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NEW FRACTIONAL INTEGRAL INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS AND THEIR APPLICATIONS

НОВІ ДРОБОВО-ІНТЕГРАЛЬНІ НЕРІВНОСТІ ДЛЯ ДИФЕРЕНЦІЙОВНИХ ОПУКЛИХ ФУНКЦІЙ ТА ЇХ ЗАСТОСУВАННЯ

We establish some new fractional integral inequalities for differentiable convex functions and give several applications for the Beta-function.

Встановлено деякі нові дробово-інтегральні нерівності для диференційовних опуклих функцій і наведено кілька застосувань для бета-функції.

1. Introduction. Throughout in this paper, let $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f\colon [a,b] \to \mathbb{R}$, is known in the literature as Hermite – Hadamard inequality [7].

For some results which generalize, improve, and extend the inequality (1.1), see [1 – 6] and [8 – 17].

In [14], Tseng et al. established the following Hermite – Hadamard-type inequality which refines the inequality (1.1).

Theorem A. *Suppose that* $f\colon [a,b] \to \mathbb{R}$ *is a convex function on* $[a,b]$. *Then we have the inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \le$$

$$\le \frac{1}{b-a}\int_a^b f(x)\,dx \le$$

$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{(a)+f(b)}{2}\right] \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The third inequality in (1.2) is known in the literature as Bullen inequality.

In [4], Dragomir and Agarwal established the following results connected with the second inequality in the inequality (1.1).

Theorem B. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b) *with* $a < b$. *If* $|f'|$ *is convex on* $[a,b]$, *then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

which is the trapezoid inequality provided $|f'|$ *is convex on* $[a,b]$.

In [11], Kirmaci and Özdemir established the following results connected with the first inequality in the inequality (1.1).

Theorem C. *Under the assumptions of Theorem B, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

which is the midpoint inequality provided $|f'|$ *is convex on* $[a,b]$.

In [12], Pearce and Pečarić established the following Hermite–Hadamard-type inequalities for differentiable functions:

Theorem D. *If* $f\colon I^o \subseteq R \to R$ *is a differentiable mapping on* I^o, $a,b \in I^o$ *with* $a<b$, $f' \in L_1[a,b]$, $q \ge 1$ *and* $|f'|^q$ *is convex on* $[a,b]$, *then the following inequalities hold*:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q},$$

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}.$$

In what follows we recall the following definition [13].

Definition 1.1. *Let* $f \in L_1[a,b]$. *The Riemann–Liouville integrals* $J^\alpha_{a^+}f$ *and* $J^\alpha_{b^-}f$ *of order* $\alpha > 0$ *with* $a \ge 0$ *are defined by*

$$J^\alpha_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a,$$

and

$$J^\alpha_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b,$$

respectively. Here $\Gamma(\alpha)$ *is the Gamma-function and* $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

In [13], Sarikaya et al. established the following Hermite–Hadamard-type inequalities for fractional integrals:

Theorem E. *Let* $f\colon [a,b] \to \mathbb{R}$ *be positive with* $0 \le a < b$ *and* $f \in L_1[a,b]$. *If f is a convex function on* $[a,b]$, *then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right] \le \frac{f(a)+f(b)}{2}$$

for $\alpha > 0$.

Theorem F. *Under the assumptions of Theorem B, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)\right]\right| \le$$

$$\le \frac{2^\alpha - 1}{2^{\alpha+1}(\alpha+1)}(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)$$

for $\alpha > 0$.

In [9], Hwang et al. established the following fractional integral inequalities:

Theorem G. *Under the assumptions of Theorem B, we have the following Hermite – Hadamard-type inequality for fractional integrals:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\leq$$
$$\leq\frac{b-a}{4(\alpha+1)}\left(\alpha-1+\frac{1}{2^{\alpha-1}}\right)\big(\left|f'(a)\right|+\left|f'(b)\right|\big)$$

for $\alpha>0$.

Theorem H. *Under the assumptions of Theorem B, we have the following inequality for fractional integrals with* $\dfrac{f\left(\dfrac{3a+b}{4}\right)+f\left(\dfrac{a+3b}{4}\right)}{2}$:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right|\leq$$
$$\leq\left(\frac{1}{8}+\frac{3^{\alpha+1}-2^{\alpha+1}+1}{4^{\alpha+1}(\alpha+1)}-\frac{1}{2(\alpha+1)}\right)(b-a)\big(\left|f'(a)\right|+\left|f'(b)\right|\big) \qquad (1.3)$$

for $\alpha>0$.

Theorem I. *Under the assumptions of Theorem B, we have the following Bullen-type inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\right.$$
$$\left.-\left[\frac{3^{\alpha}-1}{4^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{4^{\alpha}-3^{\alpha}+1}{4^{\alpha}}\frac{f(a)+f(b)}{2}\right]\right|\leq$$
$$\leq\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{3^{\alpha+1}+1}{4^{\alpha+1}}\right)(b-a)\big(\left|f'(a)\right|+\left|f'(b)\right|\big) \qquad (1.4)$$

for $\alpha>0$.

Theorem J. *Under the assumptions of Theorem B, we have the following Simpson-type inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\right.$$
$$\left.-\left[\frac{5^{\alpha}-1}{6^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\frac{f(a)+f(b)}{2}\right]\right|\leq$$
$$\leq\left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^{\alpha}-1}{12\cdot 6^{\alpha}}\right)\right](b-a)\big(\left|f'(a)\right|+\left|f'(b)\right|\big) \qquad (1.5)$$

for $\alpha>0$.

Remark **1.1.** (1) The assumptions $f\colon[a,b]\to\mathbb{R}$ is positive with $0\leq a<b$ in Theorem E can be weakened as $f\colon[a,b]\to\mathbb{R}$ with $a<b$.

(2) In Theorem D, let $q=1$. Then Theorem D reduces to Theorems B and C.

(3) In Theorems F and G, let $\alpha=1$. Then Theorems F and G reduce to Theorem B and C, respectively.

(4) In Theorem H, let $\alpha = 1$. Then the inequality (1.3) is connected with the second inequality in the inequality (1.2).

(5) In Theorem I, let $\alpha = 1$. Then the inequality (1.4) is a Bullen-type inequality.

(6) In Theorem J, let $\alpha = 1$. Then the inequality (1.5) is a Simpson-type inequality.

In this paper, we establish some new Hermite – Hadamard-type inequalities for fractional integrals which generalize Theorems D and G – J. Some applications for the Beta-function are given.

2. Main results.

Theorem 2.1. *Under the assumptions of Theorem D, then we have the following Hermite – Hadamard-type inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\le$$

$$\le\frac{b-a}{2(\alpha+1)}\left(\alpha-1+\frac{1}{2^{\alpha-1}}\right)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \tag{2.1}$$

for $\alpha > 0$.

Proof. In [9], let

$$h_1(x)=\begin{cases}(b-x)^\alpha-(x-a)^\alpha-(b-a)^\alpha, & x\in\left[a,\dfrac{a+b}{2}\right),\\[2mm] (b-x)^\alpha-(x-a)^\alpha+(b-a)^\alpha, & x\in\left[\dfrac{a+b}{2},b\right].\end{cases}$$

Then the following identities hold:

$$\frac{1}{2(b-a)^\alpha}\int\limits_a^b h_1(x)f'(x)\,dx=$$

$$=\frac{\alpha}{2(b-a)^\alpha}\int\limits_a^b\left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)\,dx-f\left(\frac{a+b}{2}\right)=$$

$$=\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha(a)\right]-f\left(\frac{a+b}{2}\right)=$$

$$=\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha(a)\right]-f\left(\frac{a+b}{2}\right). \tag{2.2}$$

Using simple computation, we have the following identities:

$$x=\frac{b-x}{b-a}a+\frac{x-a}{b-a}b,\quad x\in[a,b], \tag{2.3}$$

$$\int\limits_a^{\frac{a+b}{2}}\left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx+$$

$$+\int_{\frac{a+b}{2}}^{b}[(b-x)^{\alpha}-(x-a)^{\alpha}+(b-a)^{\alpha}]\,\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx=$$

$$=\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx+$$

$$+\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,\frac{x-a}{b-a}\left|f'(a)\right|^{q}dx=$$

$$=\left|f'(a)\right|^{q}\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,dx:=M_1, \qquad (2.4)$$

$$\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx+$$

$$+\int_{\frac{a+b}{2}}^{b}[(b-x)^{\alpha}-(x-a)^{\alpha}+(b-a)^{\alpha}]\,\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx=$$

$$=\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx+$$

$$+\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,\frac{b-x}{b-a}\left|f'(b)\right|^{q}dx=$$

$$=\left|f'(b)\right|^{q}\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,dx:=M_2, \qquad (2.5)$$

$$\int_{a}^{b}|h_1(x)|\,dx=2\int_{a}^{\frac{a+b}{2}}[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}]\,dx. \qquad (2.6)$$

Now, using power mean inequality, the identities (2.3) – (2.6) and the convexity of $|f'|^q$, we obtain the inequality

$$\left|\int_{a}^{b}h_1(x)f'(x)\,dx\right|\le\int_{a}^{b}|h_1(x)|\left|f'(x)\right|dx\le$$

$$\leq \left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^b |h_1(x)|\,\left|f'(x)\right|^q dx\right]^{1/q} =$$

$$= \left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha]\left|f'(x)\right|^q dx + \right.$$

$$\left. + \int_{\frac{a+b}{2}}^b [(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha]\left|f'(x)\right|^q dx\right]^{1/q} \leq$$

$$\leq \left[\int_a^b |h_1(x)|\,dx\right]^{\frac{q-1}{q}} (M_1+M_2)^{1/q} =$$

$$= 2\int_a^{\frac{a+b}{2}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha]\,dx \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} =$$

$$= \frac{(b-a)^{\alpha+1}}{\alpha+1}\left(\alpha - 1 + \frac{1}{2^{\alpha-1}}\right)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \qquad (2.7)$$

The inequality (2.1) follows from the identity (2.2) and the inequality (2.7).

Theorem 2.1 is proved.

***Remark* 2.1.** In Theorem 2.1, let $q = 1$. Then Theorem 2.1 reduces to Theorem G.

Theorem 2.2. *Under the assumptions of Theorem D, then we have the following inequality for fractional integrals with* $\dfrac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}$:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right] - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right| \leq$$

$$\leq \left(\frac{1}{4} + \frac{3^{\alpha+1} - 2^{\alpha+1} + 1}{2\cdot 4^\alpha(\alpha+1)} - \frac{1}{\alpha+1}\right)(b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} \qquad (2.8)$$

for $\alpha > 0$.

Proof. In [9], let

$$h_2(x) = \begin{cases} (b-x)^\alpha - (x-a)^\alpha - (b-a)^\alpha, & x \in \left[a, \dfrac{3a+b}{4}\right), \\ (b-x)^\alpha - (x-a)^\alpha, & x \in \left[\dfrac{3a+b}{4}, \dfrac{a+3b}{4}\right), \\ (b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha, & x \in \left[\dfrac{a+3b}{4}, b\right]. \end{cases}$$

Then the following identities hold:

$$\frac{1}{2(b-a)^\alpha}\int\limits_a^b h_2(x)f'(x)\,dx =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^b \left[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}\right]f(x)\,dx - \frac{f\left(\dfrac{3a+b}{4}\right)+f\left(\dfrac{a+3b}{4}\right)}{2} =$$

$$= \frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha(a)\right] - \frac{f\left(\dfrac{3a+b}{4}\right)+f\left(\dfrac{a+3b}{4}\right)}{2} =$$

$$= \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha(a)\right] - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]. \tag{2.9}$$

Using simple computation, we get the identities

$$\int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx +$$

$$+ \int\limits_{\frac{a+3b}{4}}^{b} \left[(b-x)^\alpha-(x-a)^\alpha+(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx =$$

$$= \int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx +$$

$$+ \int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(a)\right|^q dx =$$

$$= \left|f'(a)\right|^q \int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]dx := N_1, \tag{2.10}$$

$$\int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx +$$

$$+ \int\limits_{\frac{a+3b}{4}}^{b} \left[(b-x)^\alpha-(x-a)^\alpha+(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx =$$

$$= \int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha-(b-x)^\alpha+(x-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx +$$

$$+\int_{a}^{\frac{3a+b}{4}}\left[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|^{q}dx=$$

$$=\left|f'(b)\right|^{q}\int_{a}^{\frac{3a+b}{4}}\left[(b-a)^{\alpha}-(b-x)^{\alpha}+(x-a)^{\alpha}\right]dx:=N_{2},\tag{2.11}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx+$$

$$+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx=$$

$$=\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx+$$

$$+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|^{q}dx=$$

$$=\left|f'(a)\right|^{q}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]dx:=N_{3},\tag{2.12}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx+$$

$$+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx=$$

$$=\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx+$$

$$+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|^{q}dx=$$

$$= \left|f'(b)\right|^q \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] dx := N_4, \tag{2.13}$$

$$\int\limits_a^b |h_2(x)|\, dx = 2\left[\int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] dx + \right.$$

$$\left. + \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] dx\right]. \tag{2.14}$$

Now, using power mean inequality, the identities (2.3), (2.10) – (2.14) and the convexity of $|f'|^q$, we have the inequality

$$\left|\int\limits_a^b h_2(x) f'(x)\, dx\right| \le \int\limits_a^b |h_2(x)| \left|f'(x)\right| dx \le$$

$$\le \left[\int\limits_a^b |h_2(x)|\, dx\right]^{\frac{q-1}{q}} \left[\int\limits_a^b |h_2(x)| \left|f'(x)\right|^q dx\right]^{1/q} =$$

$$= \left[\int\limits_a^b |h_2(x)|\, dx\right]^{\frac{q-1}{q}} \left[\int\limits_a^{\frac{3a+b}{4}} \left[(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha\right] \left|f'(x)\right|^q dx + \right.$$

$$+ \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(b-x)^\alpha - (x-a)^\alpha\right] \left|f'(x)\right|^q dx +$$

$$+ \int\limits_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(x-a)^\alpha - (b-x)^\alpha\right] \left|f'(x)\right|^q dx +$$

$$\left. + \int\limits_{\frac{a+3b}{4}}^{b} \left[(b-x)^\alpha - (x-a)^\alpha + (b-a)^\alpha\right] \left|f'(x)\right|^q dx\right]^{1/q} \le$$

$$\le \left[\int\limits_a^b |h_2(x)|\, dx\right]^{\frac{q-1}{q}} (N_1 + N_2 + N_3 + N_4) =$$

$$= 2\left[\int\limits_a^{\frac{3a+b}{4}} [(b-a)^\alpha - (b-x)^\alpha + (x-a)^\alpha]\,dx + \right.$$

$$\left. + \int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} [(b-x)^\alpha - (x-a)^\alpha]\,dx\right]\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} =$$

$$= \left(\frac{1}{2} + \frac{3^{\alpha+1} - 2^{\alpha+1} + 1}{4^\alpha(\alpha+1)} - \frac{2}{\alpha+1}\right)(b-a)^{\alpha+1}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{2.15}$$

The inequality (2.8) follows from the identity (2.9) and the inequality (2.15).

Theorem 2.2 is proved.

***Remark* 2.2.** (1) In Theorem 2.2, let $q = 1$. Then Theorem 2.2 reduces to Theorem H.

(2) In Theorems 2.1 and 2.2, let $\alpha = 1$. Then Theorems 2.1 and 2.2 reduce to Theorem D.

Theorem 2.3. *Under the assumptions of Theorem D, then we have the following Bullen-type inequality for fractional integrals*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - \right.$$

$$\left. - \left[\frac{3^\alpha - 1}{4^\alpha} f\left(\frac{a+b}{2}\right) + \frac{4^\alpha - 3^\alpha + 1}{4^\alpha}\frac{f(a)+f(b)}{2}\right]\right| \leq$$

$$\leq \frac{1}{\alpha+1}\left(\frac{2^\alpha+1}{2^\alpha} - \frac{3^{\alpha+1}+1}{2\cdot 4^\alpha}\right)(b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} \tag{2.16}$$

for $\alpha > 0$.

Proof. Let

$$h_3(x) = \begin{cases} (b-x)^\alpha - (x-a)^\alpha - \dfrac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha, & x \in \left[a, \dfrac{a+b}{2}\right), \\ (b-x)^\alpha - (x-a)^\alpha + \dfrac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha, & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Then the following identities hold:

$$\frac{1}{2(b-a)^\alpha}\int\limits_a^b h_3(x) f'(x)\,dx =$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int\limits_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\,dx -$$

$$- \left[\frac{3^\alpha - 1}{4^\alpha} f\left(\frac{a+b}{2}\right) + \frac{4^\alpha - 3^\alpha + 1}{4^\alpha}\frac{f(a)+f(b)}{2}\right] =$$

$$= \frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha (a)\right] -$$

$$-\left[\frac{3^\alpha-1}{4^\alpha}f\left(\frac{a+b}{2}\right)+\frac{4^\alpha-3^\alpha+1}{4^\alpha}\frac{f(a)+f(b)}{2}\right]=$$

$$=\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}(a)\right]-$$

$$-\left[\frac{3^\alpha-1}{4^\alpha}f\left(\frac{a+b}{2}\right)+\frac{4^\alpha-3^\alpha+1}{4^\alpha}\frac{f(a)+f(b)}{2}\right]. \tag{2.17}$$

Using simple computation, we have the following identities:

$$\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx+$$

$$+\int\limits_{\frac{a+3b}{4}}^{b}\left[(x-a)^\alpha-(b-x)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx=$$

$$=\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx+$$

$$+\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(a)\right|^q dx=$$

$$=\left|f'(a)\right|^q\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]dx:=P_1, \tag{2.18}$$

$$\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx+$$

$$+\int\limits_{\frac{a+3b}{4}}^{b}\left[(x-a)^\alpha-(b-x)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx=$$

$$=\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx+$$

$$+\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(b)\right|^q dx=$$

$$=\left|f'(b)\right|^q\int\limits_a^{\frac{3a+b}{4}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{3^\alpha-1}{4^\alpha}(b-a)^\alpha\right]dx:=P_2, \tag{2.19}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx +$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx =$$

$$= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(a)\right|^q dx +$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(a)\right|^q dx =$$

$$= \left|f'(a)\right|^q \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx := P_3, \tag{2.20}$$

$$\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx +$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx =$$

$$= \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{x-a}{b-a} \left|f'(b)\right|^q dx +$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \frac{b-x}{b-a} \left|f'(b)\right|^q dx =$$

$$= \left|f'(b)\right|^q \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx := P_4, \tag{2.21}$$

$$\int_a^b |h_3(x)|\,dx = 2\left[\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx + \right.$$

$$\left. + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx\right]. \tag{2.22}$$

Now, using power mean inequality, the identities (2.3), (2.18)–(2.22) and the convexity of $|f'|^q$, we obtain the inequality

$$\left|\int_a^b h_3(x) f'(x) dx\right| \le \int_a^b |h_3(x)| \left|f'(x)\right| dx \le$$

$$\le \left[\int_a^b |h_3(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^b |h_3(x)| \left|f'(x)\right|^q dx\right]^{1/q} =$$

$$= \left[\int_a^b |h_3(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}\right] (b-a)^\alpha \left|f'(x)\right|^q dx + \right.$$

$$+ \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \left|f'(x)\right|^q dx +$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \left|f'(x)\right|^q dx +$$

$$\left. + \int_{\frac{a+3b}{4}}^{b} \left[(x-a)^\alpha - (b-x)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] \left|f'(x)\right|^q dx\right]^{1/q} \le$$

$$\le \left[\int_a^b |h_3(x)|\,dx\right]^{\frac{q-1}{q}} (P_1 + P_2 + P_3 + P_4) =$$

$$= 2\left[\int_a^{\frac{3a+b}{4}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{3^\alpha - 1}{4^\alpha}(b-a)^\alpha\right] dx + \right.$$

$$+\int\limits_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left[(x-a)^{\alpha} - (b-x)^{\alpha} + \frac{3^{\alpha}-1}{4^{\alpha}}(b-a)^{\alpha}\right]\Bigg] dx \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} =$$

$$= \frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha-1}} - \frac{3^{\alpha+1}+1}{4^{\alpha}}\right)(b-a)^{\alpha+1}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \qquad (2.23)$$

The inequality (2.16) follows from the identity (2.17) and the inequality (2.23).

Theorem 2.3 is proved.

***Remark* 2.3.** In Theorem 2.3, let $q = 1$. Then Theorem 2.3 reduces to Theorem I.

Theorem 2.4. *Under the assumptions of Theorem D, then we have the following Simpson-type inequality for fractional integrals*:

$$\Bigg|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] -$$

$$-\left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\frac{f(a)+f(b)}{2}\right]\Bigg| \le$$

$$\le \left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha}} - \frac{5^{\alpha+1}+1}{3\cdot 6^{\alpha}}\right) + \left(\frac{5^{\alpha}-1}{6^{\alpha+1}}\right)\right](b-a)\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} \qquad (2.24)$$

for $\alpha > 0$.

Proof. In [9], let

$$h_4(x) = \begin{cases} (b-x)^{\alpha} - (x-a)^{\alpha} - \dfrac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}, & x \in \left[a, \dfrac{a+b}{2}\right), \\ (b-x)^{\alpha} - (x-a)^{\alpha} + \dfrac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}, & x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Then, the following identities hold:

$$\frac{1}{2(b-a)^{\alpha}}\int\limits_a^b h_4(x) f'(x)\, dx =$$

$$= \frac{\alpha}{2(b-a)^{\alpha}}\int\limits_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)\, dx -$$

$$-\left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\frac{f(a)+f(b)}{2}\right] =$$

$$= \frac{\alpha\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha}(a)\right] -$$

$$-\left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right) + \frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\frac{f(a)+f(b)}{2}\right] =$$

$$= \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha}(a)\right] -$$

$$-\left[\frac{5^{\alpha}-1}{6^{\alpha}}f\left(\frac{a+b}{2}\right)+\frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\frac{f(a)+f(b)}{2}\right]. \tag{2.25}$$

Using simple computation, we have the following identities:

$$\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx+$$

$$+\int\limits_{\frac{a+5b}{6}}^{b}\left[(x-a)^{\alpha}-(b-x)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx=$$

$$=\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx+$$

$$+\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(a)\right|^{q}dx=$$

$$=\left|f'(a)\right|^{q}\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]dx:=Q_{1}, \tag{2.26}$$

$$\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx+$$

$$+\int\limits_{\frac{a+5b}{6}}^{b}\left[(x-a)^{\alpha}-(b-x)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx=$$

$$=\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{x-a}{b-a}\left|f'(b)\right|^{q}dx+$$

$$+\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(b)\right|^{q}dx=$$

$$=\left|f'(b)\right|^{q}\int\limits_{a}^{\frac{5a+b}{6}}\left[(b-x)^{\alpha}-(x-a)^{\alpha}-\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]dx:=Q_{2}, \tag{2.27}$$

$$\int\limits_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^{\alpha}-(b-x)^{\alpha}+\frac{5^{\alpha}-1}{6^{\alpha}}(b-a)^{\alpha}\right]\frac{b-x}{b-a}\left|f'(a)\right|^{q}dx+$$

$$+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left[(b-x)^\alpha-(x-a)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx+$$

$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(a)\right|^q dx+$$

$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(a)\right|^q dx=$$

$$=\left|f'(a)\right|^q\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]dx:=Q_3, \qquad (2.28)$$

$$\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx+$$

$$+\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}}\left[(b-x)^\alpha-(x-a)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx+$$

$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{x-a}{b-a}\left|f'(b)\right|^q dx+$$

$$+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]\frac{b-x}{b-a}\left|f'(b)\right|^q dx=$$

$$=\left|f'(b)\right|^q\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]dx:=Q_4, \qquad (2.29)$$

$$\int_a^b|h_4(x)|\,dx=2\left[\int_a^{\frac{5a+b}{6}}\left[(b-x)^\alpha-(x-a)^\alpha-\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]dx+\right.$$

$$\left.+\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}}\left[(x-a)^\alpha-(b-x)^\alpha+\frac{5^\alpha-1}{6^\alpha}(b-a)^\alpha\right]dx\right]. \qquad (2.30)$$

Now, using power mean inequality, the identities (2.3), (2.26) – (2.30) and the convexity of $|f'|^q$, we get the inequality

$$\left|\int_a^b h_4(x)f'(x)dx\right| \leq \int_a^b |h_4(x)|\,|f'(x)|\,dx \leq$$

$$\leq \left[\int_a^b |h_4(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^b |h_4(x)|\,|f'(x)|^q\,dx\right]^{1/q} =$$

$$= \left[\int_a^b |h_4(x)|\,dx\right]^{\frac{q-1}{q}} \left[\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]|f'(x)|^q\,dx + \right.$$

$$+ \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]|f'(x)|^q\,dx +$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]|f'(x)|^q\,dx +$$

$$\left. + \int_{\frac{a+5b}{6}}^{b} \left[(x-a)^\alpha - (b-x)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]|f'(x)|^q\,dx \right] \leq$$

$$\leq \left[\int_a^b |h_4(x)|\,dx\right]^{\frac{q-1}{q}} (Q_1 + Q_2 + Q_3 + Q_4) =$$

$$= 2\left[\int_a^{\frac{5a+b}{6}} \left[(b-x)^\alpha - (x-a)^\alpha - \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]dx + \right.$$

$$\left. + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left[(x-a)^\alpha - (b-x)^\alpha + \frac{5^\alpha - 1}{6^\alpha}(b-a)^\alpha\right]dx\right] \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q} =$$

$$= \left[\frac{1}{\alpha+1}\left(\frac{2^\alpha + 1}{2^{\alpha-1}} - \frac{5^{\alpha+1} + 1}{9 \cdot 6^{\alpha-1}}\right) + \left(\frac{5^\alpha - 1}{3 \cdot 6^\alpha}\right)\right](b-a)^{\alpha+1} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \qquad (2.31)$$

The inequality (2.24) follows from the identity (2.25) and the inequality (2.31).

Theorem 2.4 is proved.

***Remark* 2.4.** In Theorem 2.4, let $q = 1$. Then Theorem 2.4 reduces to Theorem J.

3. Applications for the Beta-functions. Throughout this section, let $\alpha > 0,\ \rho \geq 1,\ q \geq 1,\ a = 0,\ b = 1,\ \Gamma(\alpha)$ be the Gamma-function and $f(x) = x^{\rho-1}\ (x \in [0,1])$. Then $|f'|$ is convex on $[0,1]$.

Let us recall the *Beta-function*

$$B(p,r) = \int\limits_0^1 x^{p-1}(1-x)^{r-1}\,dx \quad (p, r > 0).$$

***Remark* 3.1.** Using Theorems 2.1 – 2.4, we get

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{a+}^{\alpha} f(b) = \frac{\alpha}{2}\int\limits_0^1 (1-x)^{\alpha-1} x^{\rho-1} dx = \frac{\alpha}{2} B(\rho,\alpha)$$

and

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) = \frac{\alpha}{2}\int\limits_0^1 x^{\alpha+\rho-2} dx = \frac{\alpha}{2(\alpha+\rho-1)}.$$

Using Theorems 2.1 – 2.4 and Remark 3.1, we have the following propositions:

Proposition 3.1. *In Theorem* 2.1, *the following inequality holds*:

$$\left|\frac{\alpha}{2}B(\rho,\alpha) + \frac{\alpha}{2(\alpha+\rho-1)} - \frac{1}{2^{\rho-1}}\right| \leq \left(\frac{1}{2} - \frac{2^{\alpha}-1}{2^{\alpha}(\alpha+1)}\right)\frac{\rho-1}{2^{1/q}}.$$

Proposition 3.2. *In Theorem* 2.2, *the following inequality holds*:

$$\left|\frac{\alpha}{2}B(\rho,\alpha) + \frac{\alpha}{2(\alpha+\rho-1)} - \frac{3^{\rho-1}+1}{2\cdot 4^{\rho-1}}\right| \leq$$

$$\leq \left[\frac{3^{\alpha+1}+1}{2\cdot 4^{\alpha}(\alpha+1)} + \frac{1}{4} - \frac{2^{\alpha}+1}{2^{\alpha}(\alpha+1)}\right]\frac{\rho-1}{2^{1/q}}.$$

Proposition 3.3. *In Theorem* 2.3, *the following inequality holds*:

$$\left|\frac{\alpha}{2}B(\rho,\alpha) + \frac{\alpha}{2(\alpha+\rho-1)} - \left(\frac{3^{\alpha}-1}{2^{\rho-1}4^{\alpha}} + \frac{4^{\alpha}-3^{\alpha}+1}{2\cdot 4^{\alpha}}\right)\right| \leq$$

$$\leq \frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha}} - \frac{3^{\alpha+1}+1}{2\cdot 4^{\alpha}}\right)\frac{\rho-1}{2^{1/q}}.$$

Proposition 3.4. *In Theorem* 2.4, *the following inequality holds*:

$$\left|\frac{\alpha}{2}B(\rho,\alpha) + \frac{\alpha}{2(\alpha+\rho-1)} - \left(\frac{5^{\alpha}-1}{2^{\rho-1}6^{\alpha}} + \frac{6^{\alpha}-5^{\alpha}+1}{2\cdot 6^{\alpha}}\right)\right| \leq$$

$$\leq \left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha}} - \frac{5^{\alpha+1}+1}{3\cdot 6^{\alpha}}\right) + \left(\frac{5^{\alpha}-1}{6^{\alpha+1}}\right)\right]\frac{\rho-1}{2^{1/q}}.$$

References

1. *Alomari M., Darus M.* On the Hadamard's inequality for log-convex functions on the coordinates // J. Inequal. Appl. – 2009. – Article ID 283147. – 13 p.
2. *Dragomir S. S.* Two mappings in connection to Hadamard's inequalities // J. Math. Anal. and Appl. – 1992. – **167**. – P. 49 – 56.
3. *Dragomir S. S.* On the Hadamard's inequality for convex on the co-ordinates in a rectangle from the plane // Taiwan. J. Math. – 2001. – **5**, № 4. – P. 775 – 788.
4. *Dragomir S. S., Agarwal R. P.* Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula // Appl. Math. Lett – 1998. – **11**, № 5. – P. 91 – 95.
5. *Dragomir S. S., Cho Y.-J., Kim S.-S.* Inequalities of Hadamard's type for Lipschitzian mappings and their applications // J. Math. Anal. and Appl. – 2000. – **245**. – P. 489 – 501.
6. *Fejér L.* Über die Fourierreihen, II // Math. Naturwiss. Anz Ungar. Akad. Wiss. – 1906. – **24**. – P. 369 – 390 (in Hungarian).
7. *Hadamard J.* Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann // J. math. pures et appl. – 1893. – **58**. – P. 171 – 215.
8. *Hwang S.-R., Hsu K.-C., Tseng K.-L.* Hadamard-type inequalities for Lipschitzian functions in one and two variables with their applications // J. Math. Anal. and Appl. – 2013. – **405**. – P. 546 – 554.
9. *Hwang S.-R., Tseng K.-L., Hsu K.-C.* New inequalities for fractional integrals and their applications // Turk. J. Math. – 2016. – **40**. – P. 471 – 486.
10. *Kirmaci U. S.* Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula // Appl. Math. and Comput. – 2004. – **147**. – P. 137 – 146.
11. *Kirmaci U. S., Özdemir M. E.* On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula // Appl. Math. and Comput. – 2004. – **153**. – P. 361 – 368.
12. *Pearce C. E. M., Pečarić J.* Inequalities for differentiable mappings with application to special means and quadrature formula // Appl. Math. Lett. – 2000. – **13**, № 2. – P. 51 – 55.
13. *Sarikaya M. Z., Set E., Yaldiz H., Başak N.* Hermite – Hadamard's inequalities for fractional integrals and related fractional inequalities // Math. Comput. Model. – 2013. – **57**. – P. 2403 – 2407.
14. *Tseng K.-L., Hwang S.-R., Dragomir S. S.* Fejér-type inequalities (I) // J. Inequat. Appl. – 2010. – Article ID 531976. – 7 p.
15. *Tseng K.-L., Yang G.-S., Hsu K.-C.* On some inequalities of Hadamard's type and applications // Taiwan. J. Math. – 2009. – **13**, № 6. – P. 1929 – 1948.
16. *Yang G.-S., Tseng K.-L.* On certain integral inequalities related to Hermite – Hadamard inequalities // J. Math. Anal. and Appl. – 1999. – **239**. – P. 180 – 187.
17. *Yang G.-S., Tseng K.-L.* Inequalities of Hadamard's type for Lipschitzian mappings // J. Math. Anal. and Appl. – 2001. – **260**. – P. 230 – 238.

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