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ON PRINCIPAL IDEAL MULTIPLICATION MODULES

ПРО ГОЛОВНІ ІДЕАЛЬНІ МУЛЬТИПЛІКАТИВНІ МОДУЛІ

Let R be a commutative ring with identity and let M be a unitary R-module. A submodule N of M is said to be a multiple of M if $N = rM$ for some $r \in R$. If every submodule of M is a multiple of M, then M is said to be a principal ideal multiplication module. We characterize principal ideal multiplication modules and generalize some results from [*Azizi A.* Principal ideal multiplication modules // Algebra Colloq. – 2008. – **15**. – P. 637 – 648].

Нехай R — комутативне кільце з одиницею, а M — унітарний R-модуль. Субмодуль N модуля M називається кратним M, якщо $N = rM$ для деякого $r \in R$. Якщо кожний субмодуль модуля M є кратним для M, то M називається головним ідеальним мультиплікативним модулем. У роботі наведено характеристики головних ідеальних мультиплікативних модулів та узагальнено деякі результати з роботи [*Azizi A.* Principal ideal multiplication modules // Algebra Colloq. – 2008. – **15**. – P. 637 – 648].

1. Introduction. Throughout this paper R denotes a commutative ring with identity and M denotes a unitary R-module. Also $L(R)$ (resp. $L(M)$) denotes the lattice of all ideals (resp. submodules) of R (resp. M).

A submodule N of M is *proper* if $N \neq M$. For any two submodules N and K of M, the ideal $\{a \in R \mid aK \subseteq N\}$ will be denoted by $(N : K)$. Thus $(O : M)$ is the annihilator of M. A module M is said to be *faithful* if $(O : M)$ is the zero ideal of R. We say that a module M is a *multiplication module* [7] if every submodule of M is of the form IM, for some ideal I of R.

According to [1] a submodule N of M is said to be a *multiple of* M if $N = rM$ for some $r \in R$. Also if every submodule of M is a multiple of M, then M is said to be a *principal ideal multiplication module* or a *PI-multiplication module*, for abbreviation. In [1] some properties and examples of PI-multiplication modules are given.

In this paper, we show that PI-multiplication modules are very close to cyclic modules over principal ideal rings. We will give many conditions under each of which a PI-multiplication module is a cyclic module over a principal ideal ring. Indeed our attempt for finding a noncyclic, nonzero PI-multiplication module was unsuccessful.

A proper submodule N of M is a *prime submodule*, if for any $r \in R$ and $m \in M$, $rm \in N$ implies either $m \in N$ or $r \in (N : M)$.

By a *minimal prime submodule over* a submodule N of M (or a *prime submodule minimal over* N), we mean a prime submodule which is minimal in the collection of all prime submodules containing N. Minimal prime submodules over the zero submodule are simply called the minimal prime submodules. It is well known that maximal submodules and prime submodules exist in multiplication modules (for details, see [7]). It is well known that if M is a multiplication R-module and P is a prime ideal of R containing $(O : M)$ such that $M \neq PM$, then PM is a prime submodule of M and every prime submodule of M is of the form PM for some prime ideal P of R containing

$(O\colon M)$ (see [7], Corollary 2.11). Also if M is a finitely generated multiplication R-module and P is a prime ideal of R containing $(O\colon M)$, then PM is a proper prime submodule of M (see Lemma 1.1(iii), in the following). Further PM is minimal over a submodule N of M if and only if P is minimal over the ideal $(N\colon M)$ of R.

According to [14], a submodule N of M is called *quasicyclic* if $(B \cap (K\colon N))N = BN \cap K$ and $(K + BN\colon N) = (K\colon N) + B$ for all ideals B of R and for all submodules K of M.

Note that N is quasicyclic, if and only if N is finitely generated and locally cyclic, if and only if N is a finitely generated multiplication submodule (by [14], Theorem 6 and Lemma 1.1(v)(c), in the following).

Recall that an R-module M is called a *cyclic submodule module (CSM)*, if every submodule of M is cyclic. A module M (resp. ring R) is said to be *Laskerian* [10], if every proper submodule (resp. ideal) is a finite intersection of primary submodules (resp. ideals).

A module M is said to be *finitely annihilated* if there exists a finite subset T of M with $(O\colon T) = (O\colon M)$. The finitely annihilated concept is due to G. Gabriel [8]. Evidently every finitely generated module is finitely annihilated. Also according to [1] (Proposition 4.7(i)), every semi-nontorsion module is finitely annihilated. Also note that the $\mathbb{Z}$-module $\mathbb{Q}$ is finitely annihilated, but not finitely generated.

Let M be an R-module and P a prime ideal of R. We define $T_P(M) = \{x \in M \mid (1-p)x = 0,\ \text{for some}\ p \in P\}$. It is said that M is P*-torsion* if $T_P(M) = M$. We say that M is P*-cyclic* if there exist $p_0 \in P$ and $m \in M$ such that $(1-p_0)M \subseteq Rm$.

For any $a \in R$, the principal ideal generated by a is denoted by (a). Recall that an ideal I of R is called a *multiplication ideal* if I is a multiplication R-module.

Every multiplication ideal is locally principal (see Lemma 1.1(v)(b), in the following).

An ideal I of R is called a *quasiprincipal ideal* [15, p. 147] (Exercise 10) (or a principal element of $L(R)$ [17]) if I is quasicyclic as an element of $L(R)$, that is it satisfies the identities (i) $(A \cap (B\colon I))I = AI \cap B$ and (ii) $(A + BI\colon I) = (A\colon I) + B$, for all $A,\ B \in L(R)$.

Obviously, every quasiprincipal ideal is a multiplication ideal. It should be mentioned that every quasiprincipal ideal is finitely generated and also a finite product of quasiprincipal ideals of R is again a quasiprincipal ideal [15, p. 147] (Exercise 10). In fact, an ideal I of R is quasiprincipal if and only if it is finitely generated and locally principal (see [4], Theorem 4), or [17], Theorem 2). It is well known that R is a general *ZPI*-ring if and only if every ideal is quasiprincipal [11] (Theorem 2.2).

A ring R is a π*-ring* if every principal ideal is a finite product of prime ideals of R.

If $\{P_\alpha\}$ is the collection of all minimal prime ideals of an ideal I of R, then by an *isolated* P_α*-primary component* of I we mean the intersection Q_α of all P_α-primary ideals which contain I. The $kernel$ of I is the intersection of all ${Q_\alpha}'^s$. It is well known that if R is an almost multiplication ring, then every ideal is equal to its kernel [6] (Theorem 2.9).

For the convenience of reader, some results from our references, which are used frequently in this paper, have been gathered in the following lemma.

Lemma 1.1. *Let M be a nonzero R-module.*

(i) ([1], Proposition 3.5(ii)). *If M is finitely generated and every prime submodule of M is finitely generated, then M is a Noetherian module.*

(ii) ([7], Corollary 4.8). *If M is a Noetherian multiplication module, then there exist ideals $A \subseteq B$ of R such that $M \cong B/A$.*

(iii) ([7], Theorem 3.1). *Let M be a multiplication module. Then M is finitely generated, if and only if $M \neq PM$, for each maximal ideal P of R containing $(O\colon M)$, if and only if for any ideals A, B of R containing $(O\colon M)$, the inclusion $AM \subseteq BM$ implies that $A \subseteq B$.*

(iv) ([1], Theorem 4.9(i)). *If $R/(O\colon M)$ is a principal ideal ring and M is multiplication, then M is PI-multiplication.*

(v) (a) ([7], Theorem 3.7 and [13], Lemma 1). *If M is multiplication and M (resp. R) has only finitely many minimal prime submodules (resp. ideals), then M is finitely generated.*

(b) ([7], Theorem 2.8 and [3], Proposition 4). *If M is multiplication and M (resp. R) has only finitely many maximal submodules (resp. ideals), then M is cyclic.*

(c) ([3], Proposition 5). *Let M be a finitely generated module. Then M is multiplication if and only if it is locally cyclic.*

(vi) ([13], Lemmas 4 and 3). *If M is a faithful multiplication module, then M is a Laskerian module if and only if R is a Laskerian ring. Furthermore, in this case M is finitely generated.*

(vii) ([7], Corollary 3.9). *If M is multiplication and $R/(O\colon M)$ has ACC on semiprime ideals, then M is finitely generated.*

(viii) ([13], Lemma 6). *If M is cyclic, then a submodule N of M is cyclic if and only if N is a multiple of M.*

(ix) ([5], Theorem 2). *If M is a multiplication module and every minimal prime submodule is finitely generated, then M has only finitely many minimal prime submodules.*

(x) ([13], Lemma 7). *If every cyclic submodule of M has primary decomposition, then any locally cyclic submodule of M is finitely generated.*

For general background and terminology, the reader is referred to [15] and [19].

2. *PI*-multiplication modules.

Proposition 2.1. *Let M be a PI-multiplication R-module.*

(i) *If M is finitely generated, then $R/(O\colon M)$ is a principal ideal ring.*

(ii) *$R/(O\colon M)$ is a locally principal ideal ring.*

Proof. Clearly, M is a multiplication module. Let $\bar{I}$ be an ideal of $R/(O\colon M)$. Then $\bar{I} = J/(O\colon M)$ for some $J \in L(R)$ containing $(O\colon M)$. By hypothesis, $JM = rM$ for some $r \in R$. Then $JM = (Rr + (O\colon M))M$, and according to Lemma 1.1(iii), $J = Rr + (O\colon M)$. So $\bar{I} = J/(O\colon M) = (Rr + (O\colon M))/(O\colon M) = (R/(O\colon M))(r + (O\colon M))$. Therefore $R/(O\colon M)$ is a principal ideal ring.

(ii) Evidently M is a PI-multiplication $\bar{R}$-module, where $\bar{R} = R/(O\colon M)$. Let P be a prime ideal of R containing $(O\colon M)$. Put $\bar{P} = P/(O\colon M)$. According to [1] (Lemma 4.4(ii)), $M_{\bar{P}}$ is a PI-multiplication $\bar{R}_{\bar{P}}$-module. Now since $\bar{R}_{\bar{P}}$ is a local ring, by (1.1)(v)(b), $M_{\bar{P}}$ is a PI-multiplication cyclic $\bar{R}_{\bar{P}}$-module.

Therefore by (i), $\bar{R}_{\bar{P}}$ is a principal ideal ring.

***Example* 2.1.** Any Dedekind domain is a locally principal ring, but it is not necessarily a principal ideal ring. For example, the integral closure of $\mathbb{Z}$ in $\mathbb{Q}\left[\sqrt{10}\,\right]$ is a Dedekind domain, but it is not a *PID*.

The set of all prime submodules of M is denoted by $\mathrm{Spec}M$. Let $N \in \mathrm{Spec}M$. The height of N, which is denoted by $ht\ K$, is n, if there exists a chain $N_n \subset \ldots \subset N_2 \subset N_1 \subset N_0 = N$ in $\mathrm{Spec}M$ and there does not exist such a chain of greater length (see [2]).

The following theorem generalizes most of the results in [1]. This theorem introduces many conditions under which a PI-multiplication module is a cyclic module over a principal ideal ring.

Theorem 2.1 (main theorem)**.** *Suppose M is a nonzero R-module. Then the following are equivalent*:

(i) *M is a cyclic module and $R/(O\colon M)$ is a principal ideal ring.*

(ii) *M is finitely generated and every prime submodule of M is a multiple of M.*

(iii) *M is a finitely generated PI-multiplication module.*

(iv) *M is a finitely annihilated PI-multiplication module.*

(v) *M is a multiplication module and $R/(O\colon M)$ is a principal ideal ring.*

(vi) *M is a PI-multiplication module and $(O\colon M)$ contains an ideal which has a primary decomposition.*

(vii) *M is a PI-multiplication module and a finitely generated submodule of M has a primary decomposition.*

(viii) *M is a PI-multiplication module and $R/(O\colon M)$ has ACC on semiprime ideals.*

(ix) *M is cyclic and every prime submodule of M is cyclic.*

(x) *M is a CSM* (*i.e., every submodule of M is a cyclic submodule*).

(xi) *M is a PI-multiplication module and every minimal prime submodule is finitely generated.*

(xii) *M is a PI-multiplication module and $M_P \neq O$, for each maximal (prime) ideal P containing $(O\colon M)$.*

(xiii) *M is a PI-multiplication module and M is P-cyclic for each maximal ideal P containing $(O\colon M)$.*

(xiv) *M is a PI-multiplication module and there exists a nonnegative integer n such that $\{N \in \mathrm{Spec}M \mid ht\ N = n\}$ is nonempty and finite.*

Proof. (ii) $\Longrightarrow$ (iii). By hypothesis and by Lemma 1.1(i), M is a Noetherian module. We claim that every submodule of M is a multiple of M. Suppose not. Then by Zorn's lemma, there exists a submodule N such that N is a maximal element in the set of all nonmultiples of M. By hypothesis, N is not a prime submodule. So there exist elements $a \in R$ and $x \in M$ such that $ax \in N$, $x \notin N$ and $aM \nsubseteq N$. By the maximality of N, we have $N + Rx = cM$ for some $c \in R$.

Put $L = (N\colon {}_M c)$. We show that $N = cL$. Evidently $cL \subseteq N$. Now if $y \in N$, then since $N \subseteq N + Rx = cM$, there exists $z \in M$ such that $y = cz$. Then $z \in L$ and so $y = cz \in cL$.

Obviously $N \subseteq L$. Now we show that $N \neq L$. Note that $aM \nsubseteq N$, then let $am \notin N$, where $m \in M$. We have $cm \in cM = N + Rx$, then $cm = n + rx$, for some $n \in N$ and $r \in R$. Thus $c(am) = an + rax \in N$, which implies that $am \in L \setminus N$.

Now again by the maximality of N, we have $L = dM$, for some $d \in R$. Therefore, $N = cL = (cd)M$, which is a contradiction.

(iii) $\Longrightarrow$ (i). By Proposition 2.1(i), $R/(O\colon M)$ is a principal ideal ring.

Now since M is a nonzero Noetherian multiplication $R/(O\colon M)$-module, according to Lemma 1.1(ii), $M \cong B/A$, where A, B are ideals of $R/(O\colon M)$ with $A \subseteq B$. Note that $R/(O\colon M)$ is a principal ideal ring, then the ideal B is principal, which implies that M is a cyclic $R/(O\colon M)$-module and consequently a cyclic R-module.

(i) $\Longrightarrow$ (iv). The proof is clear.

(iv) $\Longrightarrow$ (ii). By Lemma 1.1(iii), it is enough to show that $M \neq PM$, for any maximal ideal of R containing $(O \colon M)$. On the contrary let $M = mM$, where m is a maximal ideal of R containing $(O \colon M)$. Note that M_m is a multiplication R_m-module and R_m is a local ring, then by Lemma 1.1(v)(b), M_m is a cyclic module. Now since $M_m = m_m M_m$, by Nakayama's lemma there exist $r \in R$ and $s \in R \setminus m$ such that $(r/s)M_m = O$ and $1 - (r/s) \in m_m$.

Suppose that $T = \{t_1, t_2, t_3, \ldots, t_n\}$ is a finite subset of M with $(O \colon T) = (O \colon M)$. Then $(r/s)(t_i/1) = 0$, for each $1 \leq i \leq n$. So there exists $s_i \in R \setminus m$ with $s_i r t_i = 0$, for each $1 \leq i \leq n$. Put $\bar{s} = s_1 s_2 s_3 \ldots s_n$. Then $\bar{s} r t_i = 0$, for each $1 \leq i \leq n$, which implies that $\bar{s} r \in (O \colon T) = (O \colon M) \subseteq m$. Hence $r \in m$, and so $r/s \in m_m$, which is a contradiction, since $1 - (r/s) \in m_m$.

(i) $\Longrightarrow$ (v). The proof is clear.

(v) $\Longrightarrow$ (vi). Note that every principal ideal ring is a Noetherian ring and any Noetherian ring is Laskerian. Then $(O \colon M)$ has a primary decomposition. Now the proof is completed by Lemma 1.1(iv).

(vi) $\Longrightarrow$ (ii). We show that M has only finitely many minimal prime submodules, and therefore by Lemma 1.1(v)(c), M is finitely generated.

Let N be a minimal prime submodule of M and suppose that $(O \colon M)$ contains an ideal I such that I has a primary decomposition. Let $I = \cap_{i=1}^{n} Q_i$, where Q_i is a primary ideal, for each $1 \leq i \leq n$. Then $\bigcap_{i=1}^{n} Q_i = I \subseteq (O \colon M) \subseteq (N \colon M)$, and so $\bigcap_{i=1}^{n} \sqrt{Q_i} \subseteq (N \colon M)$. Now since $(N \colon M)$ is a prime ideal, $\sqrt{Q_j} \subseteq (N \colon M)$, for some $1 \leq j \leq n$, and then $\sqrt{Q_j} M \subseteq N$. Now as $\sqrt{Q_j} M$ is a prime submodule of M and N is a minimal prime submodule, $\sqrt{Q_j} M = N$, which completes the proof.

(vi) $\Longrightarrow$ (vii). Evidently M is a faithful multiplication $R/(O \colon M)$-module and since parts (vi) and (v) are equivalent, $R/(O \colon M)$ is a Noetherian ring, and so it is a Laskerian ring. Then by Lemma 1.1(vi), M is a Laskerian $R/(O \colon M)$-module and clearly a Laskerian R-module. Thus the zero submodule has a primary decomposition.

(vii) $\Longrightarrow$ (iii). Let N be a finitely generated submodule of M that has a primary decomposition. We have two cases:

Case 1. $N = 0$.

Case 2. $N \neq 0$.

First suppose that Case 1 holds. Assume that $0 = \bigcap_{i=1}^{n} Q_i$, where Q_i is a primary submodule of M, for each $1 \leq i \leq n$. Hence $(O \colon M) = \bigcap_{i=1}^{n} (Q_i \colon M)$, which is a primary decomposition of $(0 \colon M)$. Now the proof is given by (vi) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Now let Case 2 is satisfied. It is easy to see that M/N is a PI-multiplication module and since it's zero submodule has a primary decomposition, by Case 1, M/N is a finitely generated module and since N is a finitely generated module, M is finitely generated.

(v) $\Longrightarrow$ (viii). The proof follows from Lemma 1.1(iv).

(viii) $\Longrightarrow$ (ii). By Lemma 1.1(vii), M is finitely generated.

(ii) $\Longrightarrow$ (ix). Suppose (ii) holds. By (ii) and (i), M is a cyclic module. Now by Lemma 1.1(viii), every prime submodule is a cyclic submodule.

(ix) $\Longrightarrow$ (x). By Lemma 1.1(viii), every prime submodule is a multiple of M and by the proof of (ii) $\Longrightarrow$ (iii), every submodule is a multiple of M and since M is cyclic, every submodule is a cyclic submodule of M.

(x) $\Longrightarrow$ (ii). The proof is given by Lemma 1.1(viii).

(x) $\Longrightarrow$ (xi). Note that (x) and (iii) are equivalent. So M is a PI-multiplication module.

(xi) $\Longrightarrow$ (iii). By Lemma 1.1 parts (ix) and (v)(i), M finitely generated.

(iii) $\Longrightarrow$ (xii). Suppose $M_P = O$, where P is a prime ideal of R containing $(O\colon M)$. Let M be generated by $x_1, x_2, \ldots, x_n$. For each $1 \leq i \leq n$ we have $x_i/1 = 0$ in M_P, hence there exists $s_i \in R \setminus P$ such that $s_i x_i = 0$. Consequently $s_1 s_2 \ldots s_n x_i = 0$, for each $1 \leq i \leq n$, that is $s_1 s_2 \ldots s_n \in (O\colon M) \subseteq P$. Hence $s_j \in P$, for some $1 \leq j \leq n$, which is a contradiction.

(xii) $\Longrightarrow$ (iii). According to Lemma 1.1(iii), it is enough to show that $M \neq PM$, for any maximal ideal P of R containing $(O\colon M)$. If $M = PM$, where P is a maximal ideal of R containing $(O\colon M)$. Then $P_P M_P = M_P$ and since by Lemma 1.1(v)(b), M_P is a cyclic R_P-module, Nakayama's lemma implies that there exists $r \in R_P$ such that $rM_P = O$ and $r - 1 \in P_P$. Now if $r \notin P_P$, then r is a unit in R_P, which implies that $M_P = O$. Also if $r \in P_P$, then $1 \in P_P$, which is a contradiction.

(xii) $\Longrightarrow$ (xiii). Let P be a maximal ideal of R containing $(O\colon M)$. By [7] (Theorem 1.2), M is P-torsion or P-cyclic. If $T_P(M) = M$, then for each $x \in M$ there exists $p \in P$ with $(1-p)x = 0$. So $x/1 = 0$ in M_P, that is $M_P = O$, which is a contradiction. Therefore M is P-cyclic.

(xiii) $\Longrightarrow$ (xii). Let P be a maximal ideal of R containing $(O\colon M)$. By our assumption there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$. If $M_P = O$, then $m/1 = 0$, and so there exists $s \in R \setminus P$ such that $sm = 0$. Hence $s(1-p)M = O$, i.e., $s(1-p) \in (O\colon M) \subseteq P$, which is impossible.

(xiv) $\Longrightarrow$ (i). Let N be an arbitrary prime submodule of M and let $\bar{P} = (N\colon M)/(0\colon M)$. Evidentely $\bar{P}$ is a prime ideal of $\bar{R} = R/(0\colon M)$. Then $ht\ \bar{P} = \dim \bar{R}_{\bar{P}}$ and since by Proposition 2.1, $\bar{R}_{\bar{P}}$ is a principal ideal ring, we have $ht\ P = \dim\ \bar{R}_{\bar{P}} \leq 1$.

Suppose that $N_1 \subset N_2 \subset N_3 \subset \ldots \subset N_n = N$ is a chain of prime submodules of M. Evidently $(O\colon M) \subset (N_1\colon M)$ and if $(N_i : M) = (N_{i+1} : M)$, for some $1 \leq i \leq n-1$, and so $N_i = (N_i : M)M = (N_{i+1} : M)M = N_{i+1}$. Thus $(N_1\colon M)/(O\colon M) \subset (N_2\colon M)/(O\colon M) \subset (N_3\colon M)/(O\colon M) \subset \ldots \subset (N_n\colon M)/(O\colon M) = \bar{P}$ is a chain of prime ideals of $\bar{R}$. Hence $ht\ N \leq ht\ \bar{P} \leq 1$, for any prime submodule N of M. Therefore for the integer n introduced in our assumption, either $n = 0$ or $n = 1$.

If $n = 0$, that is M has only finitely many minimal prime submodules, then Lemma 1.1(v)(a) implies that M is finitely generated. So the conditions of (iii) are satisfied and as (iii) implies (i), M is a cyclic module.

Now suppose that $n = 1$. As $ht\ K \leq 1$ for any prime submodule K of M, the elements of $\{N \in \mathrm{Spec}M \mid ht\ N = 1\}$ are exactly the maximal submodules of M. Thus there are only finitely many maximal submodules. So in this case according to Lemma 1.1(v)(b), M is cyclic.

(i) $\Longrightarrow$ (xiv). It is easy to see that if N is a minimal prime submodule of M, then $(N : M)/(O\colon M)$ is a minimal prime ideal of $R/(O\colon M)$ and since $R/(O\colon M)$ is a Noetherian ring, it has only finitely many minimal prime ideals. Hence $\{N \in \mathrm{Spec}M \mid ht\ N = 0\}$ is finite. Also note that according to [7] (Theorem 2.5), every multiplication module has a maximal submodule, and so $\{N \in \mathrm{Spec}M \mid ht\ N = 0\}$ is nonempty.

As a consequence, we have the following result, which is due to I. M. Isaacs.

***Corollary* 2.1.** *A ring R is a principal ideal ring if and only if every prime ideal of R is a principal ideal.*

To illustrate the Theorem 2.1, we provide the following example.

***Example* 2.2.** Let K be a field and $R = K[x]$ and consider $M = R/Rx^n$, where $n \in \mathbb{N}$. Then M is an R-module, and $(0\colon M) = Rx^n = (0\colon T)$, where $T = \{1 + Rx^n\}$, and so M is finitely annihilated. Also $R/(0\colon M)$ is a principal ideal ring, since R is a *PID*. As $\sqrt{(0\colon M)} = \sqrt{Rx^n} = Rx$ is a maximal ideal of R, the ideal $(0\colon M)$ is a primary ideal of R, particularly $(0\colon M)$ has a primary decomposition. Every submodule N of M is of the form $N = I/Rx^n$, where I is an ideal of R, and since R is a *PID, I* is a principal ideal of R, let $I = Rr$, where $r \in R$. Thus $N = rM$, and so M is a finitely generated (cyclic) *PI*-multiplication R-module. One can easily see that if $P = p/Rx^n$ is a prime submodule of M, then p is a prime ideal of R and since $x^n \in p$, we get $x \in p$, and so $Rx \subseteq p$. Note that Rx is a maximal ideal of R, thus $Rx = p$. This shows that M has just one prime submodule, which is Rx/Rx^n.

For the proof of the following lemma, see [16] (Lemma 1.4).

Lemma 2.1. *Suppose M is a faithful quasicyclic R-module and N a submodule of M. Then the following statements are equivalent*:

(i) *N is quasicyclic.*

(ii) *$(N : M)$ is quasiprincipal.*

(iii) *$N = IM$ for some quasiprincipal ideal I of R.*

Definition 2.1. *An R-module M is said to be a general quasicyclic module if every submodule of M is quasicyclic.*

Note that if M is a CSM, then M is a general quasicyclic module. But the converse is not true since in a ring R, quasiprincipal ideals need not be principal ideals. General ZPI-rings are examples of general quasicyclics.

***Example* 2.3.** Consider $R = M = Z[\sqrt{-5}]$. Then M is a general quasicyclic R-module, but it is not a CSM, as R is not a principal ideal ring.

The following theorem, establishes several characterizations for general quasicyclic modules.

Theorem 2.2. *Suppose M is a faithful R-module. Then the following statements are equivalent*:

(i) *R is a general ZPI-ring and M is a multiplication module.*

(ii) *M is a multiplication module, locally PI-multiplication module and every cyclic submodule has only finitely many prime submodules minimal over it.*

(iii) *M is a multiplication module, locally PI-multiplication module and R is a π-ring.*

(iv) *M is a locally PI-multiplication module in which every cyclic submodule has a primary decomposition.*

(v) *M is a general quasicyclic module* (*i.e., every submodule of M is quasicyclic*).

(vi) *M is quasicyclic and every prime submodule is quasicyclic.*

(vii) *M is a multiplication module, locally PI-multiplication module and every minimal prime submodule over any cyclic submodule is finitely generated.*

Proof. (i) $\Longrightarrow$ (ii). As R is a Laskerian ring, by Lemma 1.1(vi), M is a finitely generated Laskerian module. Since R is a general ZPI-ring, it follows that R is an almost multiplication ring and so R is a locally principal ideal ring.

As M is a finitely generated faithful multiplication module, it follows that $M_P \cong R_P$, for each prime ideal P of R, and R is a locally principal ideal ring. So M is a locally PI-multiplication module and (ii) holds.

(ii) $\Longrightarrow$ (iii). Note that by Lemma 1.1(v)(a), M is finitely generated. Again since for each maximal ideal P of R, $M_P \cong R_P$, it follows that R_P is a principal ideal ring. Therefore, R is an almost multiplication ring.

We show that every quasiprincipal ideal has only finitely many minimal prime ideals. Let I be a quasiprincipal ideal. Then IM is finitely generated. Suppose $IM = \sum_{i=1}^{n} Rx_i$ for some $x_1, \ldots, x_n \in M$. Then $IM = \sum_{i=1}^{n} Rx_i = \sum_{i=1}^{n} (Rx_i : M)M = \left(\sum_{i=1}^{n} (Rx_i : M)\right) M$, so by Lemma 1.1(iii), $I = \sum_{i=1}^{n} (Rx_i : M)$. Again since each Rx_i has only finitely many minimal prime submodules and M is a faithful finitely generated multiplication module, it follows that each ideal $(Rx_i\colon M)$ has only finitely many minimal primes. Let P be a prime ideal minimal over I. Then by [4] (Theorem 1), I_P is completely join irreducible in R_P, so $I_P = (Rx_i\colon M)_P$ for some i and so P is minimal over $(Rx_i\colon M)$. Therefore, I has only finitely many minimal prime ideals. As R is an almost multiplication ring, by [6] (Theorems 2.7 and 2.9), every ideal is equal to its kernel and so I has a primary decomposition. Again by [12] (Theorem 6), R is a π-ring.

(iii) $\Longrightarrow$ (iv). As R is a π-ring, then R has finitely many minimal prime ideals, so by Lemma 1.1(v)(a), M is finitely generated. Let Rx be a cyclic submodule. Then by Lemma 2.1, $(Rx\colon M)$ is quasiprincipal, so by [12] (Theorem 6), $(Rx\colon M) = \bigcap_{i=1}^{n} Q_i$ for some primary ideals $Q_1, \ldots, Q_n$ of R. So by [7] (Theorem 1.6), $Rx = (Rx\colon M)M = \left(\bigcap_{i=1}^{n} Q_i\right) M = \bigcap_{i=1}^{n}(Q_iM)$. Again by Lemma 1.1(iii) and [20] (Corollary 1), each Q_iM is a primary submodule. Therefore, every cyclic submodule has a primary decomposition. Thus (iv) holds.

(iv) $\Longrightarrow$ (v). Note that M is locally cyclic, by Lemma 1.1(v)(b). So by Lemma 1.1(viii), every submodule is locally cyclic. Now every submodule of M is finitely generated and locally cyclic, by Lemma 1.1(x) and thus by [14] (Theorem 6), every submodule is quasicyclic.

(v) $\Longrightarrow$ (vi). The proof is obvious.

(vi) $\Longrightarrow$ (i). As M is quasicyclic, it follows that M is a finitely generated multiplication module. Let P be a prime ideal of R. Then PM is a prime submodule of R, and so by our assumption PM is quasicyclic. Now by Lemma 2.1, $PM = IM$, for some quasiprincipal ideal I of R, and according to Lemma 1.1(iii), $P = I$. Hence P is quasiprincipal. As every prime ideal is quasiprincipal, it follows that every ideal is quasiprincipal and hence R is a general ZPI-ring.

(ii) $\Longrightarrow$ (vii). According to the proof of (iv) $\Longrightarrow$ (v), every submodule of M is finitely generated.

(vii) $\Longrightarrow$ (ii). As every minimal prime submodule over any cyclic submodule is finitely generated, by Lemma 1.1(ix), every cyclic submodule has only finitely many prime submodules minimal over it.

***Corollary* 2.2.** *Suppose M is an R-module. Then the following statements are equivalent*:

(i) *$R/(O\colon M)$ is a general ZPI-ring and M is a multiplication module.*

(ii) *M is a locally PI-multiplication $R/(O\colon M)$-module in which every cyclic submodule has a primary decomposition.*

Proof. The proof is clear by Theorem 2.2.

***Corollary* 2.3.** *If M is a multiplication R-module in which every prime submodule is cyclic, then $R/(O\colon M)$ is a general ZPI-ring and M is a Noetherian locally PI-multiplication $R/(O\colon M)$-module.*

Proof. Put $\bar{R} = R/(O\colon M)$. Then M is a faithful multiplication $\bar{R}$-module in which every prime submodule is cyclic. By Lemma 1.1(v)(b), for each prime ideal $\bar{P}$ of $\bar{R}$, the $\bar{R}_{\bar{P}}$-module $M_{\bar{P}}$ is cyclic. Now let N be an arbitrary prime submodule of $M_{\bar{P}}$. Then $N^c = \{x \in M \mid x/1 \in N\}$ is a prime submodule of M and since by our assumption N^c is cyclic, $N = (N^c)_{\bar{P}}$ is cyclic. Hence by Theorem 2.1(ix), $M_{\bar{P}}$ is a PI-multiplication $\bar{R}_{\bar{P}}$-module. Now we have the conditions of Theorem 2.2(vii), which completes the proof.

If M is a noncyclic PI-multiplication R-module, then according to different parts of Theorem 2.1, we have:

(i) None of the ideals of R contained in $(0\colon M)$ has a primary decomposition, particularly R is neither a Noetherian ring nor an integral domain or a primary ring. Furthermore R does not have ACC on semiprime ideals.

(ii) M is not a finitely annihilated module, particularly it is not a finitely generated module. Also, none of the finitely generated submodules of M has a primary decomposition. Furthermore, there exists a maximal ideal P of R with $M_P = 0$, and $PM = M$, besides M has a nonfinitely generated minimal prime submodule.

Therefore, according to the above discussion, if there is any nonzero noncyclic PI-multiplication module, then it will be a very special example. This is a motivation for the following open problem.

Every nonzero PI-multiplication module is a cyclic module over a principal ideal ring.

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