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X.-Y. Xiao, H.-W. Yin (Nanchang Univ., China)

EXACT RATES IN THE DAVIS – GUT LAW OF ITERATED LOGARITHM FOR THE FIRST MOMENT CONVERGENCE OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES*

ТОЧНІ ШВИДКОСТІ В ЗАКОНІ ПОВТОРНОГО ЛОГАРИФМА ДЕВІСА – ГУТА ДЛЯ ЗБІЖНОСТІ ПЕРШОГО МОМЕНТУ НЕЗАЛЕЖНИХ ОДНАКОВО РОЗПОДІЛЕНИХ ВИПАДКОВИХ ВЕЛИЧИН

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables and let $S_n = \sum_{i=1}^n X_i$, $M_n = \max_{1\leq k\leq n} |S_k|$. For $r > 0$, let $a_n(\varepsilon)$ be a function of ε such that $a_n(\varepsilon)\log\log n \to \tau$ as $n\to\infty$ and $\varepsilon \searrow \sqrt{r}$. If $\mathbb{E}X^2 I\{|X| \geq t\} = o((\log\log t)^{-1})$ as $t\to\infty$, then, by using the strong approximation, we show that

$$\lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\log\log n}\right\}_+ = \frac{2\sigma e^{-2\tau\sqrt{r}}}{\sqrt{2\pi}r}$$

holds if and only if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$, and $\mathbb{E}X^2(\log|X|)^{r-1}(\log\log|X|)^{-\frac{1}{2}} < \infty$.

Нехай $\{X, X_n, n \geq 1\}$ — множина незалежних однаково розподілених випадкових величин та $S_n = \sum_{i=1}^n X_i$, $M_n = \max_{1\leq k\leq n} |S_k|$. Крім того, для $r > 0$ нехай $a_n(\varepsilon)$ — функція ε така, що $a_n(\varepsilon)\log\log n \to \tau$ при $n\to\infty$ та $\varepsilon \searrow \sqrt{r}$. У випадку $\mathbb{E}X^2 I\{|X| \geq t\} = o((\log\log t)^{-1})$ при $t\to\infty$ за допомогою сильної апроксимації доведено, що співвідношення

$$\lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\log\log n}\right\}_+ = \frac{2\sigma e^{-2\tau\sqrt{r}}}{\sqrt{2\pi}r}$$

виконуються тоді і тільки тоді, коли $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ та $\mathbb{E}X^2(\log|X|)^{r-1}(\log\log|X|)^{-\frac{1}{2}} < \infty$.

1.1. Introduction and main result. Throughout the paper, we suppose that $\{X, X_n, n \geq 1\}$ is a sequence of independent identically distributed (i.i.d.) random variables and let $S_n = \sum_{i=1}^n X_i$, $M_n = \max_{1\leq k\leq n} |S_k|$, for $n \geq 1$. Let $\{W(t); t \geq 0\}$ be a standard Wiener process, and N be the standard normal random variable. We denote by C a positive constant which may vary from line to line, and define $\lfloor x \rfloor = \sup\{m\colon m \leq x, m \in \mathbb{Z}^+\}$. Let $\log x = \ln(x \vee e)$ and $\varphi(x) := \log\log x = \ln(\ln(x \vee e^e))$. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n\to\infty$.

It is well known that Hsu and Robbins [8] and Erdös [4, 5] first introduced the concept of complete convergence. So far many authors have considered various extensions. Li, Wang and Rao [11], Gut and Spătaru [7] showed the precise rates in the law of the iterated logarithm (LIL) that

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n\log n} \mathbb{P}\left(|S_n| \geq \varepsilon\sqrt{n\varphi(n)}\right) = \sigma^2$$

holds if and only if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$. Jiang and Zhang [10] extended it for moment convergence. They proved, for $a_n = O(1/\varphi(n))$ and $b > -1$, that

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$$\lim_{\varepsilon\searrow 0}\varepsilon^{2(b+1)}\sum_{n=1}^{\infty}\frac{\varphi(n)^b}{n\log n}\mathbb{E}\left\{|S_n|-\sigma(\varepsilon+a_n)\sqrt{2n\varphi(n)}\right\}_+ = \frac{\sigma 2^{-b-1}\mathbb{E}|N|^{2b+3}}{(b+1)(2b+3)}$$

holds if and only if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$.

On the other hand, Davis [3] and Gut [6] showed that

$$\sum_{n=3}^{\infty}\frac{1}{n}\mathbb{P}\left(|S_n|\geq\sqrt{(2+\varepsilon)n\varphi(n)}\right)\begin{cases}<\infty, & \text{if} \quad \varepsilon>0,\\ =\infty, & \text{if} \quad -2<\varepsilon<0,\end{cases}$$

holds if and only if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$. Gut and Spătaru [7] obtained the precise rates in the LIL that for $a_n = O(\sqrt{n}(\varphi(n))^{-\gamma})$ with $\gamma > 1/2$,

$$\lim_{\varepsilon\searrow\sigma\sqrt{2}}\sqrt{\varepsilon^2-2\sigma^2}\sum_{n=3}^{\infty}\frac{1}{n}\mathbb{P}\left(|S_n|\geq\varepsilon\sqrt{n\varphi(n)}+a_n\right)=\sigma\sqrt{2}$$

holds as long as $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ and $\mathbb{E}X^2(\varphi(|X|))^{1+\delta} < \infty$ for some $\delta > 0$. They also pointed out that

$$\lim_{\varepsilon\searrow 2\sigma}\sqrt{\varepsilon^2-4\sigma^2}\sum_{n=3}^{\infty}\frac{\log n}{n}\mathbb{P}\left(|S_n|\geq\varepsilon\sqrt{n\varphi(n)}\right)=\sigma$$

holds if and only if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ and $\mathbb{E}X^2\log|X|(\varphi(|X|))^{-1} < \infty$.

Jiang, Zhang and Pang [9] provided the precise rates in the law of logarithm for moment convergence that for $r > 1$,

$$\lim_{\varepsilon\searrow\sqrt{r-1}}\frac{1}{-\log(\varepsilon^2-(r-1))}\sum_{n=1}^{\infty}n^{r-2-1/2}\mathbb{E}\left\{M_n-\varepsilon\sigma\sqrt{2n\log n}\right\}_+ = \frac{2\sigma}{(r-1)\sqrt{2\pi}}$$

holds if and only if $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ and $\mathbb{E}(X^{2r}/(\log|X|)^r) < \infty$.

Inspired by [9], we consider the precise rates in the LIL extended from Davis [3] and Gut [6] for the first moment of S_n and M_n. We obtain the following results.

Theorem 1.1. *For $r > 0$, let $a_n(\varepsilon)$ be a function of ε such that*

$$a_n(\varepsilon)\varphi(n)\to\tau, \quad \textit{as} \quad n\to\infty \quad \textit{and} \quad \varepsilon\searrow\sqrt{r}. \tag{1.1}$$

Let $\{X, X_n; n\geq 1\}$ be a sequence of i.i.d. random variables with

$$\mathbb{E}X=0, \quad \mathbb{E}X^2=\sigma^2, \quad \textit{and} \quad \mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{-1/2}<\infty. \tag{1.2}$$

Suppose that

$$\mathbb{E}X^2 I\{|X|\geq t\} = o((\varphi(t))^{-1}), \quad \textit{as} \quad t\to\infty. \tag{1.3}$$

Then we have

$$\lim_{\varepsilon\searrow\sqrt{r}}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{|S_n|-(\varepsilon+a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ = \frac{\sigma e^{-2\tau\sqrt{r}}}{\sqrt{2\pi r}} \tag{1.4}$$

and

$$\lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ = \frac{2\sigma e^{-2\tau\sqrt{r}}}{\sqrt{2\pi}r}. \tag{1.5}$$

Conversely, for $a_n(\varepsilon)$ *satisfying* (1.1), *if either* (1.4) *or* (1.5) *holds for* $r > 0$ *and some* $\sigma > 0$, *then* (1.2) *holds and*

$$\liminf_{t\to\infty} \varphi(t)\mathbb{E}X^2 I\{|X| \geq t\} = 0. \tag{1.6}$$

***Remark* 1.1.** The condition (1.3) is sharp. A sufficient condition for it is given by $\mathbb{E}X^2\varphi(|X|) < < \infty$. Obviously, when $r > 1$, condition (1.3) is implied by (1.2).

***Corollary* 1.1.** *Under conditions of* (1.1) *and* (1.2), *for any* $\varepsilon > \sqrt{r} > 0$, *we have*

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \mathbb{E}\left\{|S_n| - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ < \infty,$$

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}\sqrt{\varphi(n)}}{n} \mathbb{P}\left\{|S_n| \geq (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\} < \infty,$$

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}\sqrt{\varphi(n)}}{n} \mathbb{P}\left\{M_n \geq (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\} < \infty.$$

2. Some lemmas.

Lemma 2.1 [1, p. 79, 80]**.** *For any* $x > 0$,

$$\mathrm{P}\left\{\sup_{0\leq t\leq 1} |W(t)| \geq x\right\} = 1 - \sum_{k=-\infty}^{\infty} (-1)^k \mathrm{P}\{(2k-1)x \leq N \leq (2k+1)x\} =$$

$$= 4\sum_{k=0}^{\infty} (-1)^k \mathrm{P}\{N \geq (2k+1)x\} = 2\sum_{k=0}^{\infty} (-1)^k \mathrm{P}\{|N| \geq (2k+1)x\}.$$

In particular,

$$\mathrm{P}\left\{\sup_{0\leq t\leq 1} W(t) \geq x\right\} = 2\mathrm{P}\{N \geq x\} \sim \frac{2}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right), \quad as \quad x \to \infty.$$

Lemma 2.2 (Lemma 1.1.1 of [2])**.** *For any* $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0\leq s\leq 1-h} \sup_{0\leq t\leq h} \left|W(s+t) - W(s)\right| \geq \nu\sqrt{h}\right) \leq \frac{C}{h} e^{-\frac{\nu^2}{2+\varepsilon}}$$

holds for every $\nu > 0$ *and* $0 < h < 1$.

Lemma 2.3 (Lemma 2.4 of [9], see also [13, p. 78]). *For $q \geq 2$, let $\{\xi_k; 1 \leq k \leq n\}$ be a sequence of independent random variables with $\mathbb{E}\xi_k = 0$ and $\mathbb{E}|\xi_k|^q < \infty$. Then, for any $y > 0$,*

$$\mathbb{P}\left(\max_{k\leq n}\Big|\sum_{i=1}^{k}\xi_i\Big| \geq y\right) \leq 2\exp\left\{-\frac{y^2}{8\sum_{k=1}^{n}\mathbb{E}\xi_k^2}\right\} + (2Aq)^q y^{-q}\sum_{i=1}^{n}\mathbb{E}|\xi_i|^q,$$

where A is a universal constant.

Lemma 2.4. *There exists $N_0 > 0$ such that when $n \geq N_0$, we have*

$$n\mathbb{P}\left(|X| \geq 4\varepsilon\sqrt{n\varphi(n)}\right) \leq 4\mathbb{P}\left(|S_n| \geq \varepsilon\sqrt{n\varphi(n)}\right) \quad \textit{for} \quad \varepsilon > 0.$$

Proof. The proof is similar to that of Lemma 2.3 in [12] in which we only need to replace $\sqrt{\log n}$ by $\sqrt{\varphi(n)}$.

3. Proof of Theorem 1.1 and Corollary 1.1. Theorem 1.1 is based on the following propositions.

Proposition 3.1. *For $r > 0$, let $a_n(\varepsilon)$ be a function of ε satisfying* (1.1), *we have*

$$\lim_{\varepsilon\searrow\sqrt{r}}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n}\mathbb{E}\left\{|N| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\}_+ = \frac{e^{-2\tau\sqrt{r}}}{\sqrt{2\pi}r} \tag{3.1}$$

and

$$\lim_{\varepsilon\searrow\sqrt{r}}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n}\mathbb{E}\left\{\sup_{0\leq s\leq 1}|W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\}_+ = \frac{2e^{-2\tau\sqrt{r}}}{\sqrt{2\pi}r}. \tag{3.2}$$

Proof. By Lemma 2.1 and condition (1.1), uniformly with respect to all $x \geq 0$, we have

$$\mathbb{P}\left\{\sup_{0\leq s\leq 1}|W(s)| \geq x + (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\} \sim$$

$$\sim 2\mathbb{P}\left\{|N| \geq x + (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\} =$$

$$= 4\mathbb{P}\left\{N \geq x + (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\} \sim$$

$$\sim \frac{4}{\sqrt{2\pi}\left(x + (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right)}\exp\left\{-\frac{1}{2}\left(x + (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right)^2\right\} \sim$$

$$\sim \frac{4}{\sqrt{2\pi}\left(x + \varepsilon\sqrt{2\varphi(n)}\right)}\exp\left\{-\frac{1}{2}\left(x + \varepsilon\sqrt{2\varphi(n)}\right)^2 - a_n(\varepsilon)\sqrt{2\varphi(n)}\left(x + \varepsilon\sqrt{2\varphi(n)}\right)\right\} \sim$$

$$\sim \frac{4}{\sqrt{2\pi}\left(x + \varepsilon\sqrt{2\varphi(n)}\right)}\exp\left\{-\frac{1}{2}\left(x + \varepsilon\sqrt{2\varphi(n)}\right)^2\right\}\exp\{-2\varepsilon\tau\},$$

as $n \to \infty$ and $\varepsilon \searrow \sqrt{r}$.

Therefore, we obtain

$$\mathbb{E}\left\{\sup_{0\leq s\leq 1}|W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\}_+ \sim 2\mathbb{E}\left\{|N| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)}\right\}_+ \sim$$

$$\sim 4e^{-2\varepsilon\tau}\int\limits_0^\infty \frac{1}{\sqrt{2\pi}\left(x+\varepsilon\sqrt{2\varphi(n)}\right)}\exp\left\{-\frac{1}{2}\left(x+\varepsilon\sqrt{2\varphi(n)}\right)^2\right\}dx,$$

as $\varepsilon\searrow\sqrt{r}$ and $n\to\infty$. Next we only need to prove (3.1).

Since the limit in (3.1) does not depend on any finite terms of the infinite series, it follows that

$$\lim_{\varepsilon\searrow\sqrt{r}}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^\infty\frac{(\log n)^{r-1}}{n}\mathbb{E}\left\{|N|-(\varepsilon+a_n(\varepsilon))\sqrt{2\varphi(n)}\right\}_+ =$$

$$=\lim_{\varepsilon\searrow\sqrt{r}}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^\infty\frac{(\log n)^{r-1}}{n}2e^{-2\varepsilon\tau}\int\limits_0^\infty\frac{1}{\sqrt{2\pi}\left(x+\varepsilon\sqrt{2\varphi(n)}\right)}\times$$

$$\times\exp\left\{-\frac{1}{2}\left(x+\varepsilon\sqrt{2\varphi(n)}\right)^2\right\}dx=$$

$$=\lim_{\varepsilon\searrow\sqrt{r}}\frac{2e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{e^e}^\infty\frac{(\log y)^{r-1}}{y}dy\int\limits_0^\infty\frac{1}{\sqrt{2\pi}\left(x+\varepsilon\sqrt{2\varphi(y)}\right)}\times$$

$$\times\exp\left\{-\frac{1}{2}\left(x+\varepsilon\sqrt{2\varphi(y)}\right)^2\right\}dx=$$

$$=\lim_{\varepsilon\searrow\sqrt{r}}\frac{2e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{e^e}^\infty\frac{(\log y)^{r-1}}{y}dy\int\limits_{\varepsilon\sqrt{2\varphi(y)}}^\infty\frac{1}{z\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz=$$

$$=\lim_{\varepsilon\searrow\sqrt{r}}\frac{2e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{\sqrt{2}\varepsilon}^\infty\frac{t}{\varepsilon^2}\exp\left\{\frac{rt^2}{2\varepsilon^2}\right\}dt\int\limits_t^\infty\frac{1}{z\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz=$$

$$=\lim_{\varepsilon\searrow\sqrt{r}}\frac{2e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{\sqrt{2}\varepsilon}^\infty\frac{1}{z\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz\int\limits_{\sqrt{2}\varepsilon}^z\frac{t}{\varepsilon^2}\exp\left\{\frac{rt^2}{2\varepsilon^2}\right\}dt=$$

$$=\lim_{\varepsilon\searrow\sqrt{r}}\frac{2e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{\sqrt{2}\varepsilon}^\infty\frac{1}{z\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}\frac{1}{r}\left(\exp\left\{\frac{rz^2}{2\varepsilon^2}\right\}-e^r\right)dz=$$

$$=\frac{1}{\sqrt{2\pi}r}\lim_{\varepsilon\searrow\sqrt{r}}\frac{2e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{\sqrt{2}\varepsilon}^\infty\frac{1}{z}\exp\left\{-\frac{z^2}{2}+\frac{rz^2}{2\varepsilon^2}\right\}dz=$$

$$=\frac{1}{\sqrt{2\pi}r}\lim_{\varepsilon\searrow\sqrt{r}}\frac{e^{-2\varepsilon\tau}}{-\log(\varepsilon^2-r)}\int\limits_{\varepsilon^2-r}^\infty\frac{1}{s}e^{-s}ds\qquad\left(s=\frac{z^2}{2}\left(1-\frac{r}{\varepsilon^2}\right)\right)=$$

$$=\frac{e^{-2\tau\sqrt{r}}}{\sqrt{2\pi}r}\lim_{\varepsilon\searrow\sqrt{r}}\frac{1}{-\log(\varepsilon^2-r)}\int\limits_{\varepsilon^2-r}^1\frac{1}{s}e^{-s}ds=$$

$$= \frac{e^{-2\tau\sqrt{r}}}{\sqrt{2\pi r}}.$$

Proposition 3.1 is proved.

Next, without loss of generality, we assume that $\sigma = 1$ throughout the proof of the direct part of our main results. For each n and $1 \le j \le n,\ 1/2 < p \le 2$, let

$$X'_{nj} = X_j I\{|X_j| \le \sqrt{n}/(\varphi(n))^p\}, \quad X^{(1)}_{nj} = X'_{nj} - \mathbb{E}X'_{nj}, \quad B_n = \sum_{j=1}^{n} \mathrm{Var}\, X^{(1)}_{nj},$$

$$S^{(1)}_{nk} = \sum_{j=1}^{k} X^{(1)}_{nj}, \quad M^{(1)}_n = \max_{1\le k\le n} |S^{(1)}_{nk}|, \quad \Delta_n = \max_{1\le k\le n} |S_k - S^{(1)}_{nk}| \tag{3.3}$$

and

$$X''_{nj} = X_j I\{\sqrt{n}/(\varphi(n))^p < |X_j| \le \sqrt{n\varphi(n)}\}, \quad X^{(2)}_{nj} = X''_{nj} - \mathbb{E}X''_{nj},$$
$$X'''_{nj} = X_j I\{|X_j| > \sqrt{n\varphi(n)}\}, \quad X^{(3)}_{nj} = X'''_{nj} - \mathbb{E}X'''_{nj}.$$

Also we define $S^{(2)}_{nj},\ S^{(3)}_{nj},\ M^{(2)}_n,\ M^{(3)}_n$ similarly.

Proposition 3.2. *Let $1/2 < p' < p \le 2$. For any $\varepsilon > 0$ and $x > 0$ there exists a sequence of positive numbers p_n such that*

$$\sqrt{B_n}\mathbb{E}\left\{|N| - x - \frac{1}{(\varphi(n))^{p'}}\right\}_+ - p_n \le \mathbb{E}\left\{\Big|\sum_{j=1}^{n} X^{(1)}_{nj}\Big| - x\sqrt{B_n}\right\}_+ \le$$
$$\le \sqrt{B_n}\mathbb{E}\left\{|N| - x + \frac{1}{(\varphi(n))^{p'}}\right\}_+ + p_n \tag{3.4}$$

and

$$\sqrt{B_n}\mathbb{E}\left\{\sup_{0\le s\le 1}|W(s)| - x - \frac{1}{(\varphi(n))^{p'}}\right\}_+ - p_n \le \mathbb{E}\left\{M^{(1)}_n - x\sqrt{B_n}\right\}_+ \le$$
$$\le \sqrt{B_n}\mathbb{E}\left\{\sup_{0\le s\le 1}|W(s)| - x + \frac{1}{(\varphi(n))^{p'}}\right\}_+ + p_n, \tag{3.5}$$

where

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} p_n < \infty. \tag{3.6}$$

Proof. We can define a Wiener process

$$W_n(tB_n) = \sum_{i=1}^{\lfloor nt\rfloor} \xi_i + (nt - \lfloor nt\rfloor)\xi_{\lfloor nt\rfloor+1},$$

where $\{\xi_n; n \ge 1\}$ is a sequence of independent normal variables with $\mathbb{E}\xi_j = 0$ and $\mathbb{E}\xi_j^2 = \mathrm{Var}\,\mathrm{X}'_{nj}$. The proofs of (3.4) and (3.5) are similar to that of Proposition 2.2 in [9], and we have

$$p_n \le Cn^{(3-q)/2}(\varphi(n))^{p'(q-1)}\mathbb{E}|X|^q I\left\{|X| \le \frac{\sqrt{n}}{(\varphi(n))^p}\right\}+$$

$$+\sqrt{n}\mathbb{P}\left(\max_{0\le s\le 1}\left|W_n(s)-W_n\left(\frac{\lfloor ns\rfloor}{n}\right)\right| \ge \frac{1}{2(\varphi(n))^{p'}}\right)+$$

$$+\sqrt{n}\int\limits_1^\infty \mathbb{P}\left(\max_{0\le s\le 1}\left|W_n(s)-W_n\left(\frac{\lfloor ns\rfloor}{n}\right)\right| \ge x\right)dx =:$$

$$=: p_{n1}+D_{n1}+D_{n2}.$$

Next we only prove (3.6). On one hand, we get

$$\sum_{n=1}^\infty \frac{(\log n)^{r-1}}{n^{3/2}}p_{n1} \le C\sum_{n=1}^\infty n^{-q/2}(\log n)^{r-1}(\varphi(n))^{p'(q-1)}\mathbb{E}|X|^q I\left\{|X| \le \frac{\sqrt{n}}{(\varphi(n))^p}\right\} \le$$

$$\le C\sum_{n=1}^\infty \frac{(\varphi(n))^{p'(q-1)}}{n^{q/2}(\log n)^{1-r}}\sum_{j=1}^n \mathbb{E}|X|^q I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\varphi(j))^p}\right\} =$$

$$= C\sum_{j=1}^\infty \mathbb{E}|X|^q I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\varphi(j))^p}\right\}\sum_{n=j}^\infty \frac{(\varphi(n))^{p'(q-1)}}{n^{q/2}(\log n)^{1-r}} \le$$

$$\le C\sum_{j=1}^\infty j^{1-q/2}\frac{(\varphi(j))^{p'(q-1)}}{(\log j)^{1-r}}\mathbb{E}|X|^q I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\varphi(j))^p}\right\} \le$$

$$\le C\mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{p'-(p-p')(q-2)} \le$$

$$\le C\mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{-1/2} < \infty,$$

if $p'-(p-p')(q-2) \le -1/2$, or equivalently $q \ge 2+(p'+1/2)/(p-p')$.

On the other hand, by Lemma 2.2 and the basic inequality that $e^{-x} \le 2x^{-2}$ for $x>0$, it holds that

$$\sum_{n=1}^\infty \frac{(\log n)^{r-1}}{n^{3/2}}D_{n1} =$$

$$= \sum_{n=1}^\infty \frac{(\log n)^{r-1}}{n}\mathbb{P}\left(\max_{0\le s\le 1}\left|W_n(s)-W_n\left(\frac{\lfloor ns\rfloor}{n}\right)\right| \ge \sqrt{\frac{1}{n}}\frac{\sqrt{n}}{2(\varphi(n))^{p'}}\right) \le$$

$$\le C\sum_{n=1}^\infty (\log n)^{r-1}\exp\left\{-\frac{n}{12(\varphi(n))^{2p'}}\right\} \le C\sum_{n=1}^\infty \frac{(\log n)^{r-1}(\varphi(n))^{4p'}}{n^2} < \infty.$$

Similarly, we obtain

$$\sum_{n=1}^\infty \frac{(\log n)^{r-1}}{n^{3/2}}D_{n2} =$$

$$= \sum_{n=1}^\infty \frac{(\log n)^{r-1}}{n}\int\limits_1^\infty \mathbb{P}\left(\max_{0\le s\le 1}\left|W_n(s)-W_n\left(\frac{\lfloor ns\rfloor}{n}\right)\right| \le \sqrt{\frac{1}{n}}\sqrt{n}x\right)dx \le$$

$$\leq C\sum_{n=1}^{\infty}(\log n)^{r-1}\int_{1}^{\infty}\exp\left\{-\frac{nx^2}{3}\right\}dx \leq$$

$$\leq C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^2}\int_{1}^{\infty}x^{-4}dx \leq C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^2} < \infty.$$

Then we complete the proof of this proposition.

Proposition 3.3. *For any* $\lambda > 0,$ *we have*

$$\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\Pi_{n1} < \infty, \tag{3.7}$$

where

$$\Pi_{n1} = \int_{0}^{\infty}\mathbb{P}(M_n^{(1)} \geq \lambda\sqrt{2n\varphi(n)} + x, \qquad \Delta_n > \sqrt{n}/(\varphi(n))^p)dx,$$

Δ_n *is defined in* (3.3).

Proof. Let $\beta_n = n\mathbb{E}|X|I\{|X| > \sqrt{n}/(\varphi(n))^p\},$ then $|\mathbb{E}\sum_{i=1}^{k}X'_{ni}| \leq \beta_n,\ 1 \leq k \leq n.$ Set $H = \{n\colon \beta_n \leq \sqrt{n}/(8(\varphi(n))^2)\},$ then we get

$$\left\{\Delta_n > \frac{\sqrt{n}}{(\varphi(n))^p}\right\} \subset \bigcup_{j=1}^{n}\{X_j \neq X'_{nj}\}, \qquad n \in H.$$

Paralleling the proof of Proposition 2.3 in [9] and then using Lemma 2.3, we obtain

$$\sum_{n\in H}\frac{(\log n)^{r-1}}{n^{3/2}}\int_{0}^{\infty}\mathbb{P}\left(M_n^{(1)} \geq \lambda\sqrt{2n\varphi(n)} + x, \Delta_n > \frac{\sqrt{n}}{(\varphi(n))^p}\right)dx \leq$$

$$\leq \sum_{n\in H}\frac{(\log n)^{r-1}}{n^{3/2}}n\mathbb{P}\left(|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right)\int_{0}^{\infty}\mathbb{P}\left(M_n^{(1)} \geq \frac{\lambda}{2}\sqrt{2n\varphi(n)} + x\right)dx \leq$$

$$\leq \sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{\sqrt{n}}\mathbb{P}\left(|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right)\int_{0}^{\infty}\left(\exp\left\{-\frac{(\lambda\sqrt{2n\varphi(n)}/2 + x)^2}{8B_n}\right\} +\right.$$

$$\left. +C\left(x + \frac{\lambda}{2}\sqrt{2n\varphi(n)}\right)^{-q}\sum_{j=1}^{n}\mathbb{E}|X_{nj}^{(1)}|^q\right)dx =:$$

$$=: \sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{\sqrt{n}}\mathbb{P}\left(|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right)(F_{n1} + F_{n2}).$$

On one hand, since $\displaystyle\int_x^{\infty}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt \leq e^{-x^2/2}/2$ holds for $x \geq 0,$ it follows that

$$\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{\sqrt{n}}\mathbb{P}\left(|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right)F_{n1} =$$

$$= \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{\sqrt{n}} \mathbb{P}\left(|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right) 2\sqrt{B_n} \int\limits_{\lambda\sqrt{2n\varphi(n)}/(4\sqrt{B_n})}^{\infty} \exp\{-y^2/2\} dy \leq$$

$$\leq C \sum_{n=1}^{\infty} (\log n)^{r-1} \exp\left\{-\frac{\lambda^2 n\varphi(n)}{16 B_n}\right\} \mathbb{E} I\left\{|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right\} \leq$$

$$\leq C \sum_{n=1}^{\infty} (\log n)^{r-1-\lambda^2/16} \sum_{j=n}^{\infty} \mathbb{E} I\left\{\frac{\sqrt{j}}{(\varphi(j))^p} < |X| \leq \frac{\sqrt{j+1}}{(\varphi(j+1))^p}\right\} =$$

$$= C \sum_{j=1}^{\infty} \mathbb{E} I\left\{\frac{\sqrt{j}}{(\varphi(j))^p} < |X| \leq \frac{\sqrt{j+1}}{(\varphi(j+1))^p}\right\} \sum_{n=1}^{j} (\log n)^{r-1-\lambda^2/16} \leq$$

$$\leq C \sum_{j=1}^{\infty} \mathbb{E} I\left\{\frac{\sqrt{j}}{(\varphi(j))^p} < |X| \leq \frac{\sqrt{j+1}}{(\varphi(j+1))^p}\right\} j (\log j)^{r-1-\lambda^2/16} \leq$$

$$\leq C \mathbb{E} X^2 (\log |X|)^{r-1-\lambda^2/16} (\varphi(|X|))^{2p} < \infty.$$

On the other hand, applying Markov's inequality, we have

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{\sqrt{n}} \mathbb{P}\left(|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right) F_{n2} \leq$$

$$\leq C \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{\sqrt{n}} \frac{(\varphi(n))^{2p} \mathbb{E} X^2}{n} \int\limits_{\lambda\sqrt{2n\varphi(n)}/2}^{\infty} x^{-q} \sum_{j=1}^{n} \mathbb{E}|X_{nj}^{(1)}|^q dx \leq$$

$$\leq C \sum_{n=1}^{\infty} \frac{(\log n)^{r-1} (\varphi(n))^{2p}}{n^{3/2}} (n\varphi(n))^{(1-q)/2} n \mathbb{E}|X|^q I\left\{|X| \leq \frac{\sqrt{n}}{(\varphi(n))^p}\right\} \leq$$

$$\leq C \sum_{n=1}^{\infty} \frac{(\varphi(n))^{2p+(1-q)/2}}{n^{q/2} (\log n)^{1-r}} \sum_{j=1}^{n} \mathbb{E}|X|^q I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p} < |X| \leq \frac{\sqrt{j}}{(\varphi(j))^p}\right\} =$$

$$= C \sum_{j=1}^{\infty} \mathbb{E}|X|^q I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p} < |X| \leq \frac{\sqrt{j}}{(\varphi(j))^p}\right\} \sum_{n=j}^{\infty} \frac{(\varphi(n))^{2p+(1-q)/2}}{n^{q/2} (\log n)^{1-r}} \leq$$

$$\leq C \sum_{j=1}^{\infty} \mathbb{E}|X|^q I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p} < |X| \leq \frac{\sqrt{j}}{(\varphi(j))^p}\right\} \frac{j^{1-q/2} (\varphi(j))^{2p+(1-q)/2}}{(\log j)^{1-r}} \leq$$

$$\leq C \mathbb{E} X^2 (\log |X|)^{r-1} (\varphi(|X|))^{2p-(q-2)(p+1/2)-1/2} < \infty,$$

if $2p - (q-2)(p+1/2) \leq 0$, or equivalently $q \geq 2 + 4p/(2p+1)$. Then we get

$$\sum_{n \in H} n^{-3/2} (\log n)^{r-1} \Pi_{n1} < \infty.$$

If $n \notin H$, that is to say

$$\mathbb{E}|X|I\left\{|X|>\frac{\sqrt{n}}{(\varphi(n))^p}\right\}>\frac{1}{8\sqrt{n}(\varphi(n))^2}. \tag{3.8}$$

By Lemma 2.3, we obtain

$$\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}\Pi_{n1}\le\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}\int\limits_0^\infty\mathbb{P}\left(M_n^{(1)}\ge\lambda\sqrt{2n\varphi(n)}+x\right)dx\le$$

$$\le\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}\int\limits_0^\infty\left(\exp\left\{-\frac{(\lambda\sqrt{2n\varphi(n)}+x)^2}{8B_n}\right\}+\right.$$

$$\left.+C\left(x+\frac{\lambda}{2}\sqrt{2n\varphi(n)}\right)^{-q}\sum_{j=1}^n\mathbb{E}|X_{nj}^{(1)}|^q\right)dx=:$$

$$=:\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}(G_{n1}+G_{n2}).$$

While from (3.8), we have

$$\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}G_{n1}\le C\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}\sqrt{B_n}\exp\left\{-\frac{\lambda^2 n\varphi(n)}{4B_n}\right\}\le$$

$$\le C\sum_{n\notin H}\frac{(\log n)^{r-1-\lambda^2/4}}{n}\sqrt{n}(\varphi(n))^2\mathbb{E}|X|I\left\{|X|>\frac{\sqrt{n}}{(\varphi(n))^p}\right\}\le$$

$$\le C\sum_{n=1}^\infty\frac{(\log n)^{r-1-\lambda^2/4}}{\sqrt{n}(\varphi(n))^{-2}}\sum_{j=n}^\infty\mathbb{E}|X|I\left\{\frac{\sqrt{j}}{(\varphi(j))^p}<|X|\le\frac{\sqrt{j+1}}{(\varphi(j+1))^p}\right\}=$$

$$=C\sum_{j=1}^\infty\mathbb{E}|X|I\left\{\frac{\sqrt{j}}{(\varphi(j))^p}<|X|\le\frac{\sqrt{j+1}}{(\varphi(j+1))^p}\right\}\sum_{n=1}^j\frac{(\log n)^{r-1-\lambda^2/4}}{\sqrt{n}(\varphi(n))^{-2}}\le$$

$$\le C\sum_{j=1}^\infty\mathbb{E}|X|I\left\{\frac{\sqrt{j}}{(\varphi(j))^p}<|X|\le\frac{\sqrt{j+1}}{(\varphi(j+1))^p}\right\}\frac{\sqrt{j}(\log j)^{r-1-\lambda^2/4}}{(\varphi(j))^{-2}}\le$$

$$\le C\mathbb{E}X^2(\log|X|)^{r-1-\lambda^2/4}(\varphi(|X|))^{2+p}<\infty$$

and

$$\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{3/2}}G_{n2}\le C\sum_{n\notin H}\frac{(\log n)^{r-1}}{n^{1/2}}(n\varphi(n))^{(1-q)/2}\mathbb{E}|X|^qI\left\{|X|\le\frac{\sqrt{n}}{(\varphi(n))^p}\right\}\le$$

$$\le C\sum_{n=1}^\infty\frac{(\varphi(n))^{(1-q)/2}}{n^{q/2}(\log n)^{1-r}}\sum_{j=1}^n\mathbb{E}|X|^qI\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p}<|X|\le\frac{\sqrt{j}}{(\varphi(j))^p}\right\}=$$

$$=C\sum_{j=1}^\infty\mathbb{E}|X|^qI\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^p}<|X|\le\frac{\sqrt{j}}{(\varphi(j))^p}\right\}\sum_{n=j}^\infty\frac{(\varphi(n))^{(1-q)/2}}{n^{q/2}(\log n)^{1-r}}\le$$

$$\leq C\sum_{j=1}^{\infty}\mathbb{E}|X|^{q}I\left\{\frac{\sqrt{j-1}}{(\varphi(j-1))^{p}}<|X|\leq\frac{\sqrt{j}}{(\varphi(j))^{p}}\right\}\frac{j^{1-q/2}(\varphi(j))^{(1-q)/2}}{(\log j)^{1-r}}\leq$$

$$\leq C\mathbb{E}X^{2}(\log|X|)^{r-1}(\varphi(|X|))^{-(q-2)(p+1/2)-1/2}<\infty.$$

Proposition 3.3 is proved.

Proposition 3.4. *For any* $\lambda>0$, *we have*

$$\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\Pi_{n2}<\infty, \tag{3.9}$$

where $\Pi_{n2}=\int_{0}^{\infty}\mathbb{P}(M_n\geq\lambda\sqrt{2n\varphi(n)}+x,\ \ \Delta_n>\sqrt{n}/(\varphi(n))^{p})dx,\ \ \Delta_n$ *is defined in* (3.3).

Proof. It holds that

$$\mathbb{P}\left(M_n\geq\lambda\sqrt{2n\varphi(n)}+x,\ \Delta_n>\frac{\sqrt{n}}{(\varphi(n))^{p}}\right)\leq$$

$$\leq\mathbb{P}\left(M_n^{(1)}\geq\frac{\lambda}{3}\sqrt{2n\varphi(n)}+\frac{x}{3},\Delta_n>\frac{\sqrt{n}}{(\varphi(n))^{p}}\right)+$$

$$+\mathbb{P}\left(M_n^{(2)}\geq\frac{\lambda}{3}\sqrt{2n\varphi(n)}+\frac{x}{3}\right)+\mathbb{P}\left(M_n^{(3)}\geq\frac{\lambda}{3}\sqrt{2n\varphi(n)}+\frac{x}{3}\right).$$

From (1.3), we have

$$B_n^{(2)}:=\sum_{k=1}^{n}\operatorname{Var}X_{nk}^{(2)}\leq n\mathbb{E}X^{2}I\left\{|X|>\frac{\sqrt{n}}{(\varphi(n))^{p}}\right\}=o\left(\frac{n}{\varphi(n)}\right).$$

Then applying Lemma 2.3, we obtain

$$\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\int\limits_{0}^{\infty}\mathbb{P}\left(M_n^{(2)}\geq\frac{\lambda}{3}\sqrt{2n\varphi(n)}+\frac{x}{3}\right)dx\leq$$

$$\leq C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\int\limits_{0}^{\infty}\left(x+\lambda\sqrt{2n\varphi(n)}\right)^{-q}n\mathbb{E}|X|^{q}I\{|X|\leq\sqrt{n\varphi(n)}\}dx+$$

$$+C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\int\limits_{0}^{\infty}\exp\left\{-\frac{(\lambda\sqrt{2n\varphi(n)}+x)^{2}}{72B_n^{(2)}}\right\}dx\leq$$

$$\leq C\sum_{n=1}^{\infty}\frac{n^{-q/2}(\log n)^{r-1}}{(\varphi(n))^{(q-1)/2}}\sum_{j=1}^{n}\mathbb{E}|X|^{q}I\{\sqrt{(j-1)\varphi(j-1)}<|X|\leq\sqrt{j\varphi(j)}\}+$$

$$+C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n}\exp\left\{-\frac{\lambda^{2}n\varphi(n)}{36B_n^{(2)}}\right\}\quad(<\infty)\leq$$

$$\leq C\sum_{j=1}^{\infty}\mathbb{E}|X|^{q}I\{\sqrt{(j-1)\varphi(j-1)}<|X|\leq\sqrt{j\varphi(j)}\}\sum_{n=j}^{\infty}\frac{n^{-q/2}(\log n)^{r-1}}{(\varphi(n))^{(q-1)/2}}\leq$$

$$\leq C\sum_{j=1}^{\infty}\mathbb{E}|X|^q I\left\{\sqrt{(j-1)\varphi(j-1)}<|X|\leq\sqrt{j\varphi(j)}\right\}\frac{j^{1-q/2}(\log j)^{r-1}}{(\varphi(j))^{(q-1)/2}}\leq$$

$$\leq C\mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{-1/2}<\infty,$$

where we have used the fact that

$$\exp\left\{-\frac{\lambda^2 n\varphi(n)}{36B_n^{(2)}}\right\}=(\log n)^{-\lambda^2 n/(36B_n^{(2)})}$$

with $n/B_n^{(2)}\to+\infty$ as $n\to+\infty$.

Since $\mathbb{E}M_n^{(3)}\leq 2n\mathbb{E}|X|I\{|X|>\sqrt{n\varphi(n)}\}$, it follows that

$$\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\int\limits_0^{\infty}\mathbb{P}\left(M_n^{(3)}\geq\frac{\lambda}{3}\sqrt{2n\varphi(n)}+\frac{x}{3}\right)dx=$$

$$=\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}3\mathbb{E}\left\{M_n^{(3)}-\frac{\lambda}{3}\sqrt{2n\varphi(n)}\right\}_+\leq$$

$$\leq C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}n\mathbb{E}|X|I\{|X|>\sqrt{n\varphi(n)}\}=$$

$$=C\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{\sqrt{n}}\sum_{j=n}^{\infty}\mathbb{E}|X|I\{\sqrt{j\varphi(j)}<|X|\leq\sqrt{(j+1)\varphi(j+1)}\}=$$

$$=C\sum_{j=1}^{\infty}\mathbb{E}|X|I\{\sqrt{j\varphi(j)}<|X|\leq\sqrt{(j+1)\varphi(j+1)}\}\sum_{n=1}^{j}\frac{(\log n)^{r-1}}{\sqrt{n}}\leq$$

$$\leq C\sum_{j=1}^{\infty}\mathbb{E}|X|I\{\sqrt{j\varphi(j)}<|X|\leq\sqrt{(j+1)\varphi(j+1)}\}\sqrt{j}(\log j)^{r-1}\leq$$

$$\leq C\mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{-1/2}<\infty.$$

Then from Proposition, 3.3, we get the desired result.

***Proof of the direct part of Theorem* 1.1.** From Propositions 3.2 – 3.4, we have for large n,

$$\mathbb{E}\left\{M_n-(\varepsilon+a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+=$$

$$=\int\limits_0^{\infty}\mathbb{P}\left(M_n\geq(\varepsilon+a_n(\varepsilon))\sqrt{2B_n\varphi(n)}+x,\Delta_n\leq\frac{\sqrt{n}}{(\varphi(n))^p}\right)dx+$$

$$+\int\limits_0^{\infty}\mathbb{P}\left(M_n\geq(\varepsilon+a_n(\varepsilon))\sqrt{2B_n\varphi(n)}+x,\Delta_n>\frac{\sqrt{n}}{(\varphi(n))^p}\right)dx\leq$$

$$\leq\int\limits_0^{\infty}\mathbb{P}\left(M_n^{(1)}\geq(\varepsilon+a_n(\varepsilon))\sqrt{2B_n\varphi(n)}-\frac{\sqrt{n}}{(\varphi(n))^p}+x\right)dx+$$

$$+\int_0^\infty \mathbb{P}\left(M_n \geq \frac{\varepsilon}{2}\sqrt{2n\varphi(n)} + x, \Delta_n > \frac{\sqrt{n}}{(\varphi(n))^p}\right) dx \leq$$

$$\leq \mathbb{E}\left\{M_n^{(1)} - \sqrt{B_n}\left((\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} - \frac{2}{(\varphi(n))^p}\right)\right\}_+ + \Pi_{n2} \leq$$

$$\leq \sqrt{B_n}\mathbb{E}\left\{\sup_{0\leq s\leq 1}|W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} + \frac{2}{(\varphi(n))^p} + \frac{1}{(\varphi(n))^{p'}}\right\}_+ + p_n + \Pi_{n2}$$

and

$$\mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ \geq$$

$$\geq \int_0^\infty \mathbb{P}\left(M_n \geq (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)} + x, \Delta_n \leq \frac{\sqrt{n}}{(\varphi(n))^p}\right) dx \geq$$

$$\geq \int_0^\infty \mathbb{P}\left(M_n^{(1)} \geq (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)} + \frac{\sqrt{n}}{(\varphi(n))^p} + x, \Delta_n \leq \frac{\sqrt{n}}{(\varphi(n))^p}\right) dx \geq$$

$$\geq \int_0^\infty \mathbb{P}\left(M_n^{(1)} \geq (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)} + \frac{\sqrt{n}}{(\varphi(n))^p} + x\right) dx -$$

$$-\int_0^\infty \mathbb{P}\left(M_n^{(1)} \geq \frac{\varepsilon}{2}\sqrt{2n\varphi(n)} + x, \Delta_n > \frac{\sqrt{n}}{(\varphi(n))^p}\right) dx \geq$$

$$\geq \mathbb{E}\left\{M_n^{(1)} - \sqrt{B_n}\left((\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} + \frac{2}{(\varphi(n))^p}\right)\right\}_+ - \Pi_{n1} \geq$$

$$\geq \sqrt{B_n}\mathbb{E}\left\{\sup_{0\leq s\leq 1}|W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} - \frac{2}{(\varphi(n))^p} - \frac{1}{(\varphi(n))^{p'}}\right\}_+ -$$

$$-p_n - \Pi_{n1},$$

where Π_{n1} and Π_{n2} are defined in Propositions 3.3 and 3.4 respectively with $\lambda = \varepsilon/2$.

Therefore, for sufficiently large n, we get

$$\sqrt{B_n}\mathbb{E}\left\{\sup_{0\leq s\leq 1}|W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} - \frac{3}{(\varphi(n))^{p'}}\right\}_+ - p_n - \Pi_{n1} \leq$$

$$\leq \mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ \leq$$

$$\leq \sqrt{B_n}\mathbb{E}\left\{\sup_{0\leq s\leq 1}|W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} + \frac{3}{(\varphi(n))^{p'}}\right\}_+ + p_n + \Pi_{n2}.$$

Similarly, for n large enough, it holds that

$$\sqrt{B_n}\mathbb{E}\left\{|N| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} - \frac{3}{(\varphi(n))^{p'}}\right\}_+ - p_n - \Pi_{n1} \leq$$

$$\leq \mathbb{E}\left\{|S_n| - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ \leq$$

$$\leq \sqrt{B_n}\mathbb{E}\left\{|N| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} + \frac{3}{(\varphi(n))^{p'}}\right\}_+ + p_n + \Pi_{n2}.$$

While by Propositions 3.2 – 3.4, for $1/2 < p' < p \leq 2$ and $q \geq 2 + \max\{(p' + 1/2)/(p - p'), 4p/(2p+1)\}$, we have

$$\sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}}(p_n + \Pi_{n1} + \Pi_{n2}) < \infty.$$

Next, let $a_n'(\varepsilon) = a_n(\varepsilon) \pm 3/(\sqrt{2}(\varphi(n))^{p'+1/2})$, then we get

$$a_n'(\varepsilon)\varphi(n) \to \tau, \quad \text{as} \quad n \to \infty \quad \text{and} \quad \varepsilon \searrow \sqrt{r}.$$

Since $B_n/n \to 1$ as $n \to \infty$, then by Proposition 3.1, we obtain

$$\lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \times$$

$$\times \sqrt{B_n}\mathbb{E}\left\{\sup_{0\leq s\leq 1} |W(s)| - (\varepsilon + a_n(\varepsilon))\sqrt{2\varphi(n)} \pm \frac{3}{(\varphi(n))^{p'}}\right\}_+ =$$

$$= \lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n}\mathbb{E}\left\{\sup_{0\leq s\leq 1} |W(s)| - (\varepsilon + a_n'(\varepsilon))\sqrt{2\varphi(n)}\right\}_+ =$$

$$= \frac{2e^{-2\tau\sqrt{r}}}{\sqrt{2\pi r}}.$$

It follows that

$$\lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ = \frac{2e^{-2\tau\sqrt{r}}}{\sqrt{2\pi r}}. \tag{3.10}$$

Similarly, it holds that

$$\lim_{\varepsilon\searrow\sqrt{r}} \frac{1}{-\log(\varepsilon^2 - r)} \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{|S_n| - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ = \frac{e^{-2\tau\sqrt{r}}}{\sqrt{2\pi r}}. \tag{3.11}$$

Finally, since $\mathbb{E}X^2 I\{|X| \geq t\} = o((\varphi(t))^{-1})$ as $t \to \infty$, it follows that

$$0 \leq n - B_n \leq 2n\mathbb{E}X^2 I\left\{|X| > \frac{\sqrt{n}}{(\varphi(n))^p}\right\} = o\left(\frac{n}{\varphi(n)}\right).$$

Let

$$a_n^*(\varepsilon) = \sqrt{\frac{B_n}{n}}(\varepsilon + a_n(\varepsilon)) - \varepsilon = \sqrt{\frac{B_n}{n}}a_n(\varepsilon) - \frac{\varepsilon(n - B_n)}{\sqrt{n}(\sqrt{n} + \sqrt{B_n})}.$$

Then

$$a_n^*(\varepsilon) = a_n(\varepsilon) + o\left(\frac{1}{\varphi(n)}\right), \quad \text{as} \quad n \to \infty, \quad \varepsilon \searrow \sqrt{r},$$

and

$$\mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ = \mathbb{E}\left\{M_n - (\varepsilon + a_n^*(\varepsilon))\sqrt{2n\varphi(n)}\right\}_+,$$

$$\mathbb{E}\left\{|S_n| - (\varepsilon + a_n(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+ = \mathbb{E}\left\{|S_n| - (\varepsilon + a_n^*(\varepsilon))\sqrt{2n\varphi(n)}\right\}_+.$$

Therefore, (1.5) and (1.4) follow from (3.10) and (3.11), respectively.

***Proof of the converse part of Theorem* 1.1.** First we prove $\mathbb{E}X = 0$. From (1.2), we get, for all $\varepsilon > \sqrt{r}$, that

$$\infty > \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \mathbb{E}\left\{|S_n| - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ =$$

$$= \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \int_0^{\infty} \mathbb{P}\left\{|S_n| \geq (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)} + x\right\} dx \geq$$

$$\geq \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \int_0^{\tau(n)} \mathbb{P}\left\{|S_n| \geq (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)} + x\right\} dx \geq$$

$$\geq \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}}{n^{3/2}} \int_0^{\tau(n)} \mathbb{P}\left\{|S_n| \geq (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)} + \tau(n)\right\} dx \geq$$

$$\geq C \sum_{n=1}^{\infty} \frac{(\log n)^{r}}{n} \mathbb{P}\left\{|S_n| \geq \varepsilon\sigma\sqrt{2n}\log n\right\} \geq$$

$$\geq C \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left\{|S_n| \geq n\right\}, \tag{3.12}$$

where $\tau(n) = \sigma\varepsilon\sqrt{n}\log n$, then we have $\mathbb{E}X = 0$.

Now we show $\mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{-1/2} < \infty$. From (3.12) and Lemma 2.4, we take $\tau(n) = a\sigma\sqrt{2n\varphi(n)}$ with any $a > 0$ and $\varepsilon' = \varepsilon + a > \sqrt{r}$, then

$$\infty > C \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}\sqrt{\varphi(n)}}{n} \mathbb{P}\left\{|S_n| \geq (\varepsilon' + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\} \geq$$

$$\geq C \sum_{n=1}^{\infty} \frac{(\log n)^{r-1}\sqrt{\varphi(n)}}{n} \mathbb{P}\left\{|S_n| \geq 2\varepsilon'\sigma\sqrt{2n\varphi(n)}\right\} =$$

$$= C_1 + C \sum_{n=N_0}^{\infty} \frac{(\log n)^{r-1}\sqrt{\varphi(n)}}{n} \mathbb{P}\left\{|S_n| \geq 2\varepsilon'\sigma\sqrt{2n\varphi(n)}\right\} \geq$$

$$\geq C \sum_{n=1}^{\infty} (\log n)^{r-1}\sqrt{\varphi(n)}\,\mathbb{P}\left\{|X| \geq T\sqrt{n\varphi(n)}\right\} \quad (T \geq 8\sqrt{2}\varepsilon'\sigma) =$$

$$= C\sum_{n=1}^{\infty}\frac{\sqrt{\varphi(n)}}{(\log n)^{1-r}}\sum_{j=n}^{\infty}\mathbb{E}\left(I\left\{\sqrt{j\varphi(j)}\le\frac{|X|}{T}<\sqrt{(j+1)\varphi(j+1)}\right\}\right)\ge$$

$$\ge C\sum_{j=1}^{\infty}j\frac{\sqrt{\varphi(j)}}{(\log j)^{1-r}}\mathbb{E}\left(I\left\{\sqrt{j\varphi(j)}\le\frac{|X|}{T}<\sqrt{(j+1)\varphi(j+1)}\right\}\right)\ge$$

$$\ge C\mathbb{E}X^2(\log|X|)^{r-1}(\varphi(|X|))^{-1/2}.$$

Next we prove $\mathbb{E}X^2<\infty$. Let $\{\tilde X,\tilde X_n,n\ge1\}$ be the symmetrization of $\{X,X_n,n\ge1\}$. Set $\tilde S_n=\sum_{i=1}^n\tilde X_i$, then, for large n and $\varepsilon>\sqrt r$, it holds that

$$\mathbb{E}\left\{|\tilde S_n|-2(\varepsilon+a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+\le2\mathbb{E}\left\{|S_n|-(\varepsilon+a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+.$$

By (1.2), we have

$$\lim_{\varepsilon\searrow\sqrt r}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{|\tilde S_n|-2(\varepsilon+a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+\le\frac{2\sigma e^{-2\tau\sqrt r}}{\sqrt{2\pi r}}.$$

We define $Y=\tilde XI\{|\tilde X|<K\}$ and $Y_n=\tilde X_nI\{|\tilde X_n|<K\}$ for $K>0$. Notice that $\tilde XI\{|\tilde X|<K\}-\tilde XI\{|\tilde X|\ge K\}$ has the same distribution as $\tilde X$, and $2Y=\tilde XI\{|\tilde X|<K\}-\tilde XI\{|\tilde X|\ge K\}+\tilde X$, then by (1.2), we obtain

$$\lim_{\varepsilon\searrow\sqrt r}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{\left|\sum_{i=1}^n2Y_i\right|-4(\varepsilon+a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+\le$$

$$\le2\lim_{\varepsilon\searrow\sqrt r}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{|\tilde S_n|-2(\varepsilon+a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+\le$$

$$\le\frac{4\sigma e^{-2\tau\sqrt r}}{\sqrt{2\pi r}}.$$

Since $|Y|<K$, following the proof of the direct part, we have

$$\lim_{\varepsilon\searrow\sqrt r}\frac{1}{-\log(\varepsilon^2-r)}\sum_{n=1}^{\infty}\frac{(\log n)^{r-1}}{n^{3/2}}\mathbb{E}\left\{\left|\sum_{i=1}^n2Y_i\right|-2(\varepsilon+a_n(\varepsilon))\sqrt{(\mathbb{E}Y^2)2n\varphi(n)}\right\}_+=$$

$$=\frac{2\sqrt{\mathbb{E}Y^2}e^{-2\tau\sqrt r}}{\sqrt{2\pi r}}.$$

Therefore, $\mathbb{E}(\tilde X^2I\{|\tilde X|<K\})=\mathbb{E}Y^2\le4\sigma^2$. Then let $K\to\infty$, we get $\mathbb{E}X^2<\infty$. At last, following the proof of the direct part, we have $\mathbb{E}X^2=\sigma^2$.

Finally, we show (1.6). Suppose that (1.6) is not true. Without loss of generality, we can assume that $\sigma^{-2}\mathbb{E}X^2I\{|X|\ge\sqrt n/(\varphi(n))^2\}\ge\tau_0/\varphi(n)$, for some $\tau_0>0$ and all $n\ge1$. Then

$$n\sigma^2-B_n\ge n\mathbb{E}X^2I\left\{|X|\ge\frac{\sqrt n}{(\varphi(n))^2}\right\}\ge n\sigma^2\tau_0/\varphi(n).$$

Let

$$a_n^*(\varepsilon) = \sqrt{1 + \frac{\tau_0}{\varphi(n)}}(\varepsilon + a_n(\varepsilon)) - \varepsilon.$$

Then

$$\lim_{n\to\infty,\varepsilon\searrow\sqrt{r}} a_n^*(\varepsilon)\varphi(n) = \tau + \tau_0\sqrt{r}/2$$

and

$$\mathbb{E}\left\{M_n - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ \le \mathbb{E}\left\{M_n - (\varepsilon + a_n^*(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+,$$

$$\mathbb{E}\left\{|S_n| - (\varepsilon + a_n(\varepsilon))\sigma\sqrt{2n\varphi(n)}\right\}_+ \le \mathbb{E}\left\{|S_n| - (\varepsilon + a_n^*(\varepsilon))\sqrt{2B_n\varphi(n)}\right\}_+.$$

It follows that (1.4) and (1.5) are contradictory to (3.11) and (3.10), respectively.

By the way, the result of Corollary 1.1 is obvious from the proof of Theorem 1.1, and we omit its proof.

References

1. *Billingsley P.* Convergence of probability measures. – New York: Wiley, 1968.
2. *Csörgö M., Révész P.* Strong approximations in probability and statistics. – New York: Acad. Press, 1981.
3. *Davis J. A.* Convergence rates for the law of the iterated logarithm // Ann. Math. Statist. – 1968. – **39**. – P. 1479 – 1485.
4. *Erdös P.* On a theorem of hsu and robbins // Ann. Math. Statist. – 1949. – **20**. – P. 261 – 291.
5. *Erdös P.* Remark on my paper "on a theorem of hsu and robbins" // Ann. Math. Statist. – 1950. – **21**. – P. 138.
6. *Gut A.* Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices // Ann. Probab. – 1980. – **8**. – P. 298 – 313.
7. *Gut A., Spătaru A.* Precise asymptotics in the law of the iterated logarithm // Ann. Probab. – 2000. – **28**. – P. 1870 – 1883.
8. *Hsu P. L., Robbins H.* Complete convergence and the strong law of large numbers // Proc. Nat. Acad. Sci. USA. – 1947. – **33**. – P. 25 – 31.
9. *Jiang Y., Zhang L.-X.* Precise rates in the law of the logarithm for the moment of i.i.d. random variables // J. Math. Anal. and Appl. – 2007. – **327**. – P. 695 – 714.
10. *Jiang Y., Zhang L.-X., Pang T.-X.* Precise rates in the law of iterated logarithm for the moment of i.i.d. random variables // Acta Math. Sinica. English Ser. – 2006. – **22**. – P. 781 – 792.
11. *Li D., Wang X.-C., Rao M.-B.* Some results on convergence rates for probabilities of moderate deviations for sums of random variables // Int. J. Math. Sci. – 1992. – **15**. – P. 481 – 498.
12. *Liu W.-D., Lin Z.-Y.* Precise asymptotics for a new kind of complete moment convergence // Statist. Probab. Lett. – 2006. – **76**. – P. 1787 – 1799.
13. *Petrov V. V.* Limit theorem of probability theory. – Oxford: Oxford Univ. Press, 1995.

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