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# A REMARK ON JOHN – NIRENBERG THEOREM FOR MARTINGALES

# ЗАУВАЖЕННЯ ЩОДО ТЕОРЕМИ ДЖОНА – НІРЕНБЕРГА ДЛЯ МАРТИНГАЛІВ

This paper is mainly devoted to establishing an extension of the John – Nirenberg theorem for martingales, more precisely, let $1 < p < \infty$ and $0 < q < \infty$. If the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is regular, then $BMO_{p,q} = BMO_1$ with the equivalent norms. Our method is to use a new atomic decomposition construction of the martingale Hardy space.

Роботу, в основному, присвячено доведенню узагальнення теореми Джона – Ніренберга для мартингалів, більш точно, для $1 < p < \infty$ та $0 < q < \infty$. За умови, що стохастичний базис $(\mathcal{F}_n)_{n\geq 0}$ є регулярним, маємо $BMO_{p,q} = BMO_1$ з еквівалентними нормами. Наш метод зводиться до застосування нової конструкції атомного розкладу простору мартингалів Гарді.

**1. Introduction.** The John – Nirenberg theorem has been successfully extended to different settings in recent years. A lot of works have been done on this subject (see [5, 6, 8 – 11, 19, 20]).

This remark deals with the John – Nirenberg theorem on Lorentz space for the martingale setting. Before describing our main results, we recall the classical John – Nirenberg theorem in the martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\{\mathcal{F}_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 0} \mathcal{F}_n)$. The expectation operator and the conditioned expectation operator are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. A sequence $f = (f_n)_{n\geq 0}$ of random variables such that $f_n$ is $\mathcal{F}_n$-measurable is said to be a martingale if $\mathbb{E}(|f_n|) < \infty$ and $\mathbb{E}_n(f_{n+1}) = f_n$ for every $n \geq 0$. We always suppose that for a martingale $f$, $f_0 = 0$. The Banach spaces $BMO_p$, $1 \leq p < \infty$ are defined as follows:

$$BMO_p = \left\{ f = (f_n)_{n\geq 0} \colon \|f\|_{BMO_p} = \sup_n \|\mathbb{E}_n\left(|f - f_n|^p\right)\|_\infty^{\frac{1}{p}} < \infty \right\}.$$

Here the $f$ in $|f - f_n|^p$ means $f_\infty$. It can be shown that $\|f\|_{BMO_p}$ admits an alternative definition

$$\|f\|_{BMO_p} = \sup_{\tau\in\mathcal{T}} \frac{\left\|\left(f - f^\tau\right)\chi_{\{\tau<\infty\}}\right\|_p}{\|\chi_{\{\tau<\infty\}}\|_p},$$

where $\mathcal{T}$ denotes the set of all stopping times with respect to $\{\mathcal{F}_n\}_{n\geq 0}$. The well-known John – Nirenberg theorem (see [13, 18]) says that if the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then

$$BMO_p = BMO_1.$$

In 2014, Yi, Wu and Jiao [19] extended this result to a wider class of the rearrangement invariant Banach function space. That is, let $E$ be a rearrangement invariant Banach function space on $\Omega$ with upper Boyd indices $q_E < \infty$ and define

$$BMO_E = \left\{ f = (f_n)_{n\geq 0} \colon \|f\|_{BMO_E} < \infty \right\},$$

where

$$\|f\|_{BMO_E} = \sup_{\nu\in\mathcal{T}} \frac{\left\|(f-f^{\nu})\chi_{\{\nu<\infty\}}\right\|_E}{\left\|\chi_{\{\nu<\infty\}}\right\|_E}.$$

Then if the stochastic basis is regular,

$$BMO_E = BMO_1.$$

Hence it is natural to consider whether the John – Nirenberg theorem is true for the nonrearrangement invariant Banach function space. We will work on this problem in the present paper. Our goal is to establish the John – Nirenberg theorem in the context of Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $0 < q \leq 1$. Note that such spaces are not the rearrangement invariant Banach function spaces. The following is one of our main results:

**Theorem 1.1.** *Let* $1 < p < \infty$ *and* $0 < q < \infty$. *If the stochastic basis* $(\mathcal{F}_n)_{n\geq 0}$ *is regular, then*

$$BMO_{p,q} = BMO_1 \quad \textit{with equivalent norms},$$

*where*

$$BMO_{p,q} = \left\{f = (f_n)_{n\geq 0} \colon \|f\|_{BMO_{p,q}} < \infty\right\},$$

*and*

$$\|f\|_{BMO_{p,q}} = \sup_{\nu\in\mathcal{T}} \frac{\left\|(f-f^{\nu})\chi_{\{\nu<\infty\}}\right\|_{p,q}}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p,q}}.$$

Our main method is to use a new atomic decomposition construction of Hardy spaces by atoms associated with Lorentz spaces.

**2. Preliminaries.** In this section, we give some preliminaries necessary for the whole paper. Let us first recall some basic facts on the Lorentz spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $f$ be a measurable function defined on $\Omega$. The distribution function of $f$ is the function $\lambda_s(f)$ defined by

$$\lambda_s(f) = \mathbb{P}\big(\{\omega\in\Omega\colon |f(\omega)| > s\}\big), \quad s \geq 0.$$

And denote by $\mu_t(f)$ the decreasing rearrangement of $f$, defined by

$$\mu_t(f) = \inf\left\{s\geq 0\colon \lambda_s(f)\leq t\right\}, \quad t\geq 0,$$

with the convention that $\inf\varnothing = \infty$.

The Lorentz space $L_{p,q}(\Omega,\mathcal{F},\mathbb{P})$, $0<p<\infty$, $0<q\leq\infty$, consists of the measurable functions $f$ with finite norm or quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p}\int\limits_0^\infty \left(t^{\frac{1}{p}}\mu_t(f)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, \quad 0<q<\infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}}\mu_t(f), \quad q=\infty.$$

It will be convenient for us to use the equivalent definition of $\|f\|_{p,q}$, known as

$$\|f\|_{p,q} = \left(q\int\limits_0^\infty \left(t\mathbb{P}(|f(x)|>t)^{\frac{1}{p}}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, \quad 0<q<\infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t\mathbb{P}(|f(x)| > t)^{\frac{1}{p}}, \quad q = \infty.$$

These spaces are the generalizations of ordinary $L_p$ spaces and they coincide with $L_p$ when $q = p$. As we known, if $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, then $\|\cdot\|_{p,q}$ is equivalent to a norm. However, for the other values of $p$ and $q$, $\|\cdot\|_{p,q}$ is only a quasinorm. In particular, if $0 < q \le 1$ and $q \le p < \infty$, then $\|\cdot\|_{p,q}$ is equivalent to a $q$-norm. The following lemmas can be found in Grafakos [1].

**Lemma 2.1.** *Let* $0 < p,\ p_1,\ p_2 < \infty$ *and* $0 < q,\ p_1',\ p_2' \le \infty$ *with* $1/p = 1/p_1 + 1/p_1'$ *and* $1/q = 1/p_2 + 1/p_2'$, *then*

$$\|fg\|_{p,q} \le C\|f\|_{p_1,p_2}\|g\|_{p_1',p_2'}.$$

*Moreover, if* $p = q,\ p_1 = q_1$ *and* $p_2 = q_2$, *we have*

$$\|fg\|_p \le \|f\|_{p_1}\|g\|_{p_1'}.$$

**Lemma 2.2.** *Let* $1 < p < \infty$ *and* $0 < q \le 1$ *with* $1 = 1/p + 1/p'$, *then the dual space of* $L_{p,q}$ *is* $L_{p',\infty}$.

Now we define the Hardy martingale spaces. For a martingale $f = (f_n)_{n\ge0}$, the maximal function of martingale $f$ is defined by

$$M_n(f) = \sup_{1\le i\le n} |f_i|, \qquad M(f) = \sup_{i\ge0} |f_i|.$$

Define

$$H_p^* = \left\{ f = (f_n)_{n\ge0} \colon \|f\|_{H_p^*} = \big\|M(f)\big\|_p < \infty \right\}, \quad 0 < p < \infty,$$

$$H_{p,q}^* = \left\{ f = (f_n)_{n\ge0} \colon \|f\|_{H_{p,q}^*} = \big\|M(f)\big\|_{p,q} < \infty \right\}, \qquad 0 < p < \infty, \quad 0 < q \le \infty.$$

The stochastic basis $(\mathcal{F}_n)_{n\ge0}$ is said to be regular, if for $n \ge 1$ and $A \in \mathcal{F}_n$, there exists a $B \in \mathcal{F}_{n-1}$ such that $A \subset B$ and $\mathbb{P}(B) \le R\mathbb{P}(A)$, where $R$ is a positive constant independent of $n$. A martingale is said to be regular if it is adapted to a regular $\sigma$-algebra sequence. This amounts to saying that there exists a constant $R > 0$ such that

$$f_n \le Rf_{n-1}$$

for all non-negative martingales $(f_n)_{n\ge0}$ adapted to the stochastic basis $(\mathcal{F}_n)_{n\ge0}$. We refer to Long [13] and Weisz [18] for the theory of martingale Hardy spaces.

**3. Main results.** In this section we present the new John – Nirenberg theorem by constructing the atomic decomposition of Hardy spaces $H_p^*$ via atoms associated with $L_{q,\infty}$-space for $1 < q < \infty$. We refer to [2 – 4, 7, 14, 17] for more information on the classical atomic decompositions.

**Definition 3.1.** *Let* $0 < p < \infty$ *and* $1 < q < \infty$. *A measurable function,* $a$, *is called a* $(p, L_{q,\infty})$*-atom if there exists a stopping time* $\nu$ *such that*

(1) $a_n = E_n a = 0$ *if* $\nu \ge n$,

(2) $\big\|M(a)\big\|_{q,\infty} \le \dfrac{\big\|\chi_{\{\nu<\infty\}}\big\|_{q,\infty}}{\mathbb{P}(\nu < \infty)^{1/p}}$.

We denote the set of $(p, L_{q,\infty})$ atoms by $\mathcal{A}_{p,L_{q,\infty}}$.

**Theorem 3.1.** *Let the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ be regular and $0 < p \leq 1 < q < \infty$. Then $f \in H_p^*$ if and only if there exist a sequence $(a^k)$ of $(p, L_{q,\infty})$ atoms and a sequence $(\mu_k) \in \ell_p$ of real numbers such that*

$$f = \sum_{k\in \boldsymbol{Z}} \mu_k a^k \quad a.e.,$$

*and*

$$\|f\|_{H_p^*} \approx \inf\left\{\|(\mu_k)\|_{\ell_p}\right\},$$

*where the infimum is taken over all the preceding decompositions of $f$.*

The proof of Theorem 3.1 uses the following well known lemma which is proved in Theorem 7.1.2 of [13, p. 265].

**Lemma 3.1.** *If the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is regular, then for all non-negative adapted processes $\gamma = (\gamma_n)_{n\geq 0}$ and $\lambda \geq \|\gamma_0\|_\infty$, there exist a constant $R > 0$ and a stopping time $\tau_\lambda$ such that*

$$\{M(\gamma) > \lambda\} \subseteq \{\tau_\lambda < \infty\},$$

$$\mathbb{P}(\tau_\lambda < \infty) \leq R\mathbb{P}(M(\gamma) > \lambda),$$

$$\sup_{n\leq \tau_\lambda} \gamma_n = M_{\tau_\lambda}(\gamma) \leq \lambda,$$

$$\|\gamma_0\|_\infty \leq \lambda_1 \leq \lambda_2 \quad \textit{implies} \quad \tau_{\lambda_1} \leq \tau_{\lambda_2}.$$

***Proof of Theorem* 3.1.** Let $f \in H_p^*$. For the process $(|f_n|)_{n\geq 0}$ and $\lambda_k = 2^k$, define the stopping time $\tau_k$ associate with $\lambda_k$ satisfying the Lemma 3.1. Since $\{\tau_k\}$ is increasing and $\mathbb{P}(\tau_k < \infty) \to 0$ as $k \to \infty$, we see $\lim_{k\to\infty} \tau_k = \infty$, a.e.,

$$\lim_{k\to\infty} f_{\tau_k} = f \quad \text{a.e.} \qquad \text{and} \qquad \lim_{k\to -\infty} |f_{\tau_k}| \leq \lim_{k\to-\infty} 2^k = 0 \quad \text{a.e.}$$

Therefore, we get the following decomposition which converges pointwise:

$$f_n = \sum_{k=-\infty}^{\infty} \left(f_n^{\tau_k} - f_n^{\tau_{k-1}}\right) \quad \forall n \geq 0.$$

Set $\mu_k = 2^{k+1}\mathbb{P}(\tau_{k-1} < \infty)^{1/p}$ for all $k \in \mathbb{Z}$. When $\mu_k \neq 0$, we define

$$a_n^k = \frac{f_n^{\tau_k} - f_n^{\tau_{k-1}}}{\mu_k} \quad \forall n \geq 0.$$

If $\mu_k = 0$, then let $a_n^k = 0$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Then $(a_n^k)_{n\geq 0}$ is a martingale for each fixed $k \in \mathbb{Z}$. Since $M_{\tau_k}(f) \leq 2^k$, we obtain

$$M\left(a_n^k\right) \leq \frac{M\left(f^{\tau_k}\right) + M\left(f^{\tau_{k-1}}\right)}{\mu_k} \leq \mathbb{P}(\tau_{k-1} < \infty)^{-1/p}.$$

Hence it is easy to check that $\left(a_n^k\right)_{n\geq 0}$ is a bounded $L_2$-martingale. Consequently, there exists an element $a^k \in L_2$ such that $\mathbb{E}_n a^k = a_n^k$ If $n \leq \tau_{k-1}$, then $a_n^k = 0$, and

$$\big\|M(a^k)\big\|_{q,\infty} \le \big\|M(a^k)\big\|_\infty \|\chi_{\{\tau_{k-1}<\infty\}}\|_{q,\infty} \le \frac{\|\chi_{\{\tau_{k-1}<\infty\}}\|_{q,\infty}}{\mathbb{P}(\tau_{k-1}<\infty)^{1/p}}.$$

Thus we conclude that $\big(a^k, \tau_{k-1}\big)$ is really a $(p, L_{q,\infty})$-atom. In view of Lemma 3.1, we have

$$\sum_{k\in\mathbb{Z}} \mu_k^p = \sum_{k\in\mathbb{Z}} 2^{(k+1)p}\mathbb{P}(\tau_{k-1}<\infty) \le R\sum_{k\in\mathbb{Z}} 2^{(k+1)p}\mathbb{P}\left(M(f) > 2^{k-1}\right) \le$$

$$\le 8^p pR \sum_{k\in\mathbb{Z}} \int\limits_{2^{(k-2)p}}^{2^{(k-1)p}} t^p \mathbb{P}(M(f)>t)\,dt = 8^p pR\|f\|_{H_p^*}^p.$$

For the converse part, it suffices to prove that for any $a \in \mathcal{A}_{p,L_{q,\infty}}$,

$$\|a\|_{H_p^*} = \big\|M(a)\big\|_p \le C.$$

Indeed, for $0<p\le 1$,

$$\|f\|_{H_p^*} = \big\|M(f)\big\|_p \le \left(\sum_{k\in\mathbf{Z}} \Big\|\mu_k M(a^k)\Big\|_p^p\right)^{1/p} = \left(\sum_{k\in\mathbf{Z}} |\mu_k|^p \Big\|M(a^k)\Big\|_p^p\right)^{1/p}.$$

We first consider the case $0<p<1$. Given a $(p, L_{q,\infty})$-atom $a$, we get

$$\big\|M(a)\big\|_1 = \big\|M(a\chi_{\{\nu<\infty\}})\big\|_1 \le C\big\|M(a)\big\|_{q,\infty}\big\|\chi_{\{\nu<\infty\}}\big\|_{q',1} \le$$

$$\le C\big\|\chi_{\{\nu<\infty\}}\big\|_{q,\infty}\mathbb{P}(\nu<\infty)^{-1/p}\big\|\chi_{\{\nu<\infty\}}\big\|_{q',1} = C\mathbb{P}(\nu<\infty)^{1-1/p},$$

where $\nu$ is the stopping time corresponding to $a$. Note that $1 + \dfrac{1}{p/(1-p)} = \dfrac{1}{p}$, we obtain

$$\big\|M(a)\big\|_p = \big\|M(a\chi_{\{\nu<\infty\}})\big\|_p \le \big\|M(a)\big\|_1\big\|\chi_{\{\nu<\infty\}}\big\|_{\frac{p}{1-p}} \le$$

$$\le C\mathbb{P}(\nu<\infty)^{1-1/p}\mathbb{P}(\nu<\infty)^{\frac{1-p}{p}} = C.$$

As for the case $p=1$, we directly have

$$\big\|M(a)\big\|_1 = \big\|M(a\chi_{\{\nu<\infty\}})\big\|_1 \le C\big\|M(a)\big\|_{q,\infty}\big\|\chi_{\{\nu<\infty\}}\big\|_{q',1} \le$$

$$\le C\big\|\chi_{\{\nu<\infty\}}\big\|_{q,\infty}\mathbb{P}(\nu<\infty)^{-1}\big\|\chi_{\{\nu<\infty\}}\big\|_{q',1} = C.$$

Theorem 3.1 is proved.

**Theorem 3.2.** *Let* $1<p<\infty$ *and* $0<q\le 1$. *If the stochastic basis* $(\mathcal{F}_n)_{n\ge 0}$ *is regular, then* $BMO_{p,q} = BMO_1$ *in the sense of equivalent norm.*

Before proving Theorem 3.2, we present the maximal inequality for the martingale Lorentz – Hardy spaces.

**Lemma 3.2** (see [12])**.** *Let* $f = (f_n)_{n\ge 0} \in L_{q,\infty}$, $1<q<\infty$, *then there exists a constant* $C_q$ (*depending only on* $q$) *such that*

$$\|f\|_{q,\infty} \le \|f\|_{H_{q,\infty}^*} \le C_q\|f\|_{q,\infty}.$$

***Proof of Theorem* 3.2.** First suppose that $f \in BMO_{p,q}$, then by Lemma 2.2 we have

$$\|f\|_{BMO_1} = \sup_{\nu \in \mathcal{T}} \frac{\left\|(f - f^\nu)\chi_{\{\nu<\infty\}}\right\|_1}{\mathbb{P}(\nu < \infty)} \le \sup_{\nu \in \mathcal{T}} \frac{C\left\|(f - f^\nu)\chi_{\{\nu<\infty\}}\right\|_{p,q}\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}}{\mathbb{P}(\nu < \infty)} =$$

$$= C \sup_{\nu \in \mathcal{T}} \frac{\left\|(f - f^\nu)\chi_{\{\nu<\infty\}}\right\|_{p,q}}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p,q}} = C\|f\|_{BMO_{p,q}}.$$

On the other hand, assume that $f \in BMO_1$, then from Lemma 2.1 and the definition of supremum, there exists a function $g \in L_{p',\infty}$ with $\|g\|_{L_{p',\infty}} \le 1$ such that

$$\left\|(f - f^\nu)\chi_{\{\nu<\infty\}}\right\|_{p,q} \le C_1 \left| \int\limits_{\{\nu<\infty\}} (f - f^\nu) g \, d\mathbb{P} \right|.$$

According to Lemma 3.2, there exists a constant $C_{p'}$ such that

$$\left\|M(f)\right\|_{p',\infty} \le C_{p'}\|f\|_{p',\infty} \quad \forall f \in L_{p',\infty}.$$

Let

$$a = \frac{\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}(g - g^\nu)}{2C_{p'}\mathbb{P}(\nu < \infty)}.$$

Then we obtain

$$\left\|M(a)\right\|_{p',\infty} \le C_{p'}\|a\|_{p',\infty} = \frac{\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}}{2\mathbb{P}(\nu < \infty)}\|g - g^\nu\|_{p',\infty} \le \frac{\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}}{\mathbb{P}(\nu < \infty)},$$

which means $a \in \mathcal{A}_{1,L_{p',\infty}}$. Then it follows from Theorem 3.1 that $a \in H_1^*$ and $\|a\|_{H_1^*} = 1$. Thus

$$g - g^\nu = \frac{2C_{p'}\mathbb{P}(\nu < \infty)}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}} a \in H_1^*,$$

with its norm

$$\|g - g^\nu\|_{H_1^*} \le \frac{2C_{p'}\mathbb{P}(\nu < \infty)}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}}.$$

Since the stochastic basis $(\mathcal{F}_n)_{n\ge0}$ is regular, the dual space of $H_1^*$ is $BMO_1$ (see [17, 20]). Hence

$$\frac{\left\|(f - f^\nu)\chi_{\{\nu<\infty\}}\right\|_{p,q}}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p,q}} \le \frac{C_1 \left| \int_{\{\nu<\infty\}} (f - f^\nu) g d\mathbb{P} \right|}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p,q}} =$$

$$= \frac{C_1 \left| \int_{\{\nu<\infty\}} f\,(g - g^\nu)\, d\mathbb{P} \right|}{\|\chi_{\{\nu<\infty\}}\|_{p,q}} \le C_1 C_2 \|g - g^\nu\|_{H_1^*} \|f\|_{BMO_1} \frac{1}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p,q}} \le$$

$$\le 2C_1C_2C_{p'} \frac{\mathbb{P}(\nu < \infty)\|f\|_{BMO_1}}{\left\|\chi_{\{\nu<\infty\}}\right\|_{p,q}\left\|\chi_{\{\nu<\infty\}}\right\|_{p',\infty}} = C\|f\|_{BMO_1}.$$

Here $C = 2C_1C_2C_{p'}$. This means $\|f\|_{BMO_{p,q}} \le C\|f\|_{BMO_1}$.

***Corollary* 1.** *Let* $1 < p < \infty$ *and* $0 < q < \infty$. *If the stochastic basis* $(\mathcal{F}_n)_{n\geq 0}$ *is regular, then*

$$BMO_{p,q} = BMO_1 \tag{3.1}$$

*in the sense of equivalent norm.*

***Proof.*** Now we consider $1 < p < \infty$ and $1 < q < \infty$. As we known that $L_{p,q}$-space is a rearrangement invariant Banach function space with lower and upper Boyd indices both equal to $p$ in this case. From the Theorem 3.4 of [19], one can obtain that

$$BMO_{p,q} = BMO_1, \quad 1 < p, q < \infty. \tag{3.2}$$

Combining Theorem 3.2 and (3.2), we have the formula (3.1).

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