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JACOBI OPERATORS AND ORTHONORMAL MATRIX-VALUED POLYNOMIALS. I

ОПЕРАТОРИ ЯКОБІ ТА ОРТОГОНАЛЬНІ ОПЕРАТОРНОЗНАЧНІ ПОЛІНОМИ. I

It is shown that every self-adjoint operator in a separable Hilbert space is unitarily equivalent to a block Jacobi operator. A system of orthogonal operator-valued polynomials is constructed.

Показано, що будь-який самоспряжений оператор, заданий у сепарабельному гільбертовому просторі, унітарно еквівалентний блочному оператору Якобі. Побудовано систему ортогональних операторнозначних поліномів.

Introduction. Jacobi matrix is the canonical form of a self-adjoint operator with simple spectrum [1], spectral analysis of this matrix is tightly bound with the study of orthogonal polynomials [2, 3]. This realm of analysis has deep connections with moment problem; interpolation problems; issues of the extension of symmetrical operators, etc. [2, 3].

This work develops studies in this direction. At the beginning (Section 1), it is shown that every self-adjoint operator acting in a separable Hilbert space is realized by the block Jacobi operator (is unitarily equivalent to it), besides, sizes of the blocks correspond to the multiplicity of the spectrum of the initial operator. Section 2 is devoted to the construction of the system of orthogonal matrix-valued polynomials. These problems (in the matrix case) are studied in the works [4 – 8]. Important results in spectral analysis obtained in [4, 5] found their fruitful application in the problem of moments. Generalization of the scalar case [2] on the matrix-valued case is studied in [7, 8] and is represented in the overall survey [6]. Establishment of links between block Jacobi matrices and theory of nonself-adjoint operators with analytical analogues of the L. de Branges spaces of entire functions is the aim of the present paper. The polynomials of the first and the second kind are constructed in Section 2 using an introduced notion of the measure nondegenerateness (see an analogue in [6]), and then the operator-valued function with J-properties is constructed and its multiplicative expansion is obtained using the methods of J-theory of V. P. Potapov [9, 10].

Constructions stated in this paper refer to the so called ”truncated” problem ($n \in \mathbb{N}$), i.e., the finite block Jacobi matrix.

1. Block Jacobi operators. I. Consider the spectral resolution [1]

$$A = \int_{\mathbb{R}} \lambda dE_\lambda \tag{1}$$

of a linear self-adjoint operator A given in a separable Hilbert space H. Let us select some vector f_1 from a dense in H set (in view of the separation property) and construct the subspace

$$L(f_1) \stackrel{\text{df}}{=} \operatorname{span}\{E_\Delta f_1 : \Delta \in \mathbb{R}\}, \tag{2}$$

where Δ runs over the totality of all the intervals of the axis $\mathbb{R}$. Obviously, $L_1(f)$ and its orthogonal complement $H_1 = H \ominus L(f_1)$ are E_t-invariant ($t \in \mathbb{R}$). Since H_1 is also separable, then, selecting f_2 from a countable dense set in H_1, we define the subspace $L(f_2)$ (2) in H_1. Repeating this procedure of removal $L(f_k)$ countable number of times, we obtain

$$H = \sum_{k=1}^{\infty} \oplus L(f_k). \tag{3}$$

Note that this procedure can terminate after finite number of steps and even on the first step. The latter leads to the representation by the classical Jacobi matrix with scalar entries. Define **the generating subspace** [1] G for the operator A,

$$G \overset{\mathrm{df}}{=} \operatorname{span}\{f_k : k \in \mathbb{N}\}, \tag{4}$$

then (3) implies

$$H = \operatorname{span}\{E_\Delta g : g \in G; \Delta \in \mathbb{R}\}, \tag{5}$$

where, as usual, Δ belongs to the set of all intervals from $\mathbb{R}$. Consider some Hilbert space E ($\dim E \geq \dim G$), and let ψ be a linear bounded operator from E onto G. Define a nondecreasing operator-function in E

$$F(x) \overset{\mathrm{df}}{=} \psi^* E_x \psi, \quad x \in \mathbb{R}. \tag{6}$$

As a ψ we can take, for example, the orthoprojector P_G on G and suppose that $E = G$.

Denote by $L^2_{\mathbb{R}}\big(E, dF(x)\big)$ the Hilbert space of E-valued vector-functions on $\mathbb{R}$,

$$L^2_{\mathbb{R}}\big(E, dF(x)\big) \overset{\mathrm{df}}{=} \left\{ f(x) : \int_{\mathbb{R}} \big\langle dF(x) f(x), f(x) \big\rangle_E < \infty \right\} \tag{7}$$

which is generated as a result of the closure of linear span of finite continuous functions $f(x)$ and subsequent factorization by the kernel of metrics (7). This definition is correct (see [3, 11]). Specify the linear operator U,

$$U : L^2_{\mathbb{R}}\big(E, dF(x)\big) \to H, \qquad f = Uf(x), \quad f \overset{\mathrm{df}}{=} \int_{\mathbb{R}} dE_x \psi f(x). \tag{8}$$

Image of the operator U is dense in H since vectors

$$\int_{\mathbb{R}} dE_x \psi \chi_\Delta(x) = E_\Delta \psi f, \qquad f \in E, \quad \Delta \in \mathbb{R},$$

linear span of which is dense in H, belong to it, in view of (5) ($\psi f = g \in G$).

If $g(x) \in L^2_{\mathbb{R}}\big(E, dF(x)\big)$, and f is given by (8), then

$$\big\langle E_t f, \psi g(x) \big\rangle = \int_{-\infty}^{t} \big\langle dF(s) f(s), g(x) \big\rangle. \tag{9}$$

Let $f(x)$ from (8) be differentiable and $f'(x) \in L^2_{\mathbb{R}}\big(E, dF(x)\big)$, then

$$\|f\|^2 = \int\limits_{\mathbb{R}} \big\langle f, dE_x\psi f(x)\big\rangle = \int\limits_{\mathbb{R}} d\big\langle f, E_x\psi f(x)\big\rangle - \int\limits_{\mathbb{R}} \big\langle E_x f, \psi f'(x)\big\rangle dx =$$

$$= \int\limits_{\mathbb{R}} d \int\limits_{-\infty}^{x} \langle dF(t)f(t), f(x)\rangle - \int\limits_{\mathbb{R}} \int\limits_{-\infty}^{x} \big\langle dF(t)f(t), f'(x)\big\rangle dx = \int\limits_{\mathbb{R}} \big\langle dF(x)f(x), f(x)\big\rangle,$$

in view of (9). So, U (8) is isometrical on the dense set in $L^2_{\mathbb{R}}\big(E, dF(x)\big)$ and thus the operator U is unitary.

Let f be given by (8) where $f(x)$ belongs to the linear span of continuous functions in $L^2_{\mathbb{R}}\big(E, dF(x)\big)$. Then for all $h \in H$

$$\langle Af, h\rangle = \int\limits_{\mathbb{R}} x\big\langle f, dE_x h\big\rangle = \int\limits_{\mathbb{R}} x \int\limits_{\mathbb{R}} \big\langle dE_s\psi f(s), dE_x h\big\rangle =$$

$$= \int\limits_{\mathbb{R}} x d_x \int\limits_{\mathbb{R}} \big\langle dE_s\psi f(s), E_x h\big\rangle = \int\limits_{\mathbb{R}} x d \int\limits_{-\infty}^{x} \big\langle dE_s\psi f(s), h\big\rangle = \int\limits_{E} \big\langle dE_x\psi x f(x), h\big\rangle,$$

consequently,

$$Af = \int\limits_{\mathbb{R}} dE_x\psi(xf(x)), \tag{10}$$

and thus $AU = UQ$, where Q is the operator of multiplication by the independent variable in $L^2_{\mathbb{R}}\big(E, dF(x)\big)$,

$$(Qf)(x) \stackrel{\text{df}}{=} xf(x) \quad \big(f(x) \in L^2_{\mathbb{R}}\big(E, dF(x)\big)\big). \tag{11}$$

Isometricity of U (8) and (10) implies

$$\|Af\|^2 = \int\limits_{\mathbb{R}} x^2\big\langle dF(x)f(x), f(x)\big\rangle = \|Qf(x)\|^2.$$

So $f(x)$ belongs to the domain $\mathfrak{D}_Q$ of the operator Q (11) then and only then when f (8) belongs to the domain $\mathfrak{D}_A$ of the operator A.

Theorem 1. *An arbitrary self-adjoint operator A acting in a Hilbert space H is unitarily equivalent to the operator Q* (11) *in $L^2_{\mathbb{R}}\big(E, dF(x)\big)$* (7), *$AU = UQ$, where U is given by* (8); *E_x is the resolution of identity of the operator A*; *$F(x)$ is given by formula* (6); *and ψ is a linear bounded operator from E on the generating subspace G* (4).

II. Let A be bounded self-adjoint operator, then

$$A^n\psi g = \int\limits_{\mathbb{R}} x^n dE_x\psi g$$

make sense for all $g \in G$ and all $n \in \mathbb{Z}_+$. Show that the linear span of these vectors is dense in H. If a vector $f \in H$ is such that $f \perp A^n\psi g$ (for all $g \in G$ and all $n \in \mathbb{Z}_+$), then, using representation

(8) for f, we obtain

$$0 = \langle A^n \psi g, f \rangle = \int\limits_{\mathbb{R}} \big\langle dF(x) x^n g, f(x) \big\rangle.$$

Therefore $f(x) \perp P_n(x)$, where $P_n(x)$ is an arbitrary E-valued polynomial of the degree n. Since the set of such polynomials is dense in $L^2_{\mathbb{R}}\big(E, dF(x)\big)$ ($dF(x)$ has the dense support), then $f(x) = 0$, and so $f = 0$.

Theorem 2. *For every bounded self-adjoint operator A acting in a separable Hilbert space H,*

$$H = \operatorname{span}\big\{A^n \psi g \colon g \in E, n \in \mathbb{Z}_+\big\}. \tag{12}$$

In the case of unboundedness of A see [1, 2].

Define the subspaces

$$H_n \overset{\text{df}}{=} \operatorname{span}\big\{A^k \psi g \colon g \in E; 0 \le k \le n\big\}, \quad n \in \mathbb{Z}_+, \tag{13}$$

which are ordered by inclusion, $H_k \subseteq H_s$ as $s > k$, and let

$$G_n \overset{\text{df}}{=} H_n \ominus H_{n-1}, \quad n \in \mathbb{Z}_+, \tag{14}$$

where $H_{-1} = \{0\}$ and $G_0 = G$ (4). Then

$$H = \sum_{k=0}^{\infty} \oplus G_k. \tag{15}$$

If $g_k \in G_k$, $k \in \mathbb{Z}_+$, then $\langle A g_k, g_s \rangle = 0$ as $s > k+1$ since $A g_k \in H_{k+1}$; and, similarly, $\langle A g_k, g_s \rangle = \langle g_k, A g_s \rangle$ as $k > s+1$ ($A g_s \in H_{s+1}$). So,

$$\langle A g_k, g_s \rangle = 0 \quad \text{as} \quad s > k+1 \quad \text{and} \quad s < k-1, \quad k, s \in \mathbb{Z}_+,$$

therefore the operator A has the three-diagonal block structure corresponding to expansion (15),

$$A = \begin{bmatrix} \widetilde{A}_0 & \widetilde{B}_0 & 0 & 0 & \cdots \\ \widetilde{B}_0^* & \widetilde{A}_1 & \widetilde{B}_1 & 0 & \cdots \\ 0 & \widetilde{B}_1^* & \widetilde{A}_2 & \widetilde{B}_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{16}$$

where $\widetilde{A}_k = P_k A P_k : G_k \to G_k$, $\widetilde{B}_k = P_k A P_{k+1} \colon G_{k+1} \to G_k$ (P_k is the orthoprojector on G_k (14), $k \in \mathbb{Z}_+$). The definition of G_k (14) yields $\dim G_k \le \dim G$, $k \in \mathbb{Z}_+$. For $\dim G_k = \dim G$ we specify unitary operators $V_k \colon G_k \to G$. If $\dim G_k < \dim G$, we can define isometric operators $V_k : G_k \to G$. Consider the set of operators in G: $A_k = V_k \widetilde{A}_k V_k^*$, $B_k = V_k \widetilde{B}_k V_{k+1}^*$, $k \in \mathbb{Z}_+$, and define the **block Jacobi operator**

$$J_G \overset{\text{df}}{=} \begin{bmatrix} A_0 & B_0 & 0 & 0 & \cdots \\ B_0^* & A_1 & B_1 & 0 & \cdots \\ 0 & B_1^* & A_2 & B_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{17}$$

in the Hilbert space $l^2_{\mathbb{Z}_+}(G)$. Let $V = \operatorname{diag}[I, V_1, \ldots]$ be an isometric (by construction) operator from H (15) into $l^2_{\mathbb{Z}_+}(G)$, then $VA = J_G V$.

Theorem 3. *An arbitrary bounded self-adjoint operator acting in a separable Hilbert space H is isometrically equivalent to the Jacobi operator J_G (17) in the space $l^2_{\mathbb{Z}_+}(G)$.*

2. Matrix-valued orthogonal polynomials. III. Spectral analysis of Jacobi matrices $(\dim G = 1)$ is closely linked with properties of orthogonal polynomials [1 – 6]. We proceed to the construction of matrix-valued orthogonal polynomials.

Definition 1. *A measure $dF(x)$ is said to satisfy the nd-condition (non degenerata) if for every E-valued polynomial of the finite degree $P_n(x) = \sum_{k=0}^{n} x^k g_k$, $g_k \in E$, $1 \le k \le n$, $n \in \mathbb{Z}_+$, the estimation*

$$\int\limits_{\mathbb{R}} \big\langle dF(x)P_n(x), P_n(x)\big\rangle > \delta_n \sum_{k=0}^{n} \|g_k\|^2, \tag{18}$$

is true, besides, the number δ_n does not depend on the vectors $\{g_k\}_1^n$, and $\delta_n > 0$ for all $n \in \mathbb{Z}_+$.

Note that nd-condition is per se equivalent to the nondegenerateness of nontrivial measure in [6].

Theorem 4. *If a self-adjoint bounded operator A acting in a separable Hilbert space H is such that the measure $dF(x)$ satisfies the nd-condition, where $F(x)$ is given by (6) and $\dim E = r < \infty$, then the vector $A^n\psi g$ does not belong to the space H_{n-1} (13) for any $g \in E$ and $n \in \mathbb{N}$.*

Proof. Assuming the contrary, we suppose that there is such a vector $g \in E$ that $A^n\psi g \in H_{n-1}$ for some $n \in \mathbb{N}$, then

$$A^n\psi g + \sum_{k=1}^{n-1} A^k \psi g_k = 0, \qquad g_k \in E, \quad 0 \le k \le n-1.$$

This implies

$$\int\limits_{\mathbb{R}} dE_t \psi P_n(t) = 0,$$

where $P_n(t) = t^n g + \sum_{k=0}^{n-1} t^k g_k$ is a E-valued polynomial. Applying E_x to this equality, we obtain

$$\int\limits_{-\infty}^{x} dE_t \psi P_n(t) = 0 \quad \forall x \in \mathbb{R},$$

and thus

$$\int\limits_{-\infty}^{x} \big\langle dE_t\psi P_n(t), \psi f\big\rangle = \int\limits_{-\infty}^{x} \big\langle dF(t)P_n(t), f\big\rangle = 0 \quad \forall f \in E.$$

Consequently,

$$0 = \int\limits_{\mathbb{R}} \overline{\varphi}(x) d\int\limits_{0}^{x} \big\langle dF(t)P_n(t), f\big\rangle = \int\limits_{\mathbb{R}} \big\langle dF(x)P_n(x), \varphi(x)f\big\rangle,$$

for an arbitrary scalar function $\varphi(x)\colon \mathbb{R} \to \mathbb{C}$, therefore

$$\int\limits_{\mathbb{R}} \big\langle dF(x)P_n(x), f(x)\big\rangle = 0,$$

where $f(x)$ is any function of the form $f(x) = \sum_{k=0}^{m} \varphi_k(x)f_k,\ m \in \mathbb{Z}_+$. Assuming that $f(x) = = P_n(x)$, we obtain

$$\int\limits_{\mathbb{R}} \big\langle dF(x)P_n(x), P_n(x)\big\rangle = 0,$$

which is contrary to the nd-condition.

Theorem 4 is proved.

This theorem yields that the vector

$$f = A^n\psi g - P_{H_{n-1}}A^n\psi g$$

is nonzero for all $g \in E$ and all $n \in \mathbb{N}$ $\big(P_{H_{n-1}}$ is the orthoprojector on H_{n-1} (13)$\big)$. It is obvious that $P_{H_{n-1}}f = 0$, consequently, $f \perp H_s$ $(\forall s,\ 0 \le s \le n-1)$. Since

$$f = A^n\psi g + \sum_{k=0}^{n-1} A^k\psi g_k, \tag{19}$$

then to each $g \in E$ $(\dim E = r < \infty)$ there corresponds the set of vectors $\{g_k\}_0^{n-1}$ from E. Formula (19) follows from (13) when $\dim E < \infty$. This correspondence defines the linear operators $N_k g \overset{\text{df}}{=} g_k,\ 0 \le k \le n-1$. Since $\dim E < \infty$, N_k are bounded for all k.

Write the vector f (19) in the form

$$f = \int\limits_{\mathbb{R}} dE_t\psi\widetilde{P}_n(t)g,$$

where $\widetilde{P}_n(t) = t^n + t^{n-1}N_{n-1} + \ldots + N_0$. Orthogonality $f \perp H_s,\ 0 \le s \le n-1$ signifies that

$$\int\limits_{\mathbb{R}} \widetilde{P}_s^*(t)dF(t)\widetilde{P}_n(t) = 0, \quad 0 \le s \le n-1.$$

The operator

$$D_n = \int\limits_{\mathbb{R}} \widetilde{P}_n(t)dF(t)\widetilde{P}_n(t)$$

is nonnegative and invertible since the nd-condition (18) implies $\left\| D_n^{\frac{1}{2}} g \right\|^2 > \delta_n \|g\|^2$. Therefore the polynomial $P_n(t) = \widetilde{P}_n(t) D_n^{-\frac{1}{2}}$ is normalized to unity,

$$\int_{\mathbb{R}} P_n^*(t) dF(t) P_n(t) = I_E.$$

Theorem 5. *If a bounded self-adjoint operator A is such that the measure $dF(x)$ has the nd-property* (18) $(F(x)$ *is given by* (6) *and E_x is the resolution of identity of the operator A and* $\dim E = r < \infty)$, *then there exists the family of matrix-valued in E polynomials $\{P_n(x)\}_0^\infty$, such that*

$$\int_{\mathbb{R}} P_k^*(x) dF(x) P_n(x) = \delta_{k,n} I_E, \quad k, n \in \mathbb{Z}_+, \tag{20}$$

besides, $\deg P_n(x) = n$ *and the leading coefficient of $P_n(x)$ is invertible* $(\forall n \in \mathbb{Z}_+)$.

The expansion [2 – 6]

$$x P_n(x) = P_{n+1}(x) B_n^{(n+1)} + P_n(x) B_n^{(n)} + \ldots + P_0(x) B_n^{(0)}$$

$\big(B_n^{(s)}$ are linear bounded operators in E, $0 \le s \le n+1\big)$ and (20) imply that $B_n^{(s)} = 0$ for $0 \le s \le n-2$, besides,

$$B_n^{(n+1)} = \int_{\mathbb{R}} x P_{n+1}^*(x) dF(x) P_n(x), \qquad B_n^{(n)} = \int_{\mathbb{R}} x P_n^*(x) dF(x) P_n(x),$$

$$B_n^{(n-1)} = \int_{\mathbb{R}} x P_{n-1}^*(x) dF(x) P_n(x);$$

and thus $B_n^{(n)} = \big(B_n^{(n)}\big)^*$, $B_n^{(n-1)} = \big(B_{n-1}^{(n)}\big)^*$. So the totality $\big\{P_n(x)\big\}_1^\infty$ is the solution of the finite-difference equation

$$x P_n(x) = P_{n+1}(x) B_n + P_n(x) C_n + P_{n-1}(x) B_{n-1}^*, \quad n \in \mathbb{Z}_+, \tag{21}$$

where $P_{-1}(x) \stackrel{\text{df}}{=} 0$, $B_n = B_n^{(n+1)}$, $C_n = B_n^{(n)}$, $n \in \mathbb{Z}_+$. Invertibility of the leading coefficients of the polynomials $P_n(x)$ implies invertibility of all operators B_n. Therefore **polynomials of the first kind** $P_n(x)$ are found as the solutions of (21) unambiguously if we take into account the initial conditions

$$P_0(x) = D_0, \qquad P_1(x) = D_0 (xI - C_0) B_0^{-1}, \tag{22}$$

where $D_0 = \big(F(\infty) - F(-\infty)\big)^{-1/2}$ is an invertible positive operator. Expression

$$Q_n(x) \stackrel{\text{df}}{=} \int_{\mathbb{R}} dF(\xi) \frac{P_n(\xi) - P_n(x)}{\xi - x}, \quad n \in \mathbb{Z}_+, \tag{23}$$

defines [2] operator-valued **polynomials of the second kind** $\deg Q_n(x) = n-1$, besides, $Q_n(x)$ also satisfy the finite-difference equation (21) and the initial data

$$Q_0(x) = 0, \qquad Q_1(x) = D_0^{-1} B_0^{-1}. \tag{24}$$

Construct the Jacobi operator

$$J_E \stackrel{\text{df}}{=} \begin{bmatrix} C_0 & B_0^* & 0 & 0 & \cdots \\ B_0 & C_1 & B_1^* & 0 & \cdots \\ 0 & B_1 & C_2 & B_2^* & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{25}$$

by the coefficients $\{B_n, C_n\}_0^\infty$ from (21), then the recurrence relations (21) formally imply

$$\mathbb{P}(x) J_E = x\mathbb{P}(x), \qquad \mathbb{Q}(x) J_E = x\mathbb{Q}(x), \tag{26}$$

where $\mathbb{P}(x) = \big[P_0(x), P_1(x), \ldots\big]$, $\mathbb{Q}(x) = \big[Q_0(x), Q_1(x), \ldots\big]$.

Let $Y_n = Y_n(\lambda)$ and $Z_n = Z_n(w)$ be the solutions of (21) corresponding to λ and w accordingly ($\lambda,\ w \in \mathbb{C}$). The **Green formula** [2–4, 6]

$$(\lambda - \bar{w}) \sum_{k=m}^{n} Y_k Z_k^* = Y_{n+1} B_n Z_n^* - Y_n B_n^* Z_{n+1}^* - Y_m B_{m-1} Z_{m-1}^* + Y_{m-1} B_{m-1}^* Z_m^* \tag{27}$$

is true for all $n,\ m \in \mathbb{N}$. In particular, for $m = 1$, if $Y_k = P_k(\lambda),\ Z_k = P_k(w)$, then taking into account (22) we obtain the **Christoffel–Darboux formula** [2–4, 6]

$$(\lambda - \bar{w}) \sum_{k=0}^{n} P_k(\lambda) P_k^*(w) = P_{n+1}(\lambda) B_n P_n^*(w) - P_n(\lambda) B_n^* P_{n+1}^*(w) \tag{28}$$

for all $n \in \mathbb{Z}_+$. Assuming in (27) $m = 1,\ Y_k = P_k(\lambda),\ Z_k = Q_k(w)$ and using (22), (24), we obtain the equality

$$(\lambda - \bar{w}) \sum_{k=0}^{n} P_k(\lambda) Q_k^*(w) = P_{n+1}(\lambda) B_n Q_n^*(w) - P_n(\lambda) B_n^* Q_{n+1}^*(w) + I_E \quad \forall n \in \mathbb{Z}_+. \tag{29}$$

Finally, relation

$$(\lambda - \bar{w}) \sum_{k=0}^{n} Q_k(\lambda) Q_k^*(w) = Q_{n+1}(\lambda) B_n Q_n^*(w) - Q_n(\lambda) B_n^* Q_{n+1}^*(w) \tag{30}$$

follows from (27) as $m = 1$ and $Y_k = Q_k(\lambda),\ Z_k = Q_k(w)$.

Lemma 1. *If* $P_n(\lambda)$ *and* $Q_n(\lambda)$ *are the solutions of the finite-difference equation* (21) *which satisfy the conditions* (22), (24), *then the* **Liouville–Ostrogradsky formula** [2–4, 6]

$$\big\{P_n^*(\bar{\lambda}) Q_{n+1}(\lambda) - Q_n^*(\bar{\lambda}) P_{n+1}(\lambda)\big\} B_n = I_E \quad \forall n \in \mathbb{Z}_+ \tag{31}$$

is true, *besides*,

$$P_n^*(\bar{\lambda}) Q_n(\lambda) - Q_n^*(\bar{\lambda}) P_n(\lambda) = 0 \quad \forall n \in \mathbb{Z}_+. \tag{32}$$

Proof. We prove both equalities (31), (32) simultaneously, using induction by n. For $n = 0$, the truth of (31), (32) follows from the initial data (22), (24). Let the statement be proved for all k $(0 \le k \le n)$. Show that this implies (31), (32) for $k = n + 1$. Using (21) and invertibility of B_n, we obtain

$$B_n^*\big[P_{n+1}^*(\bar\lambda)Q_{n+1}(\lambda) - Q_{n+1}^*(\bar\lambda)P_{n+1}(\lambda)\big]B_n =$$
$$= \big[(\lambda - C_n)P_n^*(\bar\lambda) - B_{n-1}P_{n-1}^*(\bar\lambda)\big]\big[Q_n(\lambda)(\lambda - C_n) - Q_{n-1}(\lambda)B_{n-1}^*\big] -$$
$$- \big[(\lambda - C_n)Q_n^*(\bar\lambda) - B_{n-1}Q_{n-1}^*(\bar\lambda)\big]\big[P_n(\lambda)(\lambda - C_n) - P_{n-1}(\lambda)B_{n-1}^*\big] =$$
$$= (\lambda - C_n)\big[P_n^*(\bar\lambda)Q_{n-1}(\lambda) - Q_n^*(\bar\lambda)P_{n-1}(\lambda)\big]B_{n-1}^* -$$
$$- B_{n-1}\big[P_{n-1}^*(\bar\lambda)Q_n(\lambda) - Q_{n-1}^*(\bar\lambda)P_n(\lambda)\big](\lambda - C_n) = 0,$$

in view of the supposition of induction. Similarly,

$$\big[P_{n+1}^*(\bar\lambda)Q_{n+2}(\lambda) - Q_{n+1}^*(\bar\lambda)P_{n+2}(\lambda)\big]B_{n+1} =$$
$$= P_{n+1}^*(\bar\lambda)\big[Q_{n+1}(\lambda)(\lambda - C_{n+1}) - Q_n(\lambda)B_n\big] - Q_{n+1}^*(\bar\lambda)\big[P_{n+1}(\lambda)(\lambda - C_{n+1}) - P_n(\lambda)B_n\big] =$$
$$= \big\{Q_{n+1}^*(\bar\lambda)P_n(\lambda) - P_{n+1}^*(\bar\lambda)Q_n(\lambda)\big\}B_n^* = I,$$

which was to be proved.

IV. Using the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$, we construct the operator-function

$$W_n(\lambda) \stackrel{\text{df}}{=} \begin{bmatrix} P_n(\lambda) & P_{n+1}(\lambda)B_n \\ Q_n(\lambda) & Q_{n+1}(\lambda)B_n \end{bmatrix}, \tag{33}$$

besides, $\deg W_n(\lambda) = n$, and define the involution J in $E \oplus E$,

$$J \stackrel{\text{df}}{=} \begin{bmatrix} 0 & iI_E \\ -iI_E & 0 \end{bmatrix}. \tag{34}$$

From formulas (28) – (30) follows that

$$W_n(\lambda)JW_n^*(w) - J = \frac{\lambda - \bar w}{i}\sum_{n=0}^{n} \begin{bmatrix} P_k(\lambda) & 0 \\ Q_k(\lambda) & 0 \end{bmatrix} \begin{bmatrix} P_k^*(w) & Q_k^*(w) \\ 0 & 0 \end{bmatrix}, \tag{35}$$

and so $W_n(\lambda)$ (33) has J-properties [6, 7],

$$W_n(\lambda)JW_n^*(\lambda) - J = \begin{cases} \ge 0, & \lambda \in \mathbb{C}_+, \\ = 0, & \lambda \in \mathbb{R}, \\ \le 0, & \lambda \in \mathbb{C}_-. \end{cases} \tag{36}$$

Equation (21) yields

$$W_n(\lambda) = W_{n-1}(\lambda)b_n(\lambda), \tag{37}$$

where

$$b_n(\lambda) = \begin{bmatrix} 0 & B_{n-1}^{*-1} \\ B_{n-1}^{-1} & B_n^{-1}(\lambda I - C_n) \end{bmatrix}, \quad n \in \mathbb{Z}_+, \tag{38}$$

reckoning that $B_{-1} \overset{\text{df}}{=} I$. Therefore

$$W_n(\lambda) = U_0 \prod_{k=1}^{\overset{\frown}{n}} b_k(\lambda), \tag{39}$$

where U_0 is J-unitary,

$$U_0 \overset{\text{df}}{=} \begin{bmatrix} 0 & P_0(\lambda) \\ -P_0^{-1}(\lambda) & 0 \end{bmatrix},$$

and so we can regard that $W_{-1}(\lambda) = U_0$. “Primary” factors $b_k(\lambda)$ (38) also have J-properties (36), since

$$b_k(\lambda) J b_k^*(w) - J = \frac{\lambda - \bar{w}}{i} \begin{bmatrix} 0 & 0 \\ 0 & B_{k-1}^{-1}\left(B_{k-1}^{-1}\right)^* \end{bmatrix},$$

$$b_k^*(w) J b_k(\lambda) - J = \frac{\lambda - \bar{w}}{i} \begin{bmatrix} 0 & 0 \\ 0 & I_E \end{bmatrix}$$

for all $k \in \mathbb{Z}_+$. Thus factorization (39) is realized in the context of the class of operator-functions satisfying the relations (36), besides, the factors $b_k(\lambda)$ (38) are constructed by the elements of the Jacobi matrix J_E (25). Factorization, similar to (38), is obtained in [9] in somewhat different form.

(35) implies $W_n(\lambda) J W_n^*\left(\bar{\lambda}\right) J = I_{E \oplus E}$, therefore the operator

$$W_n^{-1}(\lambda) \overset{\text{df}}{=} J W_n^*\left(\bar{\lambda}\right) J = \begin{bmatrix} B_n^* Q_{n+1}^*\left(\bar{\lambda}\right) & -B_n^* P_{n+1}^*\left(\bar{\lambda}\right) \\ -Q_n^*\left(\bar{\lambda}\right) & P_n\left(\bar{\lambda}\right) \end{bmatrix}, \quad n \in \mathbb{Z}_+, \tag{40}$$

is the right inverse for $W_n(\lambda)$ (33). The fact that $W_n^{-1}(\lambda)$ (40) is also the left inverse for $W_n(\lambda)$ follows from (31), (32).

Observation 1. *The relations* (28) – (30) *provide* $W_n(\lambda)$ (33) *with the* J*-properties* (36) *and also secure the existence of the right inverse* $W_n^{-1}(\lambda)$ (40), *and the equalities* (31), (32) *are equivalent to* $W_n^{-1}(\lambda) W_n(\lambda) = I$. *So the relations* (28) – (30) *and* (31), (32) *for the polynomials* $P_n(\lambda)$ *and* $Q_n(\lambda)$ *have a natural interpretation in terms of the* J*-properties of the function* $W_n(\lambda)$.

The operator-function

$$S_n(\lambda) \overset{\text{df}}{=} W_n(\lambda) W_n^{-1}(0) \tag{41}$$

also has the J-properties (36) and

$$S_n(\lambda) J S_n^*(w) - J = \frac{\lambda - \bar{w}}{i} \sum_{k=0}^{n} \begin{bmatrix} P_k(\lambda) & 0 \\ Q_k(\lambda) & 0 \end{bmatrix} \begin{bmatrix} P_k^*(w) & Q_k^*(w) \\ 0 & 0 \end{bmatrix}, \tag{42}$$

in virtue of the J-unitarity of $W_n^{-1}(0)$. The function $S_n(\lambda)$ is such that $S_n(0) = I$. It is easy to show that $S_n(\lambda)$ is equal

$$S_n(\lambda)=\begin{bmatrix}A_n(\lambda) & B_n(\lambda)\\ C_n(\lambda) & D_n(\lambda)\end{bmatrix}, \tag{43}$$

where

$$\begin{gathered}
A_n(\lambda)\stackrel{\text{df}}{=}P_n(\lambda)B_n^*Q_{n+1}^*(0)-P_{n+1}(\lambda)B_nQ_n^*(0)=I-\lambda\sum_{k=0}^{n}P_k(\lambda)Q_k^*(0),\\
B_n(\lambda)\stackrel{\text{df}}{=}P_{n+1}(\lambda)B_nP_n^*(0)-P_n(\lambda)B_n^*P_{n+1}^*(0)=\lambda\sum_{k=0}^{n}P_k(\lambda)P_k^*(0),\\
C_n(\lambda)\stackrel{\text{df}}{=}Q_n(\lambda)B_n^*Q_{n+1}^*(0)-Q_{n+1}(\lambda)B_nQ_n^*(0)=-\lambda\sum_{k=0}^{n}Q_k(\lambda)Q_k^*(0),\\
D_n(\lambda)\stackrel{\text{df}}{=}Q_{n+1}B_nP_n^*(0)-Q_n(\lambda)B_n^*P_{n+1}^*(0)=1+\lambda\sum_{k=0}^{n}Q_k(\lambda)P_k^*(0),
\end{gathered} \tag{44}$$

in virtue of the form of $W_n(\lambda)$ (33) and $W_n^{-1}(0)$ (40), and also of (28) – (32). The functions (44) are similar to the well-known scalar [2, 3] and matrix [7] functions, besides, $\deg A_n(\lambda)=\deg B_n(\lambda)=$ $=n+1,\ \deg C_n(\lambda)=\deg D_n(\lambda)=n.$

Observation 2. *The functions* (44), *in spite of the properties following from* (42), *satisfy the equalities*

$$\begin{gathered}
A_n^*(\bar\lambda)D_n(\lambda)-C_n^*(\bar\lambda)B_n(\lambda)=I_E,\qquad D_n^*(\bar\lambda)B_n(\lambda)=B_n^*(\bar\lambda)D_n(\lambda),\\
A_n^*(\bar\lambda)C_n(\lambda)=C_n^*(\bar\lambda)A_n(\lambda),\quad n\in\mathbb{Z}_+,
\end{gathered} \tag{45}$$

which are a corollary of the fact that $JS_n^*(\bar\lambda)J$ is the left inverse for $S_n(\lambda)$. The relations (45) *can be proved directly using* (31), (32).

Observation 3. *The normalization E_t at zero for $S_n(\lambda)$* (41) *is not binding. If we consider $S_n(\lambda,\lambda_0)\stackrel{\text{df}}{=}W_n(\lambda)W_n^{-1}(\lambda_0),\ \lambda_0\in\mathbb{R}$, then*: 1) $S_n(\lambda_0,\lambda_0)=I_{E\oplus E}$; 2) (42) *take place in virtue of the J-unitarity of $W_n^{-1}(\lambda_0)$*; 3) *for $S_n(\lambda,\lambda_0)$ representation* (43) *is true with the appropriate version of the formulas* (44).

(37), (41) imply

$$S_n(\lambda)=S_{n-1}(\lambda)a_n(\lambda),\quad n\in\mathbb{Z}_+, \tag{46}$$

where the factor $a_n(\lambda)=W_{n-1}(0)b_n(\lambda)W_n^{-1}(0)$ is equal

$$a_n(\lambda)\stackrel{\text{df}}{=}I-i\lambda m_nJ,\quad n\in\mathbb{Z}_+, \tag{47}$$

besides,

$$m_n\stackrel{\text{df}}{=}\begin{bmatrix}P_n(0)P_n^*(0) & P_n(0)Q_n^*(0)\\ Q_n(0)P_n^*(0) & Q_n(0)Q_n^*(0)\end{bmatrix}\geq 0,\quad n\in\mathbb{Z}_+, \tag{48}$$

and $m_nJm_n=0$, in virtue of (32). (46) implies

$$S_n(\lambda) = \prod_{k=0}^{\overset{\frown}{n}} a_k(\lambda), \tag{49}$$

besides, $S_{-1}(\lambda) = I$ $(W_{-1}(\lambda) = U_0)$. The factors $a_k(\lambda)$ (47) have the J-properties because of

$$a_k(\lambda)Ja_k^*(w) - J = \frac{\lambda - \bar{w}}{i} m_k, \qquad a_k^*(w)Ja_k(\lambda) - J = \frac{\lambda - \bar{w}}{i} Jm_kJ.$$

$m_kJm_k = 0$ yields that $a_k(\lambda)$ (47) have the exponential representation

$$a_k(\lambda) = \exp\{-i\lambda m_k J\}, \quad k \in \mathbb{Z}_+. \tag{50}$$

Theorem 6. *The operator-function* $W_n(\lambda)$ (33) *has the* J*-properties* (36) *and the multiplicative expansion* (39), *where* J *and* $b_k(\lambda)$ *are given by* (34), (38).

The function $S_n(\lambda)$ (41) *is expressed in terms of the functions* (44) *by formula* (43), *besides*, $S_n(\lambda)$ *has the* J*-properties* (36), *and factorization* (49) *takes place*, *where the factors* $a_k(\lambda)$ *are given by* (47), (50).

In the second part of this study the connection of these constructions with L. de Branges spaces and nonself-adjoint operators will be established.

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