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B. N. Türkmen (Amasya Univ., Turkey)

A GENERALIZATION OF SEMIPERFECT MODULES
УЗАГАЛЬНЕННЯ НАПІВДОСКОНАЛИХ МОДУЛІВ

A module M is called radical semiperfect, if $\frac{M}{N}$ has a projective cover whenever $\operatorname{Rad}(M) \subseteq N \subseteq M$. We study various properties of these modules. It is proved that every left R-module is radical semiperfect if and only if R is left perfect. Moreover, radical lifting modules are defined as a generalization of lifting modules.

Модуль M називається радикальним напівдосконалим, якщо $\frac{M}{N}$ має проективне покриття, як тільки $\operatorname{Rad}(M) \subseteq$ $\subseteq N \subseteq M$. Досліджено різні властивості цих модулів. Доведено, що кожен лівий R-модуль є радикально напівдосконалим тоді і тільки тоді, коли R є лівим досконалим. Крім того, радикальні піднімаючі модулі визначено, як узагальнення піднімаючих модулів.

1. Introduction. Throughout this paper, all rings are associative with identity element and all modules are unital left R-modules. Let M be a left module over such a ring R. The notation $(N \subset M)$ $N \subseteq M$ means that N is a (proper) submodule of M. A submodule $N \subset M$ is said to be *small* in M, denoted as $N \ll M$, if $M \neq N + L$ for every proper submodule L of M. If every proper submodule N of M is small, then M is called *hollow*. A finitely generated hollow module is *local*.

For modules M and P, let $f\colon P \to M$ be an epimorphism. f is called *cover* if $\ker(f)$ is small in P. A projective module P together with a cover $f\colon P \to M$ is called a *projective cover* of M. By [1], rings whose (finitely generated) modules have a projective cover is (*semi*) *perfect*.

In [6], Kasch and Mares transferred the notions of (semi)perfect rings to modules. Let M be an R-module. M is called *semiperfect* if every factor module of M has a projective cover. They gave a characterization of semiperfect modules via supplemented modules. A module M is called *supplemented* if every submodule N of M has a supplement, that is a submodule K minimal with respect to $M = N + K$. A submodule $K \subseteq M$ is supplement of N in M if and only if $M = N + K$ and $N \cap K \ll K$. If M is a semiperfect module, then it is supplemented. Also, the converse of this fact is true in case the module is projective.

Zöschinger completely determined in [14] (Lemma 2.1) the structure of a module M with small radical over local Dedekind domains. Here the radical $\operatorname{Rad}(M)$ of a module M is the intersection of all maximal submodules of the module M, or equivalently the sum of all small submodules of M. Then, he considered the modules whose radical have a supplement. He called these modules as a *radical supplemented module*. Upon this Büyükaşık and Türkmen call a module M *strongly radical suplemented* (or briefly a *srs-module*) if every submodule containing radical has a supplement in M [2].

As motivated by the above definitions, it is natural to introduce radical semiperfect modules. We say that a module M is *radical semiperfect* if, whenever $\operatorname{Rad}(M) \subseteq N \subseteq M$, $\frac{M}{N}$ has a projective cover.

In this article, we provide some properties of these modules. We prove that every (finitely generated) left R-module is radical semiperfect if and only if R is (semi)perfect. Radical semiperfect modules are srs, and a projective module M is radical semiperfect if and only if it is $srs^{\oplus}$, i.e., for every submodule $\mathrm{Rad}(M) \subseteq N \subseteq M$, there exists a direct summand K of M such that K is a supplement of N in M (see [8]). We show that every direct summand of a radical semiperfect module is radical semiperfect. We also study radical lifting modules as a generalization of lifting modules.

2. Radical semiperfect modules. It is well known that, a ring R is left perfect if and only if every left R-module is semiperfect. We generalize this fact via radical semiperfect modules.

It is clear that every semiperfect module is radical semiperfect. But, in general, the converse is false as the following example shows.

Recall that a module M is *radical* if M has no maximal submodules, that is, $M = \mathrm{Rad}(M)$.

Lemma 2.1. *Every radical module is radical semiperfect.*

Proof. Let M be a radical module. Then $\mathrm{Rad}(M) = M$. Therefore, the factor module $\dfrac{M}{M}$ of M is zero. It follows that M is radical semiperfect.

By $P(M)$ we will denote the sum of all radical submodules of a module M. Note that $P(M)$ is the largest radical submodule of M and $P(M) \subseteq \mathrm{Rad}(M)$. Using Lemma 2.1 we obtain that $P(M)$ is radical semiperfect for every left R-module M.

***Example* 2.1.** Consider the left $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$. Since $\mathrm{Rad}(M) = M$, we obtain that M is a radical semiperfect module by Lemma 2.1. On the other hand, M is not semiperfect because it is not supplemented [12].

Proposition 2.1. *Every radical semiperfect module with small radical is semiperfect.*

Proof. Let M be a radical semiperfect module with $\mathrm{Rad}(M) \ll M$ and U be any submodule of M. Then $\mathrm{Rad}(M) \subseteq U + \mathrm{Rad}(M)$. Since M is radical semiperfect, $\dfrac{M}{U + \mathrm{Rad}(M)}$ has a projective cover. By [11] (Proposition 2.1) there exists a submodule V of M such that $M = (U + \mathrm{Rad}(M)) + V$, $(U + \mathrm{Rad}(M)) \cap V \ll V$ and V has a projective cover. Therefore, $U \cap V$ is a small submodule of V as the small submodule $(U + \mathrm{Rad}(M)) \cap V$ in V. It follows from assumption that $M = (U + \mathrm{Rad}(M)) + V = U + V$. Again by [11] (Proposition 2.1), we deduce that $\dfrac{M}{U}$ has a projective cover. Thus M is semiperfect.

Because of the above result, defining radical semiperfect rings do not make sense. That is, a ring R is semiperfect if and only if R is radical semiperfect.

Proposition 2.2. *Every radical semiperfect module is srs.*

Proof. Let M be a radical semiperfect module and N be any submodule of M with $\mathrm{Rad}(M) \subseteq N$. Then the factor module $\dfrac{M}{N}$ of M has a projective cover. Applying [11] (Proposition 2.1), N has a supplement in M. Hence M is srs.

Proposition 2.3. *Every factor module of a radical semiperfect module is radical semiperfect.*

Proof. Let M be a radical semiperfect module and $X \subseteq M$. For any submodule $\dfrac{N}{X}$ of $\dfrac{M}{X}$ with $\mathrm{Rad}\left(\dfrac{M}{X}\right) \subseteq \dfrac{N}{X}$, we can write $\mathrm{Rad}(M) \subseteq N$ because $\dfrac{\mathrm{Rad}(M) + X}{X} \subseteq \mathrm{Rad}\left(\dfrac{M}{X}\right) \subseteq \dfrac{N}{X}$. Since M is a radical semiperfect, $\dfrac{M}{N}$ has a projective cover. Note that

$$\frac{M}{N} \cong \frac{\frac{M}{X}}{\frac{N}{X}}$$

and so $\dfrac{\frac{M}{X}}{\frac{N}{X}}$ has a projective cover. Thus, $\dfrac{M}{X}$ is radical semiperfect.

The following follows from Proposition 2.3.

***Corollary* 2.1.** *Every homomorphic image of a radical semiperfect module is radical semiperfect. In particular, every direct summand of the module is radical semiperfect.*

It is known in [10] (42.3 (2)-(ii)) that if K is semiperfect and $\pi\colon M \to K$ is a cover, then M is also semiperfect. Now, we give an analogous characterization of this fact for radical semiperfect modules.

Lemma 2.2. *Let K be a radical semiperfect module and let $\pi\colon M \to K$ be a cover of K. Then M is a radical semiperfect module.*

Proof. Let N be a submodule of M such that $\mathrm{Rad}(M) \subseteq N$. We have an epimorphism $\varphi\colon \dfrac{M}{N} \to \dfrac{N}{\pi(N)}$ defined by $m + N \to \pi(m) + \pi(N)$. Since $\pi\colon M \to K$ is a cover, we have $\ker(\varphi) \ll \dfrac{M}{N}$. Moreover, $\mathrm{Rad}(K) \subseteq \pi(N)$. Since K is radical semiperfect, the factor module $\dfrac{K}{\pi(N)}$ has a projective cover, say $\psi\colon P \to \dfrac{K}{\pi(N)}$. Since P is projective, there exists a homomorphism $h\colon P \to \dfrac{M}{N}$ such that $\varphi \circ h = \psi$. So h is an epimorphism. It follows from [10] (19.3 (1)) that h is a cover. Therefore, projective module P together with a cover h is a projective cover of $\dfrac{M}{N}$. It means that M is radical semiperfect.

We have the following corollary by Proposition 2.3 and Lemma 2.2. Note that a submodule $L \subseteq M$ is small in M if and only if the canonical homomorphism $\pi\colon M \longrightarrow L$ is a cover of $\dfrac{M}{L}$.

***Corollary* 2.2.** *Let M be an R-module and $L \ll M$. Then M is radical semiperfect if and only if $\dfrac{M}{L}$ is radical semiperfect.*

Theorem 2.1. *Let M be a projective R-module. Then M is $srs^{\oplus}$ if and only if it is radical semiperfect.*

Proof. ($\Longrightarrow$) Let N be any submodule of M with $\mathrm{Rad}(M) \subseteq N$. By the assumption, we can write $M = N + K$, $N \cap K \ll K$ and $M = K \oplus K'$ for some submodules $K, K' \subseteq M$. Since M is projective, we get that K is projective as a direct summand of M. Then a homomorphism $f = \pi \circ i\colon K \to \dfrac{M}{N}$ is an epimorphism for the inclusion homomorphism $i\colon K \to M$ and the canonical homomorphism $\pi\colon M \to \dfrac{M}{N}$. Since $\mathrm{Ker}(\pi) \ll K$, K together with a cover f is a projective cover of $\dfrac{M}{N}$. Hence, M is radical semiperfect.

($\Longleftarrow$) Let N be any submodule of M with $\mathrm{Rad}(M) \subseteq N$. Since M is radical semiperfect, $\dfrac{M}{N}$ has a projective cover. Let P together with a cover φ be a projective cover of $\dfrac{M}{N}$. Since M is projective, there exists a homomorphism $f\colon M \to P$ such that $\varphi \circ f = \pi$, where $\pi\colon M \to \dfrac{M}{N}$ is the canonical homomorphism. Since φ is a cover, f is an epimorphism by [10] (19.2). Since P is projective, the exact sequence $0 \to N \to M \to P \to 0$ is decomposable. Then there exists a homomorphism $g\colon P \to M$ such that $f \circ g = 1_P$. Therefore, $M = \ker f \oplus g(P)$ and $\ker f \subseteq N$. It follows that $M = N + g(P)$. On the other hand, $\varphi = \pi|_{g(P)}\colon g(P) \to \dfrac{M}{N}$ is a cover. Thus $\ker(\pi|_{g(P)}) = U \cap g(P) \ll g(P)$. It means that M is a $srs^{\oplus}$-module.

***Corollary* 2.3.** *Let M be an R-module and let P together with a cover $f\colon P \to M$ be a projective cover of M. Then the following statements are equivalent*:

(1) *M is a radical semiperfect module*,

(2) *P is a radical semiperfect module*,

(3) *P is a $srs^{\oplus}$-module.*

Proof. (1) $\iff$ (2). The necessity is clear from Lemma 2.2. Conversely is clear from Corollary 2.1.

(2) $\iff$ (3). It follows from Theorem 2.1.

Proposition 2.4. *Let M be a radical semiperfect module. Assume that M has a projective cover. Then M is semiperfect.*

Proof. Let $f\colon P \to M$ be a projective cover. By Corollary 2.3, we obtain that P is a $srs^{\oplus}$-module. Applying [8] (Proposition 2.2) and [10] (42.1), P is semiperfect and then M is semiperfect as a factor module of the module P according to [10] (42.3 (2)-(i)).

Recall from [4] that an R-module M is *cofinitely semiperfect* if every finitely generated factor module of M has a projective cover. Now we give some relations between radical semiperfect modules and cofinitely semiperfect modules.

***Example* 2.2** (see [5], (11.3))**.** Let R be the ring $K[[x]]$ of all power series $\sum_{i=0}^{\infty} k_i x^i$ in an indeterminate x and with coefficients from a field K. Then R is semiperfect but not left perfect. So there exists an infinite index set I such that the left R-module $R^{(I)}$ is not a $srs^{\oplus}$-module. Since $R^{(I)}$ is projective, $R^{(I)}$ is not radical semiperfect by Theorem 2.1. By [4] (Theorem 2.12), $R^{(I)}$ is cofinitely semiperfect.

***Example* 2.3.** Let R be a local Dedekind domain with quotient field K. Then $\dfrac{M}{L} \cong K = \mathrm{Rad}(K)$ as an R-module, for a free R-module M and $L \subseteq M$. Since R is hereditary, M is projective. It is clear that M is not $\oplus$-supplemented. So M is not a $srs^{\oplus}$-module by [8] (Proposition 2.9). By Theorem 2.1, M is not radical semiperfect. In addition, since R is semiperfect, M is cofinitely semiperfect by [4] (Theorem 2.12).

In the following proposition, we prove that every cofinitely semiperfect module is radical semiperfect under a condition.

Proposition 2.5. *Let M be a cofinitely semiperfect module. If $\dfrac{M}{\mathrm{Rad}(M)}$ is finitely generated, then M is radical semiperfect.*

Proof. Let N be any submodule of M with $\mathrm{Rad}(M) \subseteq N$. Then $\dfrac{M}{N}$ is finitely generated as a factor module of the finitely generated module $\dfrac{M}{\mathrm{Rad}(M)}$. Since M is cofinitely semiperfect, $\dfrac{M}{N}$ has a projective cover. Thus, M is radical semiperfect.

Now, we give a characterization of perfect rings via radical semiperfect modules.

Theorem 2.2. *A ring R is left perfect if and only if every left R-module is radical semiperfect.*

Proof. ($\Longrightarrow$) It is clear.

($\Longleftarrow$) By Proposition 2.2 and [2] (Corollary 2.6).

Let R be an arbitrary ring. R is called a *left Bass ring* if every non zero left R-module has a maximal submodule [3] (2.19).

Proposition 2.6. *Let R be a ring. Then R is a left Bass ring if and only if every radical semiperfect left R-module is semiperfect.*

Proof. ($\Longrightarrow$) Let M be any radical semiperfect left R-module. Since R is a left Bass ring, it follows from [3] (2.21) that $\mathrm{Rad}(M) \ll M$. By Proposition 2.1, M is semiperfect.

($\Longleftarrow$) Let M be a left R-module which is radical. By Lemma 2.1, M is radical semiperfect. It follows from the assumption that it is semiperfect. If $M \neq 0$, then M has a maximal submodule according to [11] (Lemma 3.7), a contradiction. So $M = 0$. It means that R is a left Bass ring.

Recall from [11] that a module M is called *quasi semiperfect* if every factor module of M has a M-projective cover. As a proper generalization of this definition, we call a module M *quasi radical semiperfect* if $\dfrac{M}{N}$ has a M-projective cover for every submodule N of M containing $\mathrm{Rad}(M)$.

It is clear that every radical semiperfect module is quasi radical semiperfect. But in general the converse is not true. For example, ${}_{\mathbb{Z}}\mathbb{Q}$ is quasi radical semiperfect but not quasi semiperfect (see Lemma 2.1 and [11], Theorem 2.4).

Lemma 2.3. *Let M be a quasi radical semiperfect module. Then every submodule N of M containing* $\mathrm{Rad}(M)$ *has a supplement which has a M-projective cover.*

Proof. Let N be any submodule of M containing $\mathrm{Rad}(M)$. By the hypothesis, $\dfrac{M}{N}$ has a M-projective cover. By [11] (Proposition 2.3), N has a supplement which has a M-projective cover.

Now, we shall prove that a quasi radical semiperfect module is radical semiperfect under a certain condition.

Theorem 2.3. *Let M be a module with* $\mathrm{Rad}(M) \ll M$. *M is quasi radical semiperfect if and only if it is quasi semiperfect.*

Proof. ($\Longrightarrow$) Let N be any submodule of M. Then $\mathrm{Rad}(M) \subseteq N + \mathrm{Rad}(M)$. By the hypothesis and Lemma 2.3, there exists a submodule V of M such that $M = (N + \mathrm{Rad}(M)) + V$ and $(N + \mathrm{Rad}(M)) \cap V \ll V$. Moreover, V has a M-projective cover. So we have $M = (N + \mathrm{Rad}(M)) + V$, $(N + \mathrm{Rad}(M)) \cap V \ll V$ and V has a M-projective cover. Since $\mathrm{Rad}(M) \ll M$, then $M = N + V$ and $N \cap V \ll V$. By [11] (Proposition 2.3), $\dfrac{M}{N}$ has a M-projective cover. Thus M is quasi semiperfect.

($\Longleftarrow$) It is clear from the definition.

Proposition 2.7. *Every factor module of a quasi radical semiperfect module is quasi radical semiperfect.*

Proof. Let M be a quasi radical semiperfect module, X be any submodule of M and $\dfrac{N}{X}$ be any submodule of $\dfrac{M}{X}$ such that $\mathrm{Rad}\left(\dfrac{M}{X}\right) \subseteq \dfrac{N}{X}$. Then $\mathrm{Rad}(M) \subseteq N$. By the hypothesis, $\dfrac{M}{N}$ has a M-projective cover. Therefore there exists a cover $\varphi\colon P \to \dfrac{M}{N}$ such that P is M-projective. Let $\pi\colon M \to \dfrac{M}{X}$ be the canonical epimorphism. We have an epimorphism $f\colon \dfrac{M}{X} \to \dfrac{M}{N}$. Then $f \circ \pi\colon M \to \dfrac{M}{N}$ is an epimorphism. Since P is M-projective, there exists a homomorphism $h\colon P \to M$ such that $f \circ \pi \circ h = \varphi$. Then there exists a homomorphism $\pi \circ h\colon P \to \frac{M}{X}$ such that $f \circ \pi \circ h = \varphi$.

Therefore, P is $\dfrac{M}{X}$-projective. Since $\dfrac{M}{N}$ has a $\dfrac{M}{X}$-projective cover for every submodule $\dfrac{N}{X}$ of $\dfrac{M}{X}$ such that $\operatorname{Rad}\left(\dfrac{M}{X}\right) \subseteq \dfrac{N}{X}$, then $\dfrac{M}{X}$ is a quasi radical semiperfect module.

3. Radical lifting modules. Recall from [3] (22.2) that a module M is *lifting* if, for any submodule N of M, there exists the decomposition $M = N' \oplus K$ such that $N' \subseteq N$ and $N \cap K \ll K$. Every π-projective supplemented module is lifting. These modules are extensively studied by many authors until now. For the properties and characterizations of lifting modules, we refer to [3].

In this section, we introduce the concept of radical lifting modules as a generalization of lifting modules. We obtain the basic properties of such modules.

Definition 3.1. *Let M be a module. A module M is called radical lifting if, for every submodule N of M with $\operatorname{Rad}(M) \subseteq N$, there exists the decomposition $M = N' \oplus K$ such that $N' \subseteq N$ and $N \cap K \ll K$.*

It is easy to see that as a equivalent condition of this definition; a module M is radical lifting if and only if, for every submodule N of M with $\operatorname{Rad}(M) \subseteq N$, N contains a direct summand K of M such that $\dfrac{N}{K} \ll \dfrac{M}{K}$.

Lemma 3.1. *Let M be a radical lifting module. Then it is a $srs^{\oplus}$-module.*

Proof. Let N be any submodule of M with $\operatorname{Rad}(M) \subseteq N$. Since M is radical lifting, there are a direct summand N' of M and a submodule $K \leq M$ with $N' \leq N$, $M = N' \oplus K$ and $N \cap K \ll K$. Therefore M is a $srs^{\oplus}$-module.

Let M be a nonzero module. M is called *indecomposable* if the only direct summands of M are 0 and M.

***Corollary* 3.1.** *Let M be an indecomposable module which is not radical. Then the following statements are equivalent*:

(1) *M is radical lifting*,

(2) *M is lifting*,

(3) *M is local.*

Proof. (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear.

(1) $\Longrightarrow$ (3). It follows from Lemma 3.1 and [8] (Lemma 2.3).

Lemma 3.2. *Let M be a radical module. Then M is a radical lifting module.*

Proof. Let N be any submodule of M with $\operatorname{Rad}(M) \subseteq N$. Since $\operatorname{Rad}(M) = M$, then $N = M$. So we have $M = M \oplus 0$, $M \leq N$ and $N \cap 0 = 0 \ll 0$. Thus, M is a radical lifting module.

The next result is a direct consequence of Lemma 3.2.

***Corollary* 3.2.** *$P(M)$ is a radical lifting module for every R-module M.*

The following fact is crucial.

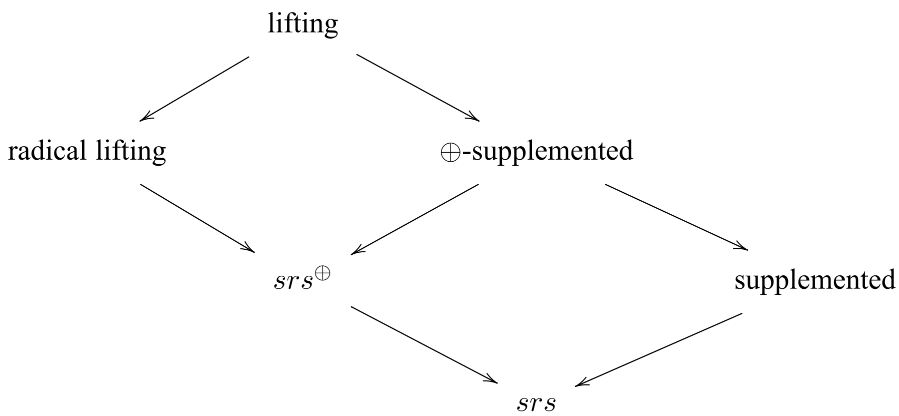
Proposition 3.1. *Let $M = X \oplus L$, where X is a radical module and L is a lifting module. Then M is radical lifting.*

Proof. For any submodule N of M with $\operatorname{Rad}(M) \subseteq N$, we have $X \subseteq N$ because X is radical. Therefore, $M = X \oplus L = N + L$. Since L is lifting, there exists the decomposition $L = L' \oplus V$ such that $L' \subseteq N \cap L$ and $(N \cap L) \cap V = N \cap V \ll V$. It follows that $M = X \oplus L = X \oplus (L' \oplus V) =$ $= (X \oplus L') \oplus V = N + V$. Note that $X \oplus L' \subseteq N$. Clearly, V is a supplement of N in M. It means that M is radical lifting.

Every lifting module is radical lifting but the converse is not always true in the following example shows.

***Example* 3.1.** Let R be a local Dedekind domain with the with quotient field K. Put $M = K^{(I)} \oplus \oplus \left(\dfrac{K}{R}\right)^{(J)} \oplus R$, where I and J are any infinite index sets. Therefore, $P(M) = K^{(I)} \oplus \left(\dfrac{K}{R}\right)^{(J)}$. Since local modules are lifting, R is lifting and so M is radical lifting by Proposition 3.1. However, it is not lifting according to [13] (Theorem 4.3).

In [7], a module M is said to be $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M. Under given definitions, we have the following diagram on modules:



***Corollary* 3.3.** *Finitely generated radical lifting modules are $\oplus$-supplemented.*

Proof. Let M be a finitely generated radical lifting module. Then, it is a $srs^{\oplus}$-module by Lemma 3.1. Since $\mathrm{Rad}(M)$ is a small submodule of M, it follows from [8] (Lemma 2.2) that M is $\oplus$-supplemented.

Theorem 3.1. *Let M be a projective module. Then M is radical lifting if and only if it is lifting.*

Proof. $(\Longrightarrow)$ Let M be a radical lifting module. By Lemma 3.1, M is $srs^{\oplus}$. So M is $\oplus$-supplemented by [8] (Lemma 2.2). Hence, M is lifting.

$(\Longleftarrow)$ It is obvious.

Recall from [3] (27.8) that a supplemented module M is called *strongly discrete* if it is self-projective.

***Corollary* 3.4.** *Let M be a projective module. Then the following statements are equivalent*:

(1) *M is strongly discrete*,

(2) *M is lifting*,

(3) *M is radical lifting*,

(4) *M is radical semiperfect.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear.

(3) $\Longrightarrow$ (4). By Lemma 3.1 and Theorem 2.1.

(4) $\Longrightarrow$ (1). It follows from Theorem 2.1 and [8] (Lemma 2.2).

A module M is said to have the *Summand Sum Property* (SSP) if the sum of two direct summands of M is again a direct summand of M.

Proposition 3.2. *Let M be a radical lifting module with the property (SSP). Suppose that X is a direct summand of M. Then $\dfrac{M}{X}$ is a radical lifting module.*

Proof. Let $\dfrac{N}{X}$ be any submodule of $\dfrac{M}{X}$ with $\operatorname{Rad}\left(\dfrac{M}{X}\right) \subseteq \dfrac{N}{X}$. Since $\dfrac{\operatorname{Rad}(M)+X}{X} \subseteq$ $\subseteq \operatorname{Rad}\left(\dfrac{M}{X}\right)$, then $\operatorname{Rad}(M) \subseteq N$. By the assumption, there exist submodules N', K of M such that $M = N' \oplus K$, $N' \leq N$ and $N \cap K \ll K$. Thus, $\dfrac{K+X}{X}$ is a supplement of $\dfrac{N}{X}$ in $\dfrac{M}{X}$ by [10] (41.1 (7)). Since M has the property (SSP), there exists a submodule Y of M such that $M = (N'+X) \oplus Y$. So we can write $\dfrac{M}{X} = \left(\dfrac{N'+X}{X}\right) \oplus \left(\dfrac{Y+X}{X}\right)$. It is clear that $\dfrac{N'+X}{X} \leq \dfrac{N}{X}$. Furthermore, $\dfrac{N}{X} \cap \dfrac{Y+X}{X} \ll \dfrac{Y+X}{X}$. Thus, the factor module $\dfrac{M}{X}$ is radical lifting.

Recall from [10] (6.4) that a submodule X of an R-module M is called *fully invariant* if $f(X)$ is contained in X for every R-endomorphism f of M. Let M be an R-module and τ be a preradical for the category of R-modules. Then, $\tau(M)$ is a fully invariant submodule of M. M is called a *duo module* if every submodule of M is fully invariant [9].

Proposition 3.3. *Let M be a radical lifting module. If X is a fully invariant submodule of M, then $\dfrac{M}{X}$ is a radical lifting module.*

Proof. Let $\dfrac{N}{X}$ be any submodule of $\dfrac{M}{X}$ with $\operatorname{Rad}\left(\dfrac{M}{X}\right) \subseteq \dfrac{N}{X}$. Since $\dfrac{\operatorname{Rad}(M)+X}{X} \subseteq$ $\subseteq \operatorname{Rad}\left(\dfrac{M}{X}\right)$, then $\operatorname{Rad}(M) \subseteq N$. By the hypothesis, there exists a direct summand N' of M such that $N' \leq N$, $M = N' \oplus K$ for a submodule $K \leq M$, and $N \cap K \ll K$. So we have $\dfrac{M}{X} = \left(\dfrac{N'+X}{X}\right) \oplus \left(\dfrac{K+X}{X}\right)$, $\dfrac{N'+X}{X} \leq \dfrac{N}{X}$ and $\dfrac{N}{X} \cap \dfrac{K+X}{X} \ll \dfrac{K+X}{X}$. Therefore the factor module $\dfrac{M}{X}$ is radical lifting.

***Corollary* 3.5.** *The following statements are equivalent for a duo module M:*

(1) *M is radical lifting*,

(2) *every direct summand of M is radical lifting*.

Let M be a module. M is called *reduced* if $P(M) = 0$. Note that $\dfrac{M}{P(M)}$ is reduced for every left R-module M.

Lemma 3.3. *Reduced radical lifting modules have a small radical.*

Proof. Let M be a reduced radical lifting module. Then, there exist submodule X and N of M such that $M = X \oplus N$, $\operatorname{Rad}(M) \cap N = \operatorname{Rad}(N) \ll N$ and $X \subseteq \operatorname{Rad}(M)$. It follows that $\operatorname{Rad}(M) = \operatorname{Rad}(X) \oplus \operatorname{Rad}(N) = X \oplus \operatorname{Rad}(N)$. So $X = \operatorname{Rad}(X)$, that is, X is radical. By the assumption, we get $X \subseteq P(M) = 0$. It means that $\operatorname{Rad}(M) \ll M$.

Recall that a module M is *coatomic* if every proper submodule of M is contained in a maximal submodule of M [12].

***Corollary* 3.6.** *Let M be a radical lifting module. Then $\dfrac{M}{P(M)}$ is coatomic and $\oplus$-supplemented.*

Proof. Since $P(M)$ is a fully invariant submodule of M, it follows from Proposition 3.3 that $\dfrac{M}{P(M)}$ is radical lifting. Applying Lemma 3.1, we obtain that $\dfrac{M}{P(M)}$ is $srs^{\oplus}$. By Lemma 3.3, $\dfrac{M}{P(M)}$ has a small radical. Thus, it is $\oplus$-supplemented by [8] (Lemma 2.2).

It follows from [10] (42.1) that $\dfrac{M}{P(M)}$ is coatomic.

Proposition 3.4. *Let M be a radical lifting module and X be a submodule with* $\mathrm{Rad}(M) \subseteq X$. *If X is a direct summand of M, then X is a radical lifting module.*

Proof. Let N be any submodule of X such that $\mathrm{Rad}(X) \subseteq N$. By [8] (Lemma 2.28), we obtain that $\mathrm{Rad}(M) = \mathrm{Rad}(X)$. Since M is a radical lifting module, there exists a direct summand N' of M such that $N' \leq N$, $M = N' \oplus K$ for a submodule $K \leq M$, and $N \cap K \ll K$. Then $X = N + (X \cap K)$. Since $M = N' \oplus K$, then $N = N' \oplus (X \cap K)$. It follows that $N \cap (X \cap K) \ll N \cap K$. Therefore N is a radical lifting module.

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