UDC 517.5

A. E. Patkowski (USA)

REMARKS ON A BAILEY PAIR WITH ONE FREE PARAMETER

ЗАУВАЖЕННЯ ЩОДО ПАРИ БЕЙЛІ З ОДНИМ ВІЛЬНИМ ПАРАМЕТРОМ

We offer a more general Bailey pair than the pair obtained in the papers [*Andrews G. E.*, Adv. Combin., Waterloo Workshop in Comput. Algebra (May 26 – 29, 2011), Springer (2013), p. 57 – 76] and [*Patkowski A. E.*, Discrete Math., **310**, 961 – 965 (2010)] by two different methods.

Запропоновано більш загальну пару Бейлі, ніж та, що була отримана в роботах [*Andrews G. E.*, Adv. Combin., Waterloo Workshop in Comput. Algebra (May 26 – 29, 2011), Springer (2013), p. 57 – 76] та [*Patkowski A. E.*, Discrete Math., **310**, 961 – 965 (2010)] двома різними методами.

1. Introduction. Recall that a Bailey pair $(\alpha_n(a,q), \beta_n(a,q)) = (\alpha_n, \beta_n)$ is a pair of sequences that satisfy (relative to a) [6]

$$\beta_n = \sum_{i\geq 0} \frac{\alpha_i}{(q)_{n-i}(aq)_{n+i}}.$$

(Refer to [10] for q-series notation.) In [22] (Lemma 2.2) we offered a new Bailey pair for

$$\beta_n = \frac{(-1)^n q^{n^2}}{(-a)_{2n}(q^2;q^2)_n}, \tag{1.1}$$

where $a = 1$, or $a = q$. As a direct consequence of this pair, we were able to offer new information on the distinct rank parity function

$$\sigma(q) = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\ldots(1+q^n)}.$$

The function $\sigma(q)$ was shown in [1] to be related to the arithmetic of $\mathbb{Q}(\sqrt{6})$, and therefore $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{3})$. The key to this observation was the use of Bailey pairs to relate $\sigma(q)$ to indefinite quadratic forms.

Andrews [5, p.72] (equation (45)) also offered a proof of the pair with the β_n given in (1.1) with $a = q$ using a recurrence approach, and subsequently establishing new partition theorems using different limiting cases of Bailey's lemma than used in [12].

2. The Bailey pair. We will apply the same proof offered in [12] but in greater generality.

Theorem 2.1. *With free parameter* x, *we have the Bailey pair* $(\alpha'_n(a^2,x,q^2), \beta'_n(a^2,x,q^2))$, *where*

$$\alpha'_n(a^2,x,q^2) = \frac{q^{n^2-n}(-x)^n(1-a^2q^{4n})(a^2/x;q^2)_n}{(1-a^2)(q^2x;q^2)_n} \sum_{0\leq j\leq n} \frac{(1-aq^{2j-1})(a)_{j-1}(x;q^2)_j}{q^{j(j-1)/2}x^j(q)_j(a^2/x;q^2)_j},$$

$$\beta'_n(a^2,x,q^2) = \frac{(-1)^n q^{n^2}(1-x)}{(-a)_{2n}(q^2;q^2)_n(1-xq^{2n})}.$$

Proof. From [2] (Theorem 2.3) we have the pair (α_n, β_n) relative to a, where

$$\alpha_n(a,b,c,q) = \frac{q^{n^2}(bc)^n(1-aq^{2n})(a/b)_n(a/c)_n}{(1-a)(bq,cq)_n} \sum_{0\le j\le n} \frac{(-1)^j(1-aq^{2j-1})(a)_{j-1}(b,c)_j}{q^{j(j-1)/2}(bc)^j(q,a/b,a/c)_j}, \quad (2.1)$$

$$\beta_n(a,b,c,q) = \frac{1}{(bq,cq)_n}. \quad (2.2)$$

Recall from [12] (equations (2.10), (2.11)) the result,

$$\bar{\alpha}_n(a^2,q^2) = \frac{(1+aq^{2n})}{(1+a)q^n}\alpha_n(a,q), \quad (2.3)$$

$$\bar{\beta}_n(a^2,q^2) = \frac{q^{-n}}{(-a;q)_{2n}} \sum_{k\ge 0} \frac{(-1)^{n-k}q^{(n-k)^2-(n-k)}}{(q^2;q^2)_{n-k}}\beta_k(a,q). \quad (2.4)$$

The equations (2.3), (2.4) allow us to change the base of a Bailey pair from q to q^2.

From Fine [9, p. 17] (equation (15.51)) we have

$$\sum_{i\ge 0} \frac{(-1)^{n-i}q^{\{(n-i)^2-(n-i)\}/2}}{(q)_{n-i}(bq)_i} = \frac{(-1)^n q^{n(n+1)/2}(1-b)}{(q)_n(1-bq^n)}. \quad (2.5)$$

Now putting $b=-c$ in (2.1), (2.2), replacing c by $\sqrt{x}$, and then inserting the resulting pair in (2.3), (2.4) gives Theorem 2.1 after noting (2.5).

We now offer some corollaries as special cases of Theorem 2.1 that will be noted in the next section.

***Corollary* 2.1.** *We have the Bailey pair*

$$\alpha'_n(q^2,-q,q^2) = \frac{(-1)^n q^{n(n-1)/2}(1-q^{2n+1})}{(1-q)},$$

$$\beta'_n(q^2,-q,q^2) = \frac{(-1)^n q^{n^2}}{(-q)_{2n+1}(q^2;q^2)_n}.$$

***Corollary* 2.2.** *We have the Bailey pair*

$$\alpha'_n(q^4,q,q^2) = \frac{(-1)^n q^{n^2}(1-q^{4n+4})}{(1-q^4)} \sum_{0\le j\le n} q^{-j(j+1)/2},$$

$$\beta'_n(q^4,q,q^2) = \frac{(-1)^n q^{n^2}}{(-q)_{2n}(q^2;q^2)_n(1-q^{4n+2})}.$$

***Corollary* 2.3.** *We have the Bailey pair*

$$\alpha'_n(q^2,q,q^2) = \frac{(-1)^n q^{n^2}(1+q^{2n+1})}{(1-q^2)} \left(\sum_{0\le j\le n} q^{-j(j+1)/2} + \sum_{0\le j\le n-1} q^{-j(j+1)/2} \right),$$

$$\beta'_n(q^2,q,q^2) = \frac{(-1)^n q^{n^2}}{(-q)_{2n}(q^2;q^2)_n(1-q^{2n+1})}.$$

3. Some partitions and q-series. We will use some special instances of Bailey's lemma [6]

$$\sum_{n\geq 0}(X)_n(Y)_n(aq/XY)^n\beta_n = \frac{(aq/X)_\infty(aq/Y)_\infty}{(aq)_\infty(aq/XY)_\infty}\sum_{n\geq 0}\frac{(X)_n(Y)_n(aq/XY)^n\alpha_n}{(aq/X)_n(aq/Y)_n}. \tag{3.1}$$

In [1] Andrews et al. considered the function

$$\sigma^*(q) = \sum_{n\geq 1}\frac{(-1)^n q^{n^2}}{(q;q^2)_n},$$

which generates $O(n)$, the number of partitions of n into odd parts with the property, that if a number appears, then all smaller numbers appear as parts as well, weighted by -1, if the largest part is congruent to $1 \pmod 4$, and $+1$, if the largest part is congruent to $3 \pmod 4$. It was noted in [1] that $\sigma^*(q)$ is also related to $\mathbb{Q}(\sqrt{6})$. Further notes on $\sigma^*(q)$ can be found in [11], and more examples related to real quadratic fields are given in [5, 7, 8, 11, 12]. We consider a similar function

$$\sum_{n\geq 1}\frac{(-1)^n q^{n^2}}{(q;q^2)_n(1+q^{2n-1})} = \sum_{n\geq 1}\frac{(-1)^n q^{1+3+\ldots+2n-1}}{(1-q)(1-q^3)\ldots(1-q^{2n-3})(1-q^{(2n-1)+(2n-1)})}. \tag{3.2}$$

The q-series in (3.2) generates the same partitions counted by $O(n)$ (and same weight function) with the additional condition that the largest part appears an odd number of times. We denote such partitions to be $O^*(n)$.

***Corollary* 3.1.** *We have*

$$2\sum_{n\geq 1}O^*(n)(-q)^n = \sum_{n\geq 1}q^{n^2}\sum_{j=-n}^{n-1}q^{-j(j+1)/2}.$$

Proof. Take the Bailey pair in Corollary 2.2 and insert it into (3.1) (with $a = q^2$, then $q \to q^2$) with $X = q^2$, $Y = -q^2$. The proof is complete after multiplying both sides by 2 upon noting that $2\sum_{0\leq j\leq n} q^{-j(j+1)/2} = \sum_{j=-n-1}^{n} q^{-j(j+1)/2}$ and then shift the resulting indefinite quadratic sum over n with $n \to n-1$.

Set $L = \mathbb{Q}(\sqrt{2})$, let O_L be the ring of integers of L, and let $\breve{a} \subset O_L$ denote an ideal. For such an ideal we denote its norm function to be $N(\breve{a})$. Corollary 3.1 may be used to relate $O^*(n)$ to the number of inequivalent elements of O_L with norm $N(\breve{a}) = 2x^2 - y^2$.

***Corollary* 3.2.** *We have*

$$2q^{-1}\sum_{n\geq 1}O^*(n)q^{8n} = \sum_{\substack{\breve{a}\subset O_L \\ N(\breve{a})\equiv -1 \pmod 8}}(-1)^{\frac{N(\breve{a})+1}{8}}q^{N(\breve{a})}.$$

Proof. We use [1] (Lemma 3) and Corollary 3.1. Since $2(-1)^n O^*(n)$ is equal to the number of solutions of $n = i^2 - j(j+1)/2$ with $-i \leq j \leq i-1$, $i \geq 1$, we may write $2(2i)^2 - (2j+1)^2 = 8n-1$. Any solution of $N(\breve{a}) = 2x^2 - y^2 = 8n-1$, $n \in \mathbb{N}$, must have y odd, and subsequently x even. Further, if we write $x = 2i$, $y = 2j+1$, we have

$$-(2i) < (2j+1) \leq (2i), \quad i > 0. \tag{3.3}$$

The solutions of (3.3) are precisely the pairs (j,i) that satisfy $-i \leq j \leq i-1$, $i \geq 1$.

***Corollary* 3.3** ([3], Entry 9.4.3). *We have*

$$\sum_{n\geq 0}\frac{q^{n(2n+1)}}{(-q)_{2n+1}}=\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1}).$$

Proof. Insert the Bailey pair in Corollary 2.1 into (3.1) with $X=q^2$, $Y\to\infty$.

We mention this result in passing only to emphasize that Theorem 2.1 contains a broad range of identities. Corollary 3.3 has appeared in [3, p. 233] (equation (9.4.4)), and was noted in [12] due to its relevance to the q-series

$$\sum_{n\geq 0}\frac{q^{n(2n+1)}}{(-q)_{2n}},$$

which was found to be lacunary and related to $\sigma(q)$ in [12], by using the $x\to 0$ instance of Theorem 2.1.

In [7] Bringmann and Kane consider the q-series

$$f_1(q)=\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(-q)_n(1-q^{2n+1})},$$

and related it to the arithmetic of $\mathbb{Q}(\sqrt{2})$. We offer some further information on $f_1(q)$.

***Corollary* 3.4.** *We have*

$$\sum_{n\geq 0}\frac{q^{n(n+1)}}{(-q^2;q^2)_n(1-q^{2n+1})}=\sum_{n\geq 0}q^{n(n+1)}(1+q^{2n+2})\sum_{j=0}^{n}q^{-j(j+1)/2}. \tag{3.4}$$

Therefore, the "even function" of $f_1(q^2)$ is a generating function for a lacunary sequence.

Proof. Let $f_1'(q)$ be the left-hand side of (3.4). Then clearly, $(f_1'(q)+f_1'(-q))/2=f_1(q^2)$. The result follows after inserting the Bailey pair in Corollary 2.2 into the $X=q^2$, $Y=-q^3$ instance of (3.1) to get (3.4).

Another proof may be obtained using Corollary 2.3, and we leave this to the reader.

***Corollary* 3.5.** *Let $s_+t(n)$ be the number of representations of n as a sum of a triangular number i and a square j weighted by $(-1)^j$ or*

$$s_+t(n)=\sum_{\substack{r\in\mathbb{Z},k\geq 0\\ n=r^2+k(k+1)/2}}(-1)^{r^2}.$$

Then

$$\sum_{n\in\mathbb{Z},m\geq 0}(-1)^n q^{n^2+m(m+1)/2}=\sum_{n\geq 0}s_+t(n)q^n=$$

$$=\sum_{n\geq 0}(-1)^n q^{n(n+1)}\left(\sum_{0\leq j\leq n}q^{-j(j+1)/2}+\sum_{0\leq j\leq n-1}q^{-j(j+1)/2}\right).$$

Proof. We use the $X = -q$, $Y = -q^2$ case of (3.1) coupled with the Bailey pair in Corollary 2.3 and invoke the identity (a special limiting case of [9, p. 18] (equation (16.3))

$$\sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)}(1-x)}{(q^2;q^2)_n(1-xq^{2n})} = \frac{(q^2;q^2)_\infty}{(xq^2;q^2)_\infty}.$$

We note in closing that Corollary 3.5 gives a mapping between the number of inequivalent elements of O_L with norm $8n+1$ to the number of inequivalent elements of $O_{L'}$, where $L' = \mathbb{Q}(\sqrt{-2})$, with norm $8n+1$.

We also mention it is possible to obtain a more general expansion for the product

$$\frac{(q)_\infty(q^2;q^2)_\infty}{(-q)_\infty(xq^2;q^2)_\infty} = \sum_{n\geq 0} q^{n^2}(-x)^n(1-q^{2n+1})\frac{(q^2/x;q^2)_n}{(q^2x;q^2)_n} \sum_{0\leq j\leq n} \frac{(1+q^j)(x;q^2)_j}{q^{j(j-1)/2}x^j(q^2/x;q^2)_j}, \quad (3.5)$$

using the same limiting case of (3.1) with the $a = q$ case of Theorem 2.1. A nice corollary of (3.5) with $x \to 0$ is the famous expansion due to Rogers [13] for the weight 1 modular form

$$\prod_{n\geq 1}(1-q^n)^2 = \sum_{n\geq 0} q^{n(2n+1)}(1-q^{2n+1}) \sum_{|j|\leq n}(-1)^j q^{-j(3j+1)/2}.$$

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