UDC 519.21

**R. Savaş, M. Öztürk** (Sakarya Univ., Turkey)

# ON GENERALIZED IDEAL ASYMPTOTICALLY STATISTICAL EQUIVALENT OF ORDER $\alpha$ FOR FUNCTIONS

# ПРО УЗАГАЛЬНЕНИЙ ІДЕАЛЬНИЙ АСИМПТОТИЧНО СТАТИСТИЧНИЙ ЕКВІВАЛЕНТ ПОРЯДКУ $\alpha$ ДЛЯ ФУНКЦІЙ

We introduce new definitions related to the notions of asymptotically $\mathcal{I}_\lambda$-statistical equivalent of order $\alpha$ to multiple $L$ and strongly $\mathcal{I}_\lambda$-asymptotically equivalent of order $\alpha$ to multiple $L$ by using two nonnegative real-valued Lebesque measurable functions in the interval $(1,\infty)$ instead of sequences. In addition, we also present some inclusion theorems.

Введено нові означення, пов'язані з поняттями асимптотично $\mathcal{I}_\lambda$-статистичного еквівалента порядку $\alpha$ для кратних $L$ та сильно $\mathcal{I}_\lambda$-асимптотичного еквівалента порядку $\alpha$ для кратних $L$ за допомогою двох невід'ємних дійснозначних функцій, вимірних за Лебегом на інтервалі $(1,\infty)$, замість послідовностей. Крім того, наведено також деякі теореми про включення.

**1. Introduction.** This paper introduces a class of summability method that can be applied to measurable functions defined on $(1,\infty)$. These methods are modeled on the methods of asymptotically statistical equivalent. As part of this paper, we establish some analogs of known results for sequential summability to the setting of real valued functions defined on $(1,\infty)$.

In 1993, Marouf [14] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 1997, Li [13] also presented and studied asymptotic equivalence of sequences and summability. In 2003, Patterson [17] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Recently, Savaş and Basarir [19] defined $(\sigma,\lambda)$-asymptotically statistical equivalent sequences. Six years later the notion of asymptotically $\mathcal{I}_\lambda$- statistical equivalent sequences was studied by Gümüs and Savaş [10] (see also Kumar and Sharma [12]).

A sequence $(x_k)$ is statistically convergent if ”almost all” of $k$ its values have a common limit point. Over the years and under different names, statistical convergence has been discussed in number theory, trigonometric series and summability theory. Statistical convergence for sequences was defined by Fast [8] in 1951 who provided an alternate proof of a result of Steinhaus [29] and then reintroduced by Schoenberg [27] independently. In the latter years it was further investigated from the sequence space point of view and linked with summability theory by Fridy [9], Connor and Savaş [4], Šalát [18], Cakalli [1] and many others.

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset $E$ of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k) \text{ exists.}$$

Note that if $K\subset\mathbb{N}$ is a finite set, then $\delta(K)=0$, and for any set $K\subset\mathbb{N}, \delta(K^c)=1-\delta(K)$.

We first recall the following definition.

**Definition 1.** *A sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$*

$$\delta\left(\{k \in \mathbb{N}\colon\ |x_k - L| \geq \varepsilon\}\right) = 0.$$

In this case we write $st - \lim x = L$ or $x_k \to L(st)$.

The concept of $\mathcal{I}$-convergence was introduced by Kostyrko et al. in a metric space [11]. Later it was further studied by Dems [7], Das, Savaş and Ghosal [5], Gümüs and Savaş [26] and Savaş [20 – 25] and many others. $\mathcal{I}$-convergence is a generalization form of statistical convergence and that is based on the notion of an ideal of the subset of positive integers $\mathbb{N}$.

On the other hand, in [2, 6] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order $\alpha$, $0 < \alpha < 1$ was introduced by replacing $n$ by $n^{\alpha}$ in the denominator in the definition of statistical convergence. One can also see [3] for related works.

In this paper we introduce new definitions to the notions of asymptotically $\mathcal{I}_\lambda$-statistical equivalent of order $\alpha$ to multiple $L$ and strongly $\mathcal{I}_\lambda$-asymptotically equivalent of order $\alpha$ to multiple $L$ by using two nonnegative real-valued Lebesque measurable functions $x\,(t)$ and $y\,(t)$ in the interval $(1, \infty)$ instead of sequences. In addition, we also present some inclusion theorems.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \qquad \lambda_1 = 1.$$

The collection of such sequences $\lambda$ will be denoted by $\Delta$.

$\lambda$-Statistical convergence was defined by Mursaleen [15]. In his examination he presented a series of critical results, beginning with the following definition.

**Definition 2.** *A sequence $x = (x_k)$ is said to be $\lambda$-statistically convergent or $S_\lambda$-convergent to the number $L$ if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n\colon\ |x_k - L| \geq \varepsilon\}\right| = 0,$$

*where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, 3, \ldots.$ and the vertical bars indicate the number of the elements in the enclosed sets. In this case we write $S_\lambda - \lim x = L$ or $x_n \to L(S_\lambda)$ and $S_\lambda$ denotes the set of all $\lambda$-statistically convergent sequences.*

Quite recently Srivastava et al. [28], studied the $S_\lambda^L(\mathcal{I})$-asymptotically statistical equivalent functions.

**2. Main definitions.** Before we present the new definitions we shall state a few known definitions.

**Definition 3** [11]. *A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of* $\mathbb{N}$, *where* $\mathbb{N}$ *will denote the set of all positive integers, if the following conditions hold*:

(a) $A,\ B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$,

(b) $A \in \mathcal{I},\ B \subset A$ *implies* $B \in \mathcal{I}$.

**Definition 4.** *A nonempty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of* $\mathbb{N}$ *if the following conditions hold*:

(a) $\phi \notin F$,

(b) $A,\ B \in F$ *implies* $A \cap B \in F$,

(c) $A \in F,\ A \subset B$ *implies* $B \in F$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ ( i.e., $\mathbb{N} \notin \mathcal{I}$ ), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N}\colon \exists\, A \in \mathcal{I}\colon M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

**Definition 5.** *A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.*

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

**Definition 6** [11]. *A sequence $(x_n)$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N}\colon |x_n - L| \geq \varepsilon\} \in \mathcal{I}$.*

Let us begin this analysis with the following preliminaries.

**Definition 7** [14]. *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(*denoted by $x \sim y$*).

**Definition 8** [9]. *The sequence $x = (x_k)$ is said to be statistically convergent to the number $L$ if for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} |\{k \leq n\colon\ |x_k - L| \geq \varepsilon\}| = 0.$$

*In this case we write $st - \lim x_k = L$.*

R. F. Patterson presented the following definition which is natural combination of Definitions 7 and 8.

**Definition 9** [17]. *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple $L$ provided that for every $\varepsilon > 0$*

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n\colon\ \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(*denoted by $x \overset{S_L}{\sim} y$*), *and simply asymptotically statistical equivalent if $L = 1$.*

The generalized La Vallée Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k.$$

A sequence $x = (x_k)$ is said to be $[V, \lambda]$-summable to a number $L$ if $t_n(x) \mapsto L$ and $n \mapsto \infty$. We write

$$[V, \lambda] = \left\{ \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \text{ for some } L \right\}$$

for the set of sequences that are strongly summable by the La Vallée Poussin method. In the special case where $\lambda_n = n$, for $n = 1, 2, 3, \ldots$, the set $[V, \lambda]$ reduces to the set $[C, 1]$-summability defined as follows:

$$[C, 1] = \left\{ \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0 \text{ for some } L \right\}.$$

We now introduce the following definitions.

**Definition 10.** *Let $\lambda \in \Delta$ and $x(t)$ be a nonnegative real-valued function which is measurable in the interval $(1,\infty)$. The function $x(t)$ is said to be $[V,\lambda](\mathcal{I})$-summable to $L$ if*

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{\lambda_n} \int\limits_{n-\lambda_n+1}^{n} |x(t) - L|\, dt = 0.$$

If $\mathcal{I} = \mathcal{I}_{\text{fin}} = \{A \subseteq \mathbb{N} \colon A \text{ is a finite subset}\}$, $[V,\lambda](\mathcal{I})$-summability becomes $[V,\lambda]$-summability, which is defined as follows (see [16]):

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int\limits_{n-\lambda_n+1}^{n} |x(t) - L|\, dt = 0.$$

**Definition 11.** *A nonnegative real-valued function $x(t)$ is said to be $\mathcal{I}_\lambda$-statistically convergent or $S_\lambda(\mathcal{I})$ convergent to $L$, if for every $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} \colon \frac{1}{\lambda_n} \left|\{t \in I_n : |x(t) - L| \geq \varepsilon\}\right| \geq \delta \right\} \in \mathcal{I}.$$

*In this case we write $S_\lambda(\mathcal{I}) - \lim x(t) = L$ or $x(t) \to L$ $(S_\lambda(\mathcal{I}))$. For $\mathcal{I} = \mathcal{I}_{\text{fin}}, S_\lambda(\mathcal{I})$-convergence again coincides with $\lambda$-statistical convergence* [16].

Following the above definitions we introduce the following new definitions related to the notions asymptotically $\mathcal{I}_\lambda$-statistical equivalent of multiple $L$, and strongly $\mathcal{I}_\lambda$-asymptotically equivalent of multiple $L$ for nonnegative real-valued functions $x(t)$ and $y(t)$.

**3. Main results.** In this section we give the main definitions and theorems of this paper.

**Definition 12.** *Let $\lambda \in \Delta$ and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ and $x(t)$, $y(t)$ be two nonnegative real-valued Lebesque measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are strongly $\mathcal{I}_\lambda$-asymptotically equivalent of order $\alpha$ to $L$, where $0 < \alpha \leq 1$, if*

$$\mathcal{I} - \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \int\limits_{n-\lambda_n+1}^{n} \left| \frac{x(t)}{y(t)} - L \right| dt = 0$$

(*denoted by $x(t) \overset{V_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$*)*, and simply asymptotically statistical equivalent equivalent of order $\alpha$ if $L = 1$. Furthermore, let $V_\lambda^L(\mathcal{I})^\alpha$ denote the set of $x(t)$ and $y(t)$ such that $x(t) \overset{V_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$.*

***Remark* 1.** If $\mathcal{I} = \mathcal{I}_{\text{fin}} = \{A \subseteq \mathbb{N} \colon A \text{ is a finite subset }\}$, strongly $\mathcal{I}_\lambda$-asymptotically equivalent of order $\alpha$ becomes strongly $\lambda$-asymptotically equivalent of order $\alpha$ which is defined as follows:

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \int\limits_{n-\lambda_n+1}^{n} \left| \frac{x(t)}{y(t)} - L \right| dt = 0.$$

Finally, for $\mathcal{I} = \mathcal{I}_{\text{fin}}$ and $\alpha = 1$ it becomes strongly $\lambda$-asymptotically equivalent of function [16].

We now have the following definitions.

**Definition 13.** *Let $x(t)$ and $y(t)$ be two nonnegative real-valued Lebesque measurable functions in the interval $(1,\infty)$ and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. We say that the functions $x(t)$ and $y(t)$ are $\mathcal{I}$-asymptotically statistical equivalent of order $\alpha$ to multiple $L$, where $0<\alpha\leq 1$, if for every $\varepsilon>0$ and $\delta>0$*

$$\left\{n\in\mathbb{N}\colon \frac{1}{n^{\alpha}}\left|\left\{k\leq n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

*In this case we write $x(t)\overset{S^L(\mathcal{I})^{\alpha}}{\sim}y(t)$.*

**Definition 14.** *Let $\lambda\in\Delta$ and $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ and $x(t)$, $y(t)$ be two nonnegative real-valued Lebesque measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are $\mathcal{I}_{\lambda}$-asymptotically statistical equivalent of order $\alpha$ to multiple $L$, where $0<\alpha\leq 1$, if for every $\varepsilon>0$ and $\delta>0$*

$$\left\{n\in\mathbb{N}\colon \frac{1}{\lambda_n^{\alpha}}\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}$$

(*denoted by $x(t)\overset{S_{\lambda}^L(\mathcal{I})^{\alpha}}{\sim}y(t)$*), *and simply asymptotically statistical equivalent of order $\alpha$ if $L=1$.*

We shall denote by $S_{\lambda}(\mathcal{I})^{\alpha}$ the collection of all $\mathcal{I}_{\lambda}$-asymptotically statistical equivalent of order $\alpha$ to multiple $L$.

For $\mathcal{I}=\mathcal{I}_{\text{fin}}$, $\mathcal{I}_{\lambda}$-asymptotically statistical equivalent of order $\alpha$ again coincides with $\lambda$-asymptotically statistical equivalent of order $\alpha$ which is defined as follows:

**Definition 15.** *Let $\lambda\in\Delta$ and $x(t)$, $y(t)$ be two nonnegative real-valued Lebesque measurable functions in the interval $(1,\infty)$. We say that the functions $x(t)$ and $y(t)$ are $\lambda$-asymptotically statistical equivalent of order $\alpha$ to multiple $L$, where $0<\alpha\leq 1$, if for every $\varepsilon>0$*

$$\left\{n\in\mathbb{N}\colon \frac{1}{\lambda_n^{\alpha}}\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|=0\right\}.$$

**4. Main theorems.**

**Theorem 1.** *Let $0<\alpha\leq\beta\leq 1$. Then $S_{\lambda}(\mathcal{I})^{\alpha}\subset S_{\lambda}(\mathcal{I})^{\beta}$.*

***Proof.*** Let $0<\alpha\leq\beta\leq 1$. Then for every $\varepsilon>0$ we have

$$\frac{\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|}{\lambda_n^{\beta}}\leq\frac{\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|}{\lambda_n^{\alpha}}$$

and so, for any $\delta>0$,

$$\left\{n\in\mathbb{N}\colon \frac{\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|}{\lambda_n^{\beta}}\geq\delta\right\}\subset\left\{n\in\mathbb{N}\colon \frac{\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|}{\lambda_n^{\alpha}}\geq\delta\right\}.$$

Hence, if the set on the right-hand side belongs to the ideal $\mathcal{I}$, then obviously the set on the left-hand side also belongs to $\mathcal{I}$. This shows that $S_{\lambda}(\mathcal{I})^{\alpha}\subset S_{\lambda}(\mathcal{I})^{\beta}$.

Theorem 1 is proved.

***Corollary* 1.** *If two the functions $x(t)$ and $y(t)$ are $\mathcal{I}_\lambda$-asymptotically statistical equivalent of order $\alpha$ to multiple $L$ for some $0<\alpha\leq 1$, then they are $\mathcal{I}_\lambda$-statistically convergent functions to $L$, i.e., $S_\lambda(\mathcal{I})^\alpha \subset S_\lambda(\mathcal{I})$.*

Similarly we can show that the following theorems are valid.

**Theorem 2.** *Let $0<\alpha\leq\beta\leq 1$. Then*

(i) $S(\mathcal{I})^\alpha \subset S(\mathcal{I})^\beta$.

(ii) $[V,\lambda]^\alpha(\mathcal{I}) \subset [V,\lambda]^\beta(\mathcal{I})$.

(iii) *In particular* $S(\mathcal{I})^\alpha \subset S(\mathcal{I})$ *and* $[V,\lambda]^\alpha(\mathcal{I}) \subset [V,\lambda](\mathcal{I})$.

**Theorem 3.** *Let $\lambda=\{\lambda_n\}_{n\in\mathbb{N}}\in\Delta$. Then*

(a) *If* $x(t) \overset{V_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$, *then* $x(t) \overset{S_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$.

(b) $V_\lambda^L(\mathcal{I})^\alpha$ *is a proper subset of* $S_\lambda^L(\mathcal{I})^\alpha$ *for every ideal* $\mathcal{I}$.

***Proof.*** (a) Let $\varepsilon>0$ and $x(t) \overset{V_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$. We have

$$\int\limits_{t\in I_n}\left|\frac{x(t)}{y(t)}-L\right|dt \geq \int\limits_{t\in I_n \& \left|\frac{x(t)}{y(t)}-L\right|>\varepsilon}\left|\frac{x(t)}{y(t)}-L\right|dt \geq \varepsilon\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|.$$

So, for given $\delta>0$,

$$\frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta \Rightarrow \frac{1}{\lambda_n^\alpha}\int\limits_{t\in I_n}\left|\frac{x(t)}{y(t)}-L\right|dt\geq\varepsilon\delta,$$

i.e.,

$$\left\{n\in\mathbb{N}\colon \frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\subset$$

$$\subset\left\{n\in\mathbb{N}\colon \frac{1}{\lambda_n^\alpha}\left\{\int\limits_{t\in I_n}\left|\frac{x(t)}{y(t)}-L\right|dt\geq\varepsilon\right\}\geq\varepsilon\delta\right\}.$$

Since right-hand belongs to $\mathcal{I}$, then left-hand also belongs to $\mathcal{I}$ and this completes the proof.

(b) To show that $x(t) \overset{S_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t) \subsetneq x(t) \overset{V_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$, take a fixed $A\in\mathcal{I}$. Define a function $x$ by

$$x(t)=\begin{cases} t & \text{for } n-[\sqrt{\lambda_n^\alpha}]+1\leq t\leq n,\ n\notin A,\\ t & \text{for } n-\lambda_n^\alpha+1\leq t\leq n, \qquad n\in A,\\ \theta & \text{otherwise,}\end{cases}$$

and $y(t)=1$. Then, for every $\varepsilon>0$ $(0<\varepsilon<1)$,

$$\frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right| = \frac{[\sqrt{\lambda_n^\alpha}]}{\lambda_n^\alpha}\to 0$$

as $n\to\infty$ and $n\notin A$, so, for every $\delta>0$,

$$\left\{n\in\mathbb{N}\colon \frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\subset A\cup\{1,2,\ldots,m\}$$

for some $m \in N$. Since $\mathcal{I}$ is admissible so it follows that it is $\mathcal{I}_\lambda$-asymptotically statistical equivalent of order $\alpha$ to multiple $L$. Obviously

$$\frac{1}{\lambda_n^\alpha} \int\limits_{t\epsilon I_n} \left| \frac{x(t)}{y(t)} - L \right| \to \infty \, (n \to \infty) \, ,$$

i.e., it is not strongly $\mathcal{I}_\lambda$-asymptotically equivalent of order $\alpha$ to multiple $L$. Note that if $A \in \mathcal{I}$ is infinite, then it is not $\mathcal{I}_\lambda$-asymptotically statistical equivalent of order $\alpha$ of multiple $L$.

**Theorem 4.** *$x(t) \overset{S^L(\mathcal{I})^\alpha}{\sim} y(t)$ implies $x(t) \overset{S_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$ if $\liminf_{n\to\infty} \dfrac{\lambda_n^\alpha}{n^\alpha} > 0$.*

***Proof.*** For given $\varepsilon > 0$,

$$\frac{1}{n^\alpha} \left| \left\{ t \leq n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| \geq \frac{1}{n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| \geq$$

$$\geq \frac{\lambda_n^\alpha}{n^\alpha} \frac{1}{\lambda_n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| .$$

If $\liminf_{n\to\infty} \dfrac{\lambda_n^\alpha}{n^\alpha} = a$, then $\left\{ n \in \mathbb{N} \colon \dfrac{\lambda_n^\alpha}{n^\alpha} < \dfrac{a}{2} \right\}$ is finite. For $\delta > 0$,

$$\left\{ n \in \mathbb{N} \colon \frac{1}{\lambda_n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \subset$$

$$\subset \left\{ n \in \mathbb{N} \colon \frac{1}{n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| \geq \frac{a}{2}\delta \right\} \cup$$

$$\cup \left\{ n \in \mathbb{N} \colon \frac{\lambda_n}{n} < \frac{a}{2} \right\} .$$

Since $\mathcal{I}$ is admissible, the set on the right-hand side belongs to $\mathcal{I}$ and this completed the proof.

**Theorem 5.** *If $\lambda \in \Delta$ be such that $\lim_n \dfrac{\lambda_n^\alpha}{n^\alpha} = 1$, then $x(t) \overset{S_\lambda^L(\mathcal{I})^\alpha}{\sim} y(t)$ implies $x(t) \overset{S^L(\mathcal{I})^\alpha}{\sim} y(t)$.*

***Proof.*** Let $\delta > 0$ be given. Since $\lim_n \dfrac{\lambda_n^\alpha}{n^\alpha} = 1$, we can choose $m \in N$ such that $\left| \dfrac{\lambda_n^\alpha}{n^\alpha} - 1 \right| < \dfrac{\delta}{2}$ for all $n \geq m$. Now observe that, for $\varepsilon > 0$,

$$\frac{1}{n^\alpha} \left| \left\{ t \leq n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| = \frac{1}{n^\alpha} \left| \left\{ t \leq n - \lambda_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| +$$

$$+ \frac{1}{n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| \leq$$

$$\leq \frac{n - \lambda_n}{n^\alpha} + \frac{1}{n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| \leq$$

$$\leq 1 - \left( 1 - \frac{\delta}{2} \right) + \frac{1}{n^\alpha} \left| \left\{ t \in I_n \colon \left| \frac{x(t)}{y(t)} - L \right| \geq \varepsilon \right\} \right| =$$

$$= \frac{\delta}{2} + \frac{1}{n^\alpha} \left| \{ t \in I_n \colon \left| x(t) - L \right| \geq \varepsilon \} \right|$$

for all $n \geq m$. Hence, for any $\delta > 0$ we write

$$\left\{n \in \mathbb{N} \colon \frac{1}{n^{\alpha}} \left|\{t \leq n \colon |x(t) - L| \geq \varepsilon\}\right| \geq \delta\right\} \subset$$

$$\subset \left\{n \in \mathbb{N} \colon \frac{1}{n^{\alpha}} \left|\{t \in I_n \colon |x(t) - L| \geq \varepsilon\}\right| \geq \frac{\delta}{2}\right\} \cup \{1, 2, 3, \ldots, m\}.$$

This shows that $x(t) \overset{S^L(\mathcal{I})^{\alpha}}{\sim} y(t)$.

Theorem 5 is proved.

***Remark* 2.** We do not know whether the condition in Theorem 5 is necessary and leave it is an open problem.

Now we shall prove a more inclusion relation theorem.

**Theorem 6.** *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two functions in $\Delta$ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let $\alpha$ and $\beta$ be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.*

(i) *If*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} > 0, \tag{1}$$

*then $S_{\mu}^L(\mathcal{I})^{\beta} \subseteq S_{\lambda}^L(\mathcal{I})^{\alpha}$.*

(ii) *If*

$$\lim_{n\to\infty} \frac{\mu_n}{\lambda_n^{\beta}} = 1, \tag{2}$$

*then $S_{\lambda}^L(\mathcal{I})^{\alpha} \subseteq S_{\mu}^L(\mathcal{I})^{\beta}$.*

***Proof.*** (i) Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and let (1) be satisfied. For given $\varepsilon > 0$ we have

$$\left\{t \in J_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\} \supseteq \left\{t \in I_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\},$$

where $I_n = [n - \lambda_n + 1, n]$ and $J_n = [n - \mu_n + 1, n]$. Therefore we can write

$$\frac{1}{\mu_n^{\beta}} \left|\left\{t \in J_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} \left|\left\{t \in I_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right|$$

and so, for all $n \in \mathbb{N}$ and for any $\delta > 0$, we obtain

$$\left\{n \in \mathbb{N} \colon \frac{1}{\lambda_n^{\alpha}} \left|\left\{t \in I_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq$$

$$\subseteq \left\{n \in \mathbb{N} \colon \frac{1}{\mu_n^{\beta}} \left|\left\{t \in J_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| \geq \delta \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right\} \in \mathcal{I}.$$

Hence, $S_{\mu}^L(\mathcal{I})^{\beta} \subseteq S_{\lambda}^L(\mathcal{I})^{\alpha}$.

(ii) Let $x = (x_k)$ and $y = (y_k) \in S_{\lambda}^L(\mathcal{I})^{\alpha}$ and (2) be satisfied. Since $I_n \subset J_n$, for $\varepsilon > 0$ we may write

$$\frac{1}{\mu_n^{\beta}} \left|\left\{t \in J_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| = \frac{1}{\mu_n^{\beta}} \left|\left\{n - \mu_n < t \leq n - \lambda_n \colon \left|\frac{x(t)}{y(t)} - L\right| \geq \varepsilon\right\}\right| +$$

$$+\frac{1}{\mu_n^\beta}\left|\left\{t\in I_n\colon\ \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\leq$$

$$\leq\frac{\mu_n-\lambda_n}{\mu_n^\beta}+\frac{1}{\lambda_n^\beta}\left|\left\{t\in I_n\colon\ \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\leq$$

$$\leq\left(\frac{\mu_n-\lambda_n^\beta}{\lambda_n^\beta}\right)+\frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon\ \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\leq$$

$$\leq\left(\frac{\mu_n}{\lambda_n^\beta}-1\right)+\frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon\ \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|$$

for all $n\in\mathbb{N}$. Hence, for any $\delta>0$, we have

$$\left\{n\in\mathbb{N}\colon\frac{1}{\mu_n^\beta}\left|\left\{t\in J_n\colon\ \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\subseteq$$

$$\subseteq\left\{n\in\mathbb{N}\colon\frac{1}{\lambda_n^\alpha}\left|\left\{t\in I_n\colon\ \left|\frac{x(t)}{y(t)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

This implies that $S_\lambda^L(\mathcal{I})^\alpha\subseteq S_\mu^L(\mathcal{I})^\beta$.

Theorem 6 is proved.

From Theorem 6 we have the following corollaries.

***Corollary* 2.** *Let* $\lambda=(\lambda_n)$ *and* $\mu=(\mu_n)$ *be two sequences in* $\Delta$ *such that* $\lambda_n\leq\mu_n$ *for all* $n\in\mathbb{N}$. *If* (1) *holds, then*

(i) $S_\mu^L(\mathcal{I})^\alpha\subseteq S_\lambda^L(\mathcal{I})^\alpha$ *for each* $\alpha\in(0,1]$,

(ii) $S_\mu^L(\mathcal{I})\subseteq S_\lambda^L(\mathcal{I})^\alpha$ *for each* $\alpha\in(0,1]$,

(iii) $S_\mu^L(\mathcal{I})\subseteq S_\lambda^L(\mathcal{I})$.

***Corollary* 3.** *Let* $\lambda=(\lambda_n)$ *and* $\mu=(\mu_n)$ *be two sequences in* $\Delta$ *such that* $\lambda_n\leq\mu_n$ *for all* $n\in\mathbb{N}$. *If* (2) *holds, then*

(i) $S_\lambda^L(\mathcal{I})^\alpha\subseteq S_\mu^L(\mathcal{I})^\alpha$ *for each* $\alpha\in(0,1]$,

(ii) $S_\lambda^L(\mathcal{I})^\alpha\subseteq S_\mu^L(\mathcal{I})$ *for each* $\alpha\in(0,1]$,

(iii) $S_\lambda^L(\mathcal{I})\subseteq S_\mu^L(\mathcal{I})$.

## References


1. *Cakalli H.* A study on statistical convergence // Funct. Anal. Approxim. and Comput. – 2009. – **1**, № 2. – P. 19–24.
2. *Colak R.* Statistical convergence of order $\alpha$ // Modern Methods in Analysis and its Appl. – New Delhi, India: Anamaya Publ., 2010. – P. 121–129.
3. *Colak R., Bektas C. A.* $\lambda$-Statistical convergence of order $\alpha$ // Acta Math. Sci. Ser. B. – 2011. – **31**, № 3. – P. 953–959.
4. *Connor J., Savas E.* Lacunary statistical and sliding window convergence for measurable functions // Acta Math. Hung. – 2015. – **145**, № 2. – P. 416–432.
5. *Das P., Savaş E., Ghosal S. Kr.* On generalizations of certain summability methods using ideals // Appl. Math. Lett. – 2011. – **24**, № 9. – P. 1509–1514.
6. *Das P., Savaş E.* On $\mathcal{I}$-statistical and $\mathcal{I}$-lacunary statistical convergence of order $\alpha$ // Bull. Iran. Math. Soc. – 2014. – **40**, № 2. – P. 459–472.
7. *Dems K.* On $\mathcal{I}$-Cauchy sequences // Real Anal. Exchance. – 2004-2005. – **30**. – P. 123–128.
8. *Fast H.* Sur la convergence ststistique // Colloq. Math. – 1951. – **2**. – P. 241–244.
9. *Fridy J. A.* On statistical convergence // Analysis. – 1985. – **5**. – P. 301–313.

10. *Gümüs H., Savaş E.* On $S^L_\lambda(\mathcal{I})$-asymptotically statistical equivalent sequences // Numer. Anal. and Appl. Math. (ICNAAM-2012): AIP Conf. Proc. – 2012. – **1479**. – P. 936 – 941.
11. *Kostyrko P., Salat T., Wilczynski* W. $\mathcal{I}$-convergence // Real Anal. Exchange. – 2000/2001. – **26**, № 2. – P. 669 – 686.
12. *KumarV., Sharma A.* On asymptotically generalized statistical equivalent sequences via ideal // Tamkang J. Math. – 2012. – **43**, № 3. – P. 469 – 478.
13. *Li J.* Asymptotic equivalence of sequences and summability // Internat. J. Math. and Math. Sci. – 1997. – **20**, № 4. – P. 749 – 758.
14. *Marouf M.* Asymptotic equivalence and summability // Internat. J. Math. Sci. – 1993. – **16**, № 4. – P. 755 – 762.
15. *Mursaleen M.* $\lambda$-Statistical convergence // Math. Slovaca. – 2000. – **50**. – P. 111 – 115.
16. *Nuray F.* $\lambda$-Strongly summable and $\lambda$ statistically convergent functions // Iran. J. Sci. and Technol. Trans. A. – 2010. – **34**, № 4. – P. 335 – 339.
17. *Patterson R. F.* On asymptotically statistically equivalent sequences // Demostr. Math. – 2003. – **36**, № 1. – P. 149 – 153.
18. *Šalát T.* On statistically convergent sequences of real numbers // Math. Slovaca. – 1980. – **30**. – P. 139 – 150.
19. *Savaş R., Basarir M.* $(\sigma, \lambda)$-Asymptotically statistical equivalent sequences // Filomat. – 2006. – **20**, № 1. – P. 35 – 42.
20. *Savaş E.* $\triangle^m$-Strongly summable sequences spaces in 2-normed spaces defined by ideal convergence and an Orlicz function // Appl. Math. and Comput. – 2010. – **217**. – P. 271 – 276.
21. *Savaş E.* A-sequence spaces in 2-normed space defined by ideal convergence and an Orlicz function // Abst. and Appl. Anal. – 2011. – **2011**. – Article ID 741382.
22. *Savaş E.* On some new sequence spaces in 2-normed spaces using ideal convergence and an Orlicz function // J. Inequal. and Appl. – 2010. – Article No. 482392.
23. *Savaş E., Das P.* A generalized statistical convergence via ideals // Appl. Math. Lett. – 2011. – **24**. – P. 826 – 830.
24. *Savaş E.* On $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences // Adv. Difference Equat. – 2013. – № 111.
25. *Savaş E.* On asymptotically $\mathcal{I}$-statistical equivalent sequences of order $\alpha$ // Indian J. Math. – 2014. – **56**, № 2.
26. *Savaş E., Gümüs H.* A generalization on $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences // J. Inequal. and Appl. – 2013. – **2013**. – № 270.
27. *Schoenberg I. J.* The integrability methods // Amer. Math. Monthly. – 1959. – **66**. – P. 361 – 375.
28. *Srivastava H. M., Savaş E., Patterson R. F.* A class of $S^L_\lambda(\mathcal{I})$-asymptotically statistically equivalent functions // J. Nonlinear Sci. and Appl. – 2018. – **11**. – P. 1161 – 1170.
29. *Steinhaus H.* Sur la convergence ordinaire et la convergence asymptotique // Colloq. Math. – 1951. – **2**. – P. 73 – 74.

Received 27.12.16