UDC 512.5

T. S. Ma, H. Y. Li, L. H. Dong (School Math. and Inform. Sci., Henan Normal Univ., Xinxiang, China)

A CLASS OF DOUBLE CROSSED BIPRODUCTS*
ПРО ОДИН КЛАС ПОДВІЙНИХ ПЕРЕХРЕСНИХ БІДОБУТКІВ

Let H be a bialgebra, let A be an algebra and a left H-comodule coalgebra, let B be an algebra and a right H-comodule coalgebra. Also let $f: H\otimes H \longrightarrow A\otimes H$, $R: H\otimes A \longrightarrow A\otimes H$, and $T: B\otimes H \longrightarrow H\otimes B$ be linear maps. We present necessary and sufficient conditions for the one-sided Brzeziński's crossed product algebra $A\#^f_R H_T\# B$ and the two-sided smash coproduct coalgebra $A\times H\times B$ to form a bialgebra, which generalizes the main results from [On Ranford biproduct // Communs Algebra. – 2015. – **43**, № 9. – P. 3946 – 3966]. It is clear that both Majid's double biproduct [Double-bosonization of braided groups and the construction of $U_q(g)$ // Math. Proc. Cambridge Phil. Soc. – 1999. – **125**, № 1. – P. 151 – 192] and the Wang – Jiao – Zhao's crossed product [Hopf algebra structures on crossed products // Communs Algebra. – 1998. – **26**. – P. 1293 – 1303] are obtained as special cases.

Нехай H — біалгебра, A — алгебра та водночас ліва H-комодульна коалгебра, а B — алгебра та водночас права H-комодульна коалгебра. Крім того, нехай $f: H\otimes H \longrightarrow A\otimes H$, $R: H\otimes A \longrightarrow A\otimes H$ та $T: B\otimes H \longrightarrow H\otimes B$ — лінійні відображення. Наведено необхідні та достатні умови для того, щоб одностороння алгебра Бжезінського $A\#^f_R H_T\# B$ з перехресним добутком та двостороння коалгебра $A\times H\times B$ зі схрещеним кодобутком утворювали біалгебру, що узагальнює основні результати, отримані в [On Ranford biproduct // Communs Algebra. – 2015. – **43**, № 9. – P. 3946 – 3966]. Очевидно, що як подвійний бідобуток Маджіда [Double-bosonization of braided groups and the construction of $U_q(g)$ // Math. Proc. Cambridge Phil. Soc. – 1999. – **125**, № 1. – P. 151 – 192], так і перехресний добуток Ванга – Джао – Жао [Hopf algebra structures on crossed products // Communs Algebra. – 1998. – **26**. – P. 1293 – 1303] можна отримати як частинні випадки.

1. Introduction and preliminaries. Let H be a Hopf algebra over a field K. S. Majid [10, 11] made the following conclusion: A is a bialgebra in Yetter – Drinfeld category ${}^H_H\mathcal{YD}$ if and only if $A\star H$ is a Radford biproduct [14]. The Radford biproduct plays an important role in the lifting method for the classification of finite dimensional pointed Hopf algebras [2]. Let A be a bialgebra in ${}^H_H\mathcal{YD}$ and B a bialgebra in $\mathcal{YD}^H_H$. In [9], S. Majid gave the sufficient conditions for a two-sided smash product algebra $A\#H\#B$ and a two-sided smash coproduct coalgebra $A\times H\times B$ to be a bialgebra, named the double biproduct and denoted by $A\diamondsuit H\diamondsuit B$. Some related results about the double biproduct were recently given in the literature [6, 7, 9, 13].

Let A be an associative and unitary algebra and H a vector space endowed with a distinguished element 1_H. Let $f: H\otimes H \longrightarrow A\otimes H$ and $R: H\otimes A \longrightarrow A\otimes H$ be two linear maps. Let $A\#^f_R H$ be an associative and unitary algebra, with underlying vector space $A\otimes H$. Following [4], T. Brzeziński gave the necessary and sufficient conditions for the crossed product $A\#^f_R H$ to be an algebra, called Brzeziński's crossed product. Brzeziński's crossed product is an extensive definition that includes the crossed product $A\#_\sigma H$ in [3] and the twisted tensor product $A\#_R H$ in [5]. In [6], the authors replaced the left smash product by the crossed product $A\#_\sigma H$ in the double biproduct $A\diamondsuit H\diamondsuit B$ and obtained a generalized version of $A\diamondsuit H\diamondsuit B$. And in [7], the authors gave a further extension of $A\diamondsuit H\diamondsuit B$ via Brzeziński's crossed product. When the twisted tensor product takes the place of the right smash product in $A\diamondsuit H\diamondsuit B$, we want to know under what conditions the resulting structure will inherit a bialgebra structure. In this paper, we will derive the necessary and sufficient conditions for

* This work was partially supported by China Postdoctoral Science Foundation (No. 2017M611291), Foundation for Young Key Teacher by Henan Province (No. 2015GGJS-088), Natural Science Foundation of Henan Province (No. 17A110007), and National Natural Science Foundation of China (No. 11801150).

the one-sided Brzeziński's crossed product algebra $A\#_R^f H_T\#B$ and the two-sided smash coproduct coalgebra $A\times H\times B$ to be a bialgebra, which we call the Brzeziński's double biproduct. The main results in [6, 7] will be included, of course, the celebrated Radford biproduct [14], Majid's double biproduct [9], Agore and Militaru's unified product [1] and Wang – Jiao – Zhao's crossed product [16] are all examples of the Brzeziński's double biproduct.

Throughout the paper, we follow the definitions and terminologies in [12, 15] and all algebraic systems are over a field K. Let C be a coalgebra. Then we use the simple Sweedler's notation for the comultiplication, $\Delta(c)=c_1\otimes c_2,\ c\in C$. We denote the category of left H-comodules by ${}^H\mathcal{M}$, for $(M,\rho)\in {}^H\mathcal{M}$ and write $\rho(x)=x_{(-1)}\otimes x_{(0)}\in H\otimes M$ for all $x\in M$. We denote the category of right H-comodules by $\mathcal{M}^H$ for $(M,\psi)\in\mathcal{M}^H$, write $\psi(x)=x_{[0]}\otimes x_{[1]}\in M\otimes H$, for all $x\in M$. We denote the left-left Yetter – Drinfeld category by ${}^H_H\mathcal{YD}$ and the right-right Yetter – Drinfeld category by $\mathcal{YD}^H_H$. Given a K-space M, we write id_M for the identity map on M.

Next we recall [4, 9, 12, 14] some basic definitions and results which will be used later.

Brzeziński's crossed product. Let A be an algebra, H a vector space and $1_H\in H$. The vector space $A\otimes H$ is an algebra with unit $1_A\otimes 1_H$ and a product such that

$$(a\otimes 1_H)(a'\otimes x')=aa'\otimes x'$$

if and only if there exist linear maps $f\colon H\otimes H\longrightarrow A\otimes H$ $\big($write $f(x\otimes x')=x^f\otimes x'_f$ for all $x,x'\in H\big)$ and $R\colon H\otimes A\longrightarrow A\otimes H$ (write $R(x\otimes a)=a_R\otimes x_R$ for all $x\in H$ and $a\in A$) that satisfy the following conditions:

(A_1) $a_R\otimes 1_{HR}=a\otimes 1_H,\ 1_{AR}\otimes x_R=1_A\otimes x;$

(A_2) $(aa')_R\otimes x_R=a_Ra'_r\otimes x_{Rr};$

(A_3) $x^f\otimes 1_{Hf}=1_H{}^f\otimes x_f=1_A\otimes x;$

(A_4) $x'^f{}_R x_R{}^g\otimes x''_{fg}=x^f x'_f{}^g\otimes x''_g;$

(A_5) $a_{Rr}x_r{}^f\otimes x'_{Rf}=x^f a_R\otimes x'_{fR}$

for all $a,a'\in A,\ x,x',x''\in H$, where $g=f$ and $r=R$.

The product $\mu_{A\otimes H}$ in $A\otimes H$ explicitly reads

$$(a\otimes x)(a'\otimes x')=aa'_Rx_R{}^f\otimes x'_f$$

for all $a,a'\in A$ and $x,x'\in H$. In this case, we call the algebra *Brzeziński's crossed product* [4] and denote it by $A\#_R^f H$.

– Let A be a bialgebra and H a coalgebra with $1_H\in H$. A Brzeziński's crossed product, $A\#_R^f H$, equipped with the usual tensor product coalgebra structure is a bialgebra if and only if the following conditions hold:

(B_1) $\Delta_H(1_H)=1_H\otimes 1_H$ and $\varepsilon_H(1_H)=1;$

(B_2) f is a coalgebra map;

(B_3) R is a coalgebra map.

Double biproduct. We recall, from [9], the construction of the so-called double biproduct. Let H be a bialgebra, A a bialgebra in ${}^H_H\mathcal{YD}$, and B a bialgebra in $\mathcal{YD}^H_H$. Adopt the following notation for the structure maps: the counits are ε_A and ε_B, the comultiplications are $\Delta_A(a)=a_1\otimes a_2$ and $\Delta_B(b)=b_1\otimes b_2$, and the actions and coactions are

$$H\otimes A\longrightarrow A,\quad x\otimes a\mapsto x\triangleright a,$$

$$A \longrightarrow H \otimes A, \quad a \mapsto a_{(-1)} \otimes a_{(0)},$$

$$B \otimes H \longrightarrow B, \quad b \otimes x \mapsto b \triangleleft x,$$

$$B \longrightarrow B \otimes H, \quad b \mapsto b_{[0]} \otimes b_{[1]}$$

for all $x \in H$, $a \in A$, $b \in B$. Let $A \diamondsuit H \diamondsuit B$ denote the vector space $A \otimes H \otimes B$, which becomes an algebra (called the two-sided smash product, $A \# H \# B$) with unit $1_A \otimes 1_H \otimes 1_B$ and multiplication

$$(a \otimes x \otimes b)(a' \otimes x' \otimes b') = a(x_1 \triangleright a') \otimes x_2 x'_1 \otimes (b \triangleleft x'_2)b',$$

and a coalgebra (called the two-sided smash coproduct, $A \times H \times B$) with counit $\varepsilon(a \otimes x \otimes b) = = \varepsilon_A(a)\varepsilon_H(x)\varepsilon_B(b)$ and comultiplication

$$\Delta \colon A \diamondsuit H \diamondsuit B \longrightarrow (A \diamondsuit H \diamondsuit B) \otimes (A \diamondsuit H \diamondsuit B),$$

$$\Delta(a \otimes x \otimes b) = a_1 \otimes a_{2(-1)} x_1 \otimes b_{1[0]} \otimes a_{2(0)} \otimes x_2 b_{1[1]} \otimes b_2.$$

Moreover, assume that the following condition holds:

(DB) $b_{[1]} \triangleright a_{(0)} \otimes b_{[0]} \triangleleft a_{(-1)} = a \otimes b,\ a \in A,\ b \in B.$

It follows that $A \diamondsuit H \diamondsuit B$ is a bialgebra, called the *double biproduct*.

Remark **1.1.** When $A = K$ (or $B = K$), the double biproduct is exactly the right (or left) variant of Radford biproduct.

2. Main results and its consequence. In this section, we give an extended version of the structure of the Majid's double biproduct.

First, we list the right version of twisted tensor product.

Proposition 2.1. *Let B and H be two algebras, $T \colon B \otimes H \longrightarrow H \otimes B$ a linear map. Then $H_T \# B$ $(= H \otimes B$ as a linear space$)$ with the multiplication*

$$(x \otimes b)(x' \otimes b') = x x'_T \otimes b_T b',$$

where $x, x' \in H, b, b' \in B$, and unit $1_H \otimes 1_B$ becomes an algebra if and only if the following conditions hold:

(RT_1) $b_T \otimes 1_{HT} = b \otimes 1_H,\ 1_{BT} \otimes x_T = 1_B \otimes x,$

(RT_2) $x_T \otimes (bb')_T = x_{Tt} \otimes b_t b'_T,$

(RT_3) $(xx')_T \otimes b_T = x_T x'_t \otimes b_{Tt},$

where $x, x' \in H$, $b, b' \in B$, and $t = T$. We call this algebra right twisted tensor product algebra and denote it by $H_T \# B$.

Proof. Straightforward.

Lemma 2.1. *Let H be a vector space and $1_H \in H$ and A, B be two algebras. Let $f \colon H \otimes H \longrightarrow A \otimes H$, $R \colon H \otimes A \longrightarrow A \otimes H$ and $T \colon B \otimes H \longrightarrow H \otimes B$ be linear maps. If conditions (RT_1), (RT_2), $(A_1)-(A_3)$ and*

(BT_1) ${x'}^f{}_R x_R{}^F \otimes x''{}_{fTF} \otimes b_T = x^f {x'}_{Tf}{}^F \otimes x''{}_{tF} \otimes b_{Tt};$

(BT_2) $a_{Rr} x_r{}^f \otimes x'{}_{RTf} \otimes b_T = x^f a_R \otimes x'{}_{TfR} \otimes b_T$

are satisfied for all $a \in A,\ b \in B,\ x, x', x'' \in H$ *and* $F = f,\ r = R,\ t = T$, *then* $A\#^f_R H_T\#B$ ($= A \otimes H \otimes B$ *as a vector space*) *is an associative algebra with unit* $1_A \otimes 1_H \otimes 1_B$ *and multiplication given by*

$$(a \otimes x \otimes b)(a' \otimes x' \otimes b') = aa'_R x_R{}^f \otimes x'_{Tf} \otimes b_T b',$$

where $a, a' \in A,\ x, x' \in H$ *and* $b, b' \in B$. *In this case, we call* $A\#^f_R H_T\#B$ *the one-sided Brzeziński crossed product.*

Proof. We check associativity as follows. For $a, a', a'' \in A,\ x, x', x'' \in H,\ b, b', b'' \in B$ and $F = f,\ \bar{R} = r = \bar{r} = R$,

$$(a \otimes x \otimes b)\big((a' \otimes x' \otimes b')(a'' \otimes x'' \otimes b'')\big) =$$

$$= a\underline{(a'a''_R x'_R{}^f)_r}\underline{x_r}{}^F \otimes x''_{TftF} \otimes b_t b'_T b'' \overset{(A_2)}{=}$$

$$\overset{(A_2)}{=} aa'_r a''_{R\bar{R}} \underline{x'_R{}^f{}_{\bar{r}} x_{r\bar{R}\bar{r}}}{}^F \otimes \underline{x''_{TftF}} \otimes \underline{b_t} b'_T b'' \overset{(BT_1)}{=}$$

$$\overset{(BT_1)}{=} aa'_r \underline{a''_{R\bar{R}} x_{r\bar{R}}}{}^f \underline{x'_{Rtf}}{}^F \otimes x''_{T\bar{t}F} \otimes \underline{b_{t\bar{t}}} b'_T b'' \overset{(BT_2)}{=}$$

$$\overset{(BT_2)}{=} aa'_r x_r{}^f a''_R x'_{tfR}{}^F \otimes \underline{x''_{T\bar{t}F}} \otimes \underline{b_{t\bar{t}} b'_T} b'' \overset{(RT_2)}{=}$$

$$\overset{(RT_2)}{=} aa'_r x_r{}^f a''_R x'_{TfR}{}^F \otimes x''_{tF} \otimes (b_T b')_t b'' =$$

$$= \big((a \otimes x \otimes b)(a' \otimes x' \otimes b')\big)(a'' \otimes x'' \otimes b'').$$

It's obvious that $1_A \otimes 1_H \otimes 1_B$ is a unit by conditions (A_1), (A_3) and (RT_1).

***Remark* 2.1.** (1) If there is an element $\varepsilon_B \in \mathrm{Hom}(B, K)$ such that $\varepsilon_B(1_B) = 1$ and T is trivial, i.e., T is the flip map, then the conditions (A_4) and (A_5) can be obtained by setting $b = 1_B$ and applying $\mathrm{id}_A \otimes \mathrm{id}_H \otimes \varepsilon_B$ to the condition (BT_1) and (BT_2), respectively.

(2) If H is an algebra, and there is an element $\varepsilon_A \in \mathrm{Hom}(A, K)$ such that $\varepsilon_A(1_A) = 1$, and $f,\ F$ are trivial, then the condition (RT_3) can be obtained by setting $x = 1_H$ and applying $\varepsilon_A \otimes \mathrm{id}_H \otimes \mathrm{id}_B$ to the condition (BT_1).

(3) Taking either $A = K$ or $B = K$, we obtain the right twisted tensor product and Brzeziński crossed product, respectively.

Theorem 2.1. *Let* H *be a bialgebra*, A *an algebra and a left* H*-comodule coalgebra such that* $\varepsilon_A(1_A) = 1$, B *an algebra and a right* H*-comodule coalgebra such that* $\varepsilon_B(1_B) = 1$. *Let* $f\colon H \otimes H \longrightarrow A \otimes H$, $R\colon H \otimes A \longrightarrow A \otimes H$, $T\colon B \otimes H \longrightarrow H \otimes B$ *be linear maps such that*

(BP) $a_{(-1)}x'_{T1} \otimes b_{T1[0]}b'_{[0]} \otimes a_{(0)R}x_R{}^f b_{T1[1]}{}^F \otimes x'_{T2f}b'_{[1]tF} \otimes b_{T2t} =$

$$= (a_{(-1)}x'_1)_T \otimes b_{1[0]T}b'_{[0]} \otimes a_{(0)R}(xb_{1[1]})_R{}^f \otimes (x'_2 b'_{[1]})_{tf} \otimes b_{2t}$$

holds for all $a \in A,\ x, x' \in H,\ b, b' \in B$ *and* $F = f,\ t = T$. *Then the one-sided Brzeziński crossed product* $A\#^f_R H_T\#B$ *equipped with the two-sided smash coproduct* $A \times H \times B$ *becomes a bialgebra if and only if the following conditions hold* ($a, a' \in A,\ x, x' \in H,\ b, b' \in B,\ F = f$ *and* $r = R$):

(C_1) $\varepsilon_A,\ \varepsilon_B$ *are algebra maps*, $\varepsilon_A(x^f)\varepsilon_H(x'_f) = \varepsilon_H(x)\varepsilon_H(x')$;

(C_2) $\varepsilon_A(a_R)\varepsilon_H(x_R) = \varepsilon_A(a)\varepsilon_H(x),\ \varepsilon_B(b_T)\varepsilon_H(x_T) = \varepsilon_B(b)\varepsilon_H(x)$;

(C_3) $\Delta_A(1_A) = 1_A \otimes 1_A,\ \Delta_B(1_B) = 1_B \otimes 1_B$;

(C_4) $1_{A(-1)} \otimes 1_{A(0)} = 1_H \otimes 1_A,\ 1_{B[0]} \otimes 1_{B[1]} = 1_B \otimes 1_H$;

(C_5) $(bb')_{1[0]} \otimes 1_A \otimes (bb')_{1[1]} \otimes (bb')_2 = b_{1[0]}b'_{1[0]} \otimes {b_{1[1]}}^f \otimes {b'_{1[1]}}_{Tf} \otimes b_{2T}b'_2$;

(C_6) $a_1 \otimes a_{2(-1)}x \otimes a_{2(0)} = a_1{a_{2(-1)}}^f \otimes x_f \otimes a_{2(0)}$;

(C_7) $(aa')_1 \otimes (aa')_{2(-1)} \otimes (aa')_{2(0)} = a_1a'_{1R}{a_{2(-1)R}}^f \otimes a'_{2(-1)f} \otimes a_{2(0)}a'_{2(0)}$;

(C_8) ${x^f}_1 \otimes {x^f}_{2(-1)}x'_{f1} \otimes {x^f}_{2(0)} \otimes x'_{f2} = {x_1}^f \otimes x'_{1f} \otimes {x_2}^F \otimes x'_{2F}$;

(C_9) $a_{R1} \otimes a_{R2(-1)}x_{R1} \otimes a_{R2(0)} \otimes x_{R2} = a_{1R}{x_{1R}}^f \otimes a_{2(-1)f} \otimes a_{2(0)r} \otimes x_{2r}$.

In this case, we call the bialgebra *Brzeziński double biproduct*, and denote it by $A\diamondsuit_R^f H_T \diamondsuit B$.

Proof. *Sufficiency*. It is easy to prove that $\varepsilon_{A\times H\times B}$ is an algebra map. Here we check only that $\Delta_{A\times H\times B}$ is an algebra map. We have

$$\Delta_{A\times H\times B}((a\otimes x\otimes b)(a'\otimes x'\otimes b')) =$$

$$= (aa'_R{x_R}^f)_1 \otimes (aa'_R{x_R}^f)_{2(-1)}x'_{Tf1} \otimes (b_Tb')_{1[0]} \otimes (aa'_R{x_R}^f)_{2(0)} \otimes x'_{Tf2}(b_Tb')_{1[1]}\otimes$$

$$\otimes (b_Tb')_2 \overset{(C_6)}{=} (aa'_R{x_R}^f)_1{(aa'_R{x_R}^f)_{2(-1)}}^F \otimes x'_{Tf1F} \otimes (b_Tb')_{1[0]} \otimes (aa'_R{x_R}^f)_{2(0)}\otimes$$

$$\otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(C_7)}{=}$$

$$\overset{(C_7)}{=} a_1(a'_R{x_R}^f)_{1\bar R}{a_{2(-1)\bar R}}^{\bar f}{(a'_R{x_R}^f)_{2(-1)\bar f}}^F \otimes x'_{Tf1F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}(a'_R{x_R}^f)_{2(0)}\otimes$$

$$\otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_4)}{=}$$

$$\overset{(A_4)}{=} a_1(a'_R{x_R}^f)_{1\bar R}{(a'_R{x_R}^f)_{2(-1)}}^{\bar f}{}_r{a_{2(-1)\bar Rr}}^F \otimes x'_{Tf1\bar fF} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}(a'_R{x_R}^f)_{2(0)}\otimes$$

$$\otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_2)}{=}$$

$$\overset{(A_2)}{=} a_1((a'_R{x_R}^f)_1{(a'_R{x_R}^f)_{2(-1)}}^{\bar f})_r{a_{2(-1)r}}^F \otimes x'_{Tf1\bar fF} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}(a'_R{x_R}^f)_{2(0)}\otimes$$

$$\otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(C_7)}{=}$$

$$\overset{(C_7)}{=} a_1(a'_{R1}{x_R}^f{}_{1\bar R}{a'_{R2(-1)\bar R}}^{\bar F}{x_R}^f{}_{2(-1)\bar F}{}^{\bar f})_r{a_{2(-1)r}}^F \otimes x'_{Tf1\bar fF} \otimes (b_Tb')_{1[0]}\otimes$$

$$\otimes a_{2(0)}a'_{R2(0)}{x_R}^f{}_{2(0)} \otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_4)}{=}$$

$$\overset{(A_4)}{=} a_1(a'_{R1}{x_R}^f{}_{1\bar R}{x_R}^f{}_{2(-1)}{}^{\bar F}{}_{\bar r}{a'_{R2(-1)\bar R\bar r}}^{\bar f})_r{a_{2(-1)r}}^F \otimes x'_{Tf1\bar F\bar fF} \otimes (b_Tb')_{1[0]}\otimes$$

$$\otimes a_{2(0)}a'_{R2(0)}{x_R}^f{}_{2(0)} \otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_2)}{=}$$

$$\overset{(A_2)}{=} a_1(a'_{R1}({x_R}^f{}_1{x_R}^f{}_{2(-1)}{}^{\bar F})_{\bar R}{a'_{R2(-1)\bar R}}^{\bar f})_r{a_{2(-1)r}}^F \otimes x'_{Tf1\bar F\bar fF} \otimes (b_Tb')_{1[0]}\otimes$$

$$\otimes a_{2(0)}a'_{R2(0)}{x_R}^f{}_{2(0)} \otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(C_6)}{=}$$

$$\overset{(C_6)}{=} a_1(a'_{R1}{x_R}^f{}_{1\bar R}{a'_{R2(-1)\bar R}}^{\bar f})_r{a_{2(-1)r}}^F \otimes ({x_R}^f{}_{2(-1)}x'_{Tf1})_{\bar fF} \otimes (b_Tb')_{1[0]}\otimes$$

$$\otimes a_{2(0)}a'_{R2(0)}x_R{}^f{}_{2(0)} \otimes x'_{Tf2}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(C_8)}{=}$$

$$\overset{(C_8)}{=} a_1(a'_{R1}x_{R1}{}^f{}_{\bar{R}}a'_{R2(-1)\bar{R}}{}^{\bar{f}})_r a_{2(-1)r}{}^F \otimes x'_{T1f\bar{f}F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{R2(0)}x_{R2}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_4)}{=}$$

$$\overset{(A_4)}{=} a_1(a'_{R1}a'_{R2(-1)}{}^f x_{R1f}{}^{\bar{f}})_r a_{2(-1)r}{}^F \otimes x'_{T1\bar{f}F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{R2(0)}x_{R2}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_2)}{=}$$

$$\overset{(A_2)}{=} a_1(a'_{R1}a'_{R2(-1)}{}^f)_r x_{R1f}{}^{\bar{f}}{}_{\bar{R}} a_{2(-1)r\bar{R}}{}^F \otimes x'_{T1\bar{f}F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{R2(0)}x_{R2}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(C_6)}{=}$$

$$\overset{(C_6)}{=} a_1a'_{R1r}(a'_{R2(-1)}x_{R1})^{\bar{f}}{}_{\bar{R}} a_{2(-1)r\bar{R}}{}^F \otimes x'_{T1\bar{f}F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{R2(0)}x_{R2}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_4)}{=}$$

$$\overset{(A_4)}{=} a_1a'_{R1r}a_{2(-1)r}{}^{\bar{f}}(a'_{R2(-1)}x_{R1})_{\bar{f}}{}^F \otimes x'_{T1F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{R2(0)}x_{R2}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(C_9)}{=}$$

$$\overset{(C_9)}{=} a_1(a'_{1R}x_{1R}{}^f)_r a_{2(-1)r}{}^{\bar{f}} a'_{2(-1)f\bar{f}}{}^F \otimes x'_{T1F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{2(0)\bar{R}}x_{2\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_2)}{=}$$

$$\overset{(A_2)}{=} a_1a'_{1Rr}x_{1R}{}^f{}_{\bar{r}} a_{2(-1)r\bar{r}}{}^{\bar{f}} a'_{2(-1)f\bar{f}}{}^F \otimes x'_{T1F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{2(0)\bar{R}}x_{2\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_4)}{=}$$

$$\overset{(A_4)}{=} a_1a'_{1Rr}a_{2(-1)r}{}^f x_{1Rf}{}^{\bar{f}} a'_{2(-1)\bar{f}}{}^F \otimes x'_{T1F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{2(0)\bar{R}}x_{2\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_5)}{=}$$

$$\overset{(A_5)}{=} a_1a_{2(-1)}{}^f a'_{1R}x_{1fR}{}^{\bar{f}} a'_{2(-1)\bar{f}}{}^F \otimes x'_{T1F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{2(0)\bar{R}}x_{2\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_4)}{=}$$

$$\overset{(A_4)}{=} a_1a_{2(-1)}{}^f a'_{1R}a'_{2(-1)}{}^{\bar{f}}{}_r x_{1fRr}{}^F \otimes x'_{T1\bar{f}F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{2(0)\bar{R}}x_{2\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_Tb')_{1[1]} \otimes (b_Tb')_2 \overset{(A_2)}{=}$$

$$\overset{(A_2)}{=} a_1a_{2(-1)}{}^f (a'_1a'_{2(-1)}{}^{\bar{f}})_R x_{1fR}{}^F \otimes x'_{T1\bar{f}F} \otimes (b_Tb')_{1[0]} \otimes a_{2(0)}a'_{2(0)\bar{R}}x_{2\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes x'_{T2\bar{F}}(b_T b')_{1[1]} \otimes (b_T b')_2 \overset{(C_5)}{=}$$

$$\overset{(C_5)}{=} a_1 a_{2(-1)}{}^f (a'_1 a'_{2(-1)}{}^{\bar{f}})_R x_{1fR}{}^F \otimes x'_{T1\bar{f}F} \otimes b_{T1[0]} b'_{1[0]} \otimes a_{2(0)} a'_{2(0)\bar{R}} x_{2\bar{R}}{}^{\bar{F}} b_{T1[1]}{}^{\bar{\bar{F}}} \otimes$$

$$\otimes x'_{T2\bar{F}} b'_{1[1]t\bar{\bar{F}}} \otimes b_{T2t} b'_2 \overset{(C_6)}{=}$$

$$\overset{(C_6)}{=} a_1 a_{2(-1)}{}^f a'_{1R} x_{1fR}{}^F \otimes (a'_{2(-1)} x'_{T1})_F \otimes b_{T1[0]} b'_{1[0]} \otimes a_{2(0)} a'_{2(0)\bar{R}} x_{2\bar{R}}{}^{\bar{F}} b_{T1[1]}{}^{\bar{\bar{F}}} \otimes$$

$$\otimes x'_{T2\bar{F}} b'_{1[1]t\bar{\bar{F}}} \otimes b_{T2t} b'_2 \overset{(BP)}{=}$$

$$\overset{(BP)}{=} a_1 a_{2(-1)}{}^f a'_{1R} x_{1fR}{}^F \otimes (a'_{2(-1)} x'_1)_{TF} \otimes b_{1[0]T} b'_{1[0]} \otimes a_{2(0)} a'_{2(0)\bar{R}} (x_2 b_{1[1]})_{\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes (x'_2 b'_{1[1]})_{t\bar{F}} \otimes b_{2t} b'_2 \overset{(C_6)}{=}$$

$$\overset{(C_6)}{=} a_1 a'_{1R} (a_{2(-1)} x_1)_R{}^F \otimes (a'_{2(-1)} x'_1)_{TF} \otimes b_{1[0]T} b'_{1[0]} \otimes a_{2(0)} a'_{2(0)\bar{R}} (x_2 b_{1[1]})_{\bar{R}}{}^{\bar{F}} \otimes$$

$$\otimes (x'_2 b'_{1[1]})_{t\bar{F}} \otimes b_{2t} b'_2 = \Delta_{A\times H\times B}(a \otimes x \otimes b)\Delta_{A\times H\times B}(a' \otimes x' \otimes b').$$

Necessity. Since $\varepsilon_{A\Diamond^f_R H \Diamond B}$ is an algebra map, we get

(BA_1) $\varepsilon_A(aa'_R x_R{}^f)\varepsilon_H(x'_{Tf})\varepsilon_B(b_T b') = \varepsilon_A(a)\varepsilon_A(a')\varepsilon_H(x)\varepsilon_H(x')\varepsilon_B(b)\varepsilon_B(b')$.

Let $x = x = 1_H,\ b = b' = 1_B,\ x = x = 1_H,\ a = a' = 1_A$ and $a = a' = 1_A,\ b = b' = 1_B$ in Eq. (BA_1), respectively, we obtain (C_1). Similarly, (C_2) holds.

Apply $\mathrm{id}_A \otimes \varepsilon_H \otimes \varepsilon_B \otimes \mathrm{id}_A \otimes \varepsilon_H \otimes \varepsilon_B$ (respectively $\varepsilon_A \otimes \varepsilon_H \otimes \mathrm{id}_B \otimes \varepsilon_A \otimes \varepsilon_H \otimes \mathrm{id}_B$) to

(BA_2) $1_{A1} \otimes 1_{A2(-1)} \otimes 1_{B1[0]} \otimes 1_{A2(0)} \otimes 1_{B1[1]} \otimes 1_{B2} = 1_A \otimes 1_H \otimes 1_B \otimes 1_A \otimes 1_H \otimes 1_B$,

we have (C_3). Likewise, we get (C_4).

Since $\Delta_{A\times H\times B}\big((a \otimes x \otimes b)(a' \otimes x' \otimes b')\big) = \Delta_{A\times H\times B}(a \otimes x \otimes b)\Delta_{A\times H\times B}(a' \otimes x' \otimes b')$, we obtain

(BA_3) $(aa'_R x_R{}^f)_1 \otimes (aa'_R x_R{}^f)_{2(-1)} x'_{1f1} \otimes (b_T b')_{1[0]} \otimes (aa'_R x_R{}^f)_{2(0)} \otimes$

$$\otimes x'_{Tf2}(b_T b')_{1[1]} \otimes (b_T b')_2 = a_1 a'_{1R}(a_{2(-1)} x_1)_R{}^f \otimes (a'_{2(-1)} x'_1)_{Tf} \otimes$$

$$\otimes b_{1[0]T} b'_{1[0]} \otimes a_{2(0)} a'_{2(0)r} (x_2 b_{1[1]})_r{}^F \otimes (x'_2 b'_{1[1]})_{tF} \otimes b_{2t} b'_2.$$

Let $x = x' = 1_H$ and $a = a' = 1_A$ in Eq. (BA_3), we have

$$1_A \otimes 1_H \otimes (bb')_{1[0]} \otimes 1_A \otimes (bb')_{1[1]} \otimes (bb')_2 =$$

$$= 1_A \otimes 1_H \otimes b_{1[0]} b'{}_{1[0]} \otimes b_{1[1]}{}^f \otimes b'{}_{1[1]_{tf}} \otimes b_{2t} b'{}_2.$$

Apply $\varepsilon_A \otimes \varepsilon_H \otimes \mathrm{id}_B \otimes \mathrm{id}_A \otimes \mathrm{id}_H \otimes \mathrm{id}_B$ to the above equation, we get (C_5). The conditions (C_6)–(C_9) can be derived by the similar method.

Remark 2.2. 1. Let $x = x' = 1_H,\ b' = 1_B,\ R(x \otimes a) = x_1 \triangleright a \otimes x_2$ and $T(b \otimes x) = x_1 \otimes b \triangleleft x_2$ in Eq. (BP), we can obtain the condition (DB). Then Brzeziński double biproduct $A\Diamond^f_R H_T \Diamond B$ is the double biproduct $A\Diamond H \Diamond B$ when $f(x \otimes y) = 1_A \otimes xy,\ R(x \otimes a) = x_1 \triangleright a \otimes x_2$ and $T(b \otimes x) = x_1 \otimes b \triangleleft x_2$ in Theorem 2.1.

2. Setting $A = K$ and $B = K$, we obtain the right version of Radford biproduct in [14] and Brzeziński crossed biproduct in [6], respectively. Furthermore, if the left comodule coaction is trivial, and $f(x, x') = \sigma(x_1, x'_1) \otimes x_2 x'_2$, $R(x \otimes a) = x_1 \triangleright a_1 \otimes x_2 \triangleleft a_2$ in Brzeziński crossed biproduct, then we can get Agore and Militaru's unified product [1].

3. Taking $f(x, x') = \sigma(x_1, x'_1) \otimes x_2 x'_2$ in Brzeziński crossed biproduct, we obtain Wang – Jiao – Zhao's crossed product in [16].

4. Setting $T(b \otimes x) = x_1 \otimes b \triangleleft x_2$ in Theorem 2.1, we can get the main result in [7] (Theorem 3.2). And the condition (BP) here implies the condition (C_{11}) there.

5. Let $A = K$, the maps $f\colon H \otimes H \longrightarrow A \otimes H$ and $R\colon H \otimes A \longrightarrow A \otimes H$ be trivial in Theorem 2.1, we can obtain that the right twisted tensor product $H_T \# B$ equipped with the right smash coproduct $H \times B$ becomes a bialgebra if and only if the following conditions hold ($x \in H$, $b, b' \in B$ and $T = t$):

(D_1) ε_B are algebra maps, $\varepsilon_B(b_T)\varepsilon_H(x_T) = \varepsilon_B(b)\varepsilon_H(x)$;

(D_2) $1_{B[0]} \otimes 1_{B[1]} = 1_B \otimes 1_H$, $\Delta_B(1_B) = 1_B \otimes 1_B$;

(D_3) $(bb')_{1[0]} \otimes (bb')_{1[1]} \otimes (bb')_2 = b_{1[0]}b'_{1[0]} \otimes b_{1[1]}{b'_{1[1]}}_T \otimes b_{2T}b'_2$;

(D_4) $x_{T1} \otimes b_{T1[0]} \otimes x_{T2}b_{T1[1]} \otimes b_{T2} = x_{1T} \otimes b_{1[0]T} \otimes b_{1[1]}x_{2t} \otimes b_{2t}$.

This exactly is the right version of [8] (Corollary 2.5).

References

1. *Agore A. L., Militaru G.* Extending structures II: The quantum version // J. Algebra. – 2011. – **336**. – P. 321 – 341.
2. *Andruskiewitsch N., Schneider H.-J.* On the classification of finite-dimensional pointed Hopf algebras // Ann. Math. – 2010. – **171**, № 1. – P. 375 – 417.
3. *Blattner R. J., Cohen M., Montgomery S.* Crossed products and inner actions of Hopf algebras // Trans. Amer. Math. Soc. – 1986. – **289**. – P. 671 – 711.
4. *Brzeziński T.* Crossed products by a coalgebra // Communs Algebra. – 1997. – **25**. – P. 3551 – 3575.
5. *Caenepeel S., Ion B., Militaru G., Zhu S. L.* The factorization problem and the smash biproduct of algebras and coalgebras // Algebra. Represent Theory. – 2000. – **3**. – P. 19 – 42.
6. *Ma T. S., Jiao Z. M., Song Y. N.* On crossed double biproduct // J. Algebra and Appl. – 2013. – **12**, № 5. – 17 p.
7. *Ma T. S., Li H. Y.* On Radford biproduct // Communs Algebra. – 2015. – **43**, № 9. – P. 3946 – 3966.
8. *Ma T. S., Wang S. H.* General double quantum groups // Communs Algebra. – 2010. – **38**, № 2. – P. 645 – 672.
9. *Majid S.* Double-bosonization of braided groups and the construction of $U_q(g)$ // Math. Proc. Cambridge Phil. Soc. – 1999. – **125**, № 1. – P. 151 – 192.
10. *Majid S.* Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group // Communs Math. Phys. – 1993. – **156**. – P. 607 – 638.
11. *Majid S.* Algebras and Hopf algebras in braided categories // Adv. Hopf algebras (Chicago, IL, 1992): Lect. Notes Pure and Appl. Math. – 1994. –**158**. – P. 55 – 105.
12. *Montgomery S.* Hopf algebras and their actions on rings // CBMS Lect. Math. – 1993. – **82**.
13. *Panaite F., Van Oystaeyen F.* L-R-smash biproducts, double biproducts and a braided category of Yetter – Drinfeld – Long bimodules // Rocky Mountain J. Math. – 2010. – **40**, № 6. – P. 2013 – 2024.
14. *Radford D. E.* The structure of Hopf algebra with a projection // J. Algebra. – 1985. – **92**. – P. 322 – 347.
15. *Radford D. E.* Hopf algebras // KE Ser. Knots and Everything. – New Jersey: World Sci., 2012. – **49**.
16. *Wang S. H., Jiao Z. M., Zhao W. Z.* Hopf algebra structures on crossed products // Communs Algebra. – 1998. – **26**. – P. 1293 – 1303.

Received 21.11.14,
after revision — 18.08.18