Then

$$\sum_{n=2}^{\infty}\lambda_n\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)}\,\frac{\widetilde{[n]}_q(k+1)-(k+\alpha)}{1-\alpha}=\sum_{n=2}^{\infty}\lambda_n=\sum_{n=1}^{\infty}\lambda_n-\lambda_1=1-\lambda_1\le 1,$$

and we have $f\in k\text{-}\widetilde{\mathcal{ST}}_q^{-}(\alpha)$.

Conversely, suppose that $f\in k\text{-}\widetilde{\mathcal{ST}}_q^{-}(\alpha)$. Since $|a_n|\le(1-\alpha)/\big[\widetilde{[n]}_q(k+1)-(k+\alpha)\big]$, we may set

$$\lambda_n=\frac{\widetilde{[n]}_q(k+1)-(k+\alpha)}{1-\alpha}\,|a_n| \qquad \text{and} \qquad \lambda_1=1-\sum_{n=2}^{\infty}\lambda_n.$$

Then

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n=z+\sum_{n=2}^{\infty}\lambda_n\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)}z^n=$$

$$=z+\sum_{n=2}^{\infty}\lambda_n(z+f_n(z))=z+\sum_{n=2}^{\infty}\lambda_nz+\sum_{n=2}^{\infty}\lambda_nf_n(z)=$$

$$=\left(1-\sum_{n=2}^{\infty}\lambda_n\right)z+\sum_{n=2}^{\infty}\lambda_nf_n(z)=\lambda_1z+\sum_{n=2}^{\infty}\lambda_nf_n(z)=\sum_{n=1}^{\infty}\lambda_nf_n(z).$$

Theorem 2.5 is proved.

3. Hankel determinant. Let n and s be the natural numbers, such that $n\ge 0$ and $s\ge 1$. In 1976 Noonan and Thomas [**?**] defined the s^{th} Hankel determinant of f as

$$H_s(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+s-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+s}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+s-1} & a_{n+s} & \cdots & a_{n+2s-2}\end{vmatrix} \quad (a_1=1). \tag{3.1}$$

This determinant has been considered by several authors. For example, Noor [**?**] determined the rate of growth of $H_s(n)$ as $n\to\infty$ for functions f given by (1.2) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$, known as a second Hankel determinant, were obtained in [**?**, **?**] for different classes of functions.

Note that

$$H_2(1)=\begin{vmatrix} a_1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2^2, \qquad H_2(2)=\begin{vmatrix} a_2 & a_3\\ a_3 & a_4\end{vmatrix}=a_2a_4-a_3^2,$$

and the first Hankel determinant $H_2(1)=a_3-a_2^2$ is known as a special case of the Fekete – Szegö functional.

In this section will look more closely at the behaviour of the first and second Hankel determinant in the class $k\text{-}\widetilde{\mathcal{ST}}_q(\alpha)$, additionally we find a bound of the Fekete – Szegö functional and, as a special case, we obtain a bound of $|H_2(1)|$. For convenience, in the sequel we use the abbreviations

$$q_2=\widetilde{[2]}_q-1, \qquad q_3=\widetilde{[3]}_q-1, \qquad q_4=\widetilde{[4]}_q-1, \qquad \text{where}\quad 0<q<1.$$

Theorem 3.1. *Let* $0 \le k < \infty,\ 0 < q < 1,\ 0 \le \alpha < 1$, *and let* $f \in k\text{-}\widetilde{\mathcal{ST}}_q(\alpha)$.

1. *If*

$$U - P_1 q_2(q_2 q_4 - 1) \le 0, \qquad V - P_1^2 q_2^2 q_4 \le 0,$$

then the second Hankel determinant satisfies

$$\left|a_2 a_4 - a_3^2\right| \le \frac{P_1^2}{q_3^2}.$$

2. *If*

$$U - P_1 q_2(q_2 q_4 - 1) \ge 0, \qquad 2S - U - P_1^2 q_2(1 + q_2 q_4) \ge 0,$$

or

$$U - P_1 q_2(q_2 q_4 - 1) \le 0, \qquad V - P_1^2 q_2^2 q_4 \ge 0,$$

then the second Hankel determinant satisfies

$$\left|a_2 a_4 - a_3^2\right| \le \frac{V}{q_2^2 q_3^2 q_4}.$$

3. *If*

$$U - P_1 q_2(q_2 q_4 - 1) > 0, \qquad 2V - U - P_1^2 q_2(1 + q_2 q_4) \le 0,$$

then

$$\left|a_2 a_4 - a_3^2\right| \le \frac{4\, P_1^2 q_2^2 q_4 V - 2P_1^2 q_2(1 + q_2 q_4)U - U^2 - P_1^4 q_2^2(1 + q_2 q_4)^2}{4\big(V - U - P_1^2 q_2\big) q_2^2 q_3^2 q_4},$$

where U, V, *and* M, N, S *are given by*

$$U = |M + 2P_1^2 q_2 + 2P_1 q_2 q_4 S|, \qquad V = |M + N + P_1^2 q_2 - q_4 S^2 + 2P_1 q_2 q_4 S|,$$
$$N = P_1 q_3\big[P_1^3 + (P_3 - 2P_2) q_2 q_3 + P_1(P_2 - P_1)(q_2 + q_3) + P_1 q_2 q_3\big], \tag{3.2}$$
$$M = P_1 q_3\big[2q_2 q_3(P_2 - P_1) + P_1^2(q_2 + q_3)\big], \qquad S = P_1^2 + q_2(P_2 - P_1).$$

Proof. Let $f \in k\text{-}\widetilde{\mathcal{ST}}_q(\alpha)$. Then, there exists a Schwarz function w, $w(0) = 1, |w(z)| < 1$ for $z \in \mathbb{D}$, such that

$$\frac{z(\widetilde{D}_q f)(z)}{f(z)} = p_{k,\alpha}(w(z)).$$

Let

$$p_0(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + B_1 z + B_2 z^2 + \ldots, \tag{3.3}$$

or, equivalently,

$$w(z) = \frac{p_0(z) - 1}{p_0(z) + 1} = \frac{1}{2}\left(B_1 z + \left(B_2 - \frac{B_1^2}{2}\right) z^2 + \ldots\right).$$

Such function p_0 is analytic in the unit disk, and has a positive real part there. By using the Taylor expansion of $p_{k,\alpha}$ and w, we obtain

$$p_{k,\alpha}(w(z)) = 1 + \frac{P_1 B_1}{2} z + \left(\frac{P_1 B_2}{2} + \frac{B_1^2(P_2 - P_1)}{4}\right) z^2 +$$

$$+\left(\frac{P_1B_3+(P_2-P_1)B_1B_2}{2}+\frac{B_1^3(P_3+P_1)}{8}-\frac{P_2B_1^3}{4}\right)z^3+\ldots. \tag{3.4}$$

Since

$$\frac{z(\widetilde{D}_qf)(z)}{f(z)}=1+q_2a_2z+\left[q_3a_3-q_2a_2^2\right]z^2+\left[q_4a_4-(q_2+q_3)a_2a_3+q_2a_2^3\right]z^3+\ldots,$$

then, combining (3.3) with (3.4), we have

$$a_2=\frac{P_1B_1}{2q_2},\qquad a_3=\frac{1}{4q_2q_3}\left[P_1^2B_1^2-P_1B_1^2q_2+P_2B_1^2q_2+2P_1B_2q_2\right],$$

$$a_4=\frac{B_1^3\big(P_1^3+(P_3-2P_2+P_1)q_2q_3+P_1(P_2-P_1)(q_2+q_3)\big)}{8q_2q_3q_4}+ \tag{3.5}$$

$$+\frac{2B_1B_2\big(P_1^2(q_2+q_3)+2q_2q_3(P_2-P_1)\big)+4B_3P_1q_2q_3}{8q_2q_3q_4}.$$

From the above we find that

$$H_2(2)=a_2a_4-a_3^2=\frac{B^4N+(2B_2)B^2M+(4B_3)BP_1^2q_2q_3^2-\left[(2B_2)P_1q_2+B^2S\right]^2q_4}{16q_2^2q_3^2q_4},$$

where, without loss of generality, we set $B:=B_1>0$, and N, M, S are given by (3.2). Applying Lemma 1.1 and performing the necessary computations, we obtain

$$H_2(2)=\frac{B^4\left[N+M+P_1^2q_2-q_4S^2+2P_1q_2q_4S\right]+xB^2(4-B^2)\left[M+2P_1^2q_2-2P_1q_2q_4S\right]}{16q_2^2q_3^2q_4}+$$

$$+\frac{-x^2(4-B^2)\left[B^2P_1^2q_2+4P_1^2q_2^2q_4\right]+2B(4-B^2)(1-|x|^2)zP_1^2q_2q_3^2}{16q_2^2q_3^2q_4}.$$

Set now $\rho=|x|$, where $\leq\rho\leq 1$, and take an absolute value of $H_2(2)$. Applying additionally $|z|\leq 1$, we have $|H_2(2)|\leq\Phi(\rho,B)=W(\alpha\rho^2+\beta\rho+\gamma)$, where

$$\alpha=(4-B^2)\left[B^2P_1^2q_2+4P_1^2q_2^2q_4\right]-2B(4-B^2)^2P_1^2q_2q_3^2,$$

$$\beta=B^2(4-B^2)\left|M+2P_1^2q_2+2P_1q_2q_4S\right|,$$

$$\gamma=2B(4-B^2)P_1^2q_2q_3^2+B^4\left|N+M+P_1^2q_2-q_4S^2+2P_1q_2q_4S\right|,$$

and $W=1/(16q_2^2q_3^2q_4)$. We note that $\alpha\geq 0$, $\beta\geq 0$. Indeed, an inequality $\beta\geq 0$ is obvious, and we get $\alpha=(4-B^2)P_1^2q_2\left[B^2-2Bq_3^2+4q_2q_4\right]$. The expression in a square brackets $\Psi(B)=B^2-$ $-2Bq_3^2+4q_2q_4$ is a quadratic function of B $(0\leq B\leq 2)$ with roots at $B=2$, and $B=2(q_3^2-1)>$ >2. Since $\Psi(0)=4q_2q_4>0$, then $\Psi(B)>0$ for $0\leq B\leq 2$. Hence $\partial\Phi/\partial\rho=W(2\alpha\rho+\beta)\geq 0$, and from this fact we conclude that Φ is increasing function of ρ. Therefore, for fixed $B\in[0,2]$, the maximum of $\Phi(\rho,B)$ is attained at $\rho=1$, that is $\max\Phi(\rho,B)=\Phi(1,B)=:G(B)$. We note that

$$G(B) = \frac{1}{16q_2^2q_3^2q_4}\Bigg(B^4\Big[\big|M+N+P_1^2q_2-q_4S^2+2P_1q_2q_4S\big|-$$

$$-\big|M+2P_1^2q_2+2P_1q_2q_4S\big|-P_1^2q_2\Big]+B^2\Big[4\big|M+2P_1^2q_2+2P_1q_2q_4S\big|+$$

$$+4P_1^2q_2(1-q_2q_4)\Big]+16P_1^2q_2^2q_4\Bigg).$$

Let

$$\begin{aligned} &P = \big|M+N+P_1^2q_2-q_4S^2+2P_1q_2q_4S\big|-\big|M+2P_1^2q_2+2P_1q_2q_4S\big|-P_1^2q_2,\\ &Q = 4\big|M+2P_1^2q_2+2P_1q_2q_4S\big|+4P_1^2q_2(1-q_2q_4),\\ &R = 16P_1^2q_2^2q_4. \end{aligned} \tag{3.6}$$

Now, analyzing the maximum of a Pt^2+Qt+R, over $0\le t\le 4$, we conclude that

$$|H_2(2)| \le \frac{1}{16q_2^2q_3^2q_4}\begin{cases} R & \text{for}\quad Q\le 0,\quad P\le -Q/4,\\ 16P+4Q+R & \text{for}\quad Q\ge 0, P\ge -Q/8 \quad\text{or}\quad Q\le 0, P\ge -Q/4,\\ R-Q^2/(4P) & \text{for}\quad Q>0,\quad P\le -Q/8, \end{cases}$$

where P, Q, R are given by (3.6).

Theorem 3.1 is proved.

***Corollary* 3.1.** *Let $q\to 1^-$. Then k-$\widetilde{\mathcal{ST}}_q(\alpha)\to k$-$\mathcal{ST}(\alpha)$, for which $P_1=\dfrac{8}{\pi^2}$. Then we get*

$$\big|a_2a_4-a_3^2\big| \le \frac{16}{\pi^2}.$$

Theorem 3.2. *Let $0\le k<\infty$, $0<q<1$, $0\le\alpha<1$, and let $f\in k$-$\widetilde{\mathcal{ST}}_q(\alpha)$. Then for complex μ it holds*

$$\big|a_3-\mu a_2^2\big| \le \frac{P_1^2|q_2-\mu q_3|+P_2q_2^2}{q_2^2q_3}.$$

In the case, when μ is real, then

$$\big|a_3-\mu a_2^2\big| \le \begin{cases} \dfrac{P_2q^2}{q^4+1}+P_1^2q^2\dfrac{q(q^2-q+1)-\mu(q^4+1)}{(q^4+1)(q^2-q+1)^2} & \textit{for}\quad \mu\le\dfrac{q(q^2-q+1)}{q^4+1},\\[2ex] \dfrac{P_2q^2}{q^4+1}+P_1^2q^2\dfrac{\mu(q^4+1)-q(q^2-q+1)}{(q^4+1)(q^2-q+1)^2} & \textit{for}\quad \mu\ge\dfrac{q(q^2-q+1)}{q^4+1}. \end{cases}$$

Proof. We apply a form of a_2, a_3, given by (3.5), and assume as in the proof of the first part that $B:=B_1>0$. Then, for complex μ, we have

$$a_3-\mu a_2^2 = \frac{B^2\big(P_1^2q_2+q_2^2(P_2-P_1)-\mu P_1^2q_3\big)+(2B_2)P_1q_2^2}{4q_2^2q_3}.$$

Making use of Lemma 1.1, we obtain

$$a_3 - \mu a_2^2 = \frac{B^2\big(P_1^2 q_2 + q_2^2(P_2 - P_1) - \mu P_1^2 q_3\big) + (B^2 + x(4 - B^2))P_1 q_2^2}{4 q_2^2 q_3},$$

where x is a complex number satisfying $|x| \leq 1$. Hence

$$a_3 - \mu a_2^2 = \frac{B^2\big[q_2(P_1^2 + P_2 q_2) - \mu P_1^2 q_3\big] + (4 - B^2)P_1 q_2^2}{4 q_2^2 q_3}.$$

After simplification and using $B \leq 2$, we get

$$|a_3 - \mu a_2^2| = \frac{\big|P_1^2(q_2 - \mu q_3) + P_2 q_2^2\big|}{q_2^2 q_3}.$$

We note also that P_1, P_2 are nonnegative, and q_2, q_3 are positive real number, therefore

$$|a_3 - \mu a_2^2| = \frac{P_1^2\big|q_2 - \mu q_3\big| + P_2 q_2^2}{q_2^2 q_3},$$

that establishes our first assertion. For real μ our claim is deduced by the observation that $q_2 = q + 1/q - 1$, and $q_3 = q^2 + 1/q^2$, where $0 < q < 1$.

Theorem 3.2 is proved.

A trivial computation gives the bound for the first Hankel derivative, and for the third coefficient, below.

***Corollary* 3.2.** *Let* $0 \leq k < \infty,\ 0 < q < 1,\ 0 \leq \alpha < 1$, *and let* $f \in k\text{-}\widetilde{\mathcal{ST}}_q(\alpha)$. *Then the first Hankel determinant satisfy*

$$\big|a_3 - a_2^2\big| \leq \frac{q^2(P_2 + P_1^2 q)}{q^4 + 1} - \frac{P_1^2 q^2}{q^2 - q + 1}.$$

***Corollary* 3.3.** *Under the assumption the same as in the Corollary* 3.2 *we have*

$$|a_3| \leq \frac{q^2\big(P_2 + P_1^2 q\big)}{q^4 + 1}.$$

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