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CONVERGENCE OF FOURIER SERIES OF FUNCTIONS Lip 1 WITH RESPECT TO GENERAL ORTHONORMAL SYSTEMS*

ЗБІЖНІСТЬ РЯДІВ ФУР'Є ФУНКЦІЙ Lip 1 ВІДНОСНО ЗАГАЛЬНИХ ОРТОНОРМОВАНИХ СИСТЕМ

We establish sufficient conditions that should be satisfied by functions of a general orthonormal system (ONS) $\{\varphi_n(x)\}$ in order that the Fourier series in this system for any function from the class $\mathrm{Lip}\,1$ be convergent almost everywhere on $[0,1]$. It is shown that the obtained conditions are best possible in a certain sense.

Встановлено достатні умови, які повинні задовольняти функції загальної ортонормованої системи (ЗОС) $\{\varphi_n(x)\}$, для того, щоб ряд Фур'є відносно вказаної системи для будь-якої функції з класу $\mathrm{Lip}\,1$ збігався майже скрізь на $[0,1]$. Показано, що отримані розв'язки є, в деякому розумінні, найкращими.

1. Introduction. Let $\{\varphi_n(x)\}$ be ONS on $[0,1]$,

$$\widehat{\varphi}_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \ldots,$$

are Fourier coefficients of the function $f(x) \in L(0,1)$.

Suppose

$$S_n(a) = \frac{1}{n}\sum_{i=1}^{n-1}\left|\int_0^{i/n} \Phi_n(a,x)\,dx\right|, \tag{1}$$

where

$$\Phi_n(a,x) = \sum_{k=1}^{n} a_k\lambda_k\varphi_k(x)$$

and (a_k) and (λ_k) are some sequences of numbers.

Besides,

$$L_n(x) = \int_0^1 \left|\sum_{k=1}^{n} \varphi_k(x)\varphi_k(t)\right| dt \tag{2}$$

is a Lebesgue function.

We have (see [1, p. 180 and 207]) the following theorem.

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Theorem A (Kačmarž). *Let* (λ_n) *be a nondecreasing sequence of positive numbers,* $\lim_{n\to\infty}\lambda_n = +\infty$ *and* $L_n(x) = O(\lambda_n)$. *If*

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n < +\infty, \tag{3}$$

then the series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{4}$$

converges a.e. on $[0,1]$.

***Remark* 1.** It is well known (see [2, p. 202]) that for any ONS a.e.

$$L_n(x) = O(\gamma_n^{1/2}),$$

where γ_n is a positive monotone increasing sequence satisfying the condition

$$\sum_{n=1}^{\infty} \gamma_n^{-1} < \infty.$$

Setting $\gamma_n = n^2$ we get $L_n(x) = O(n)$.[1]

Let now

$$\lambda_n' = \min(\lambda_n, n),$$

where (λ_n) is a sequence from Theorem A. Then λ_n' will satisfy all conditions of Theorem A. In addition,

$$\lambda_n' \leq n.$$

Thus in Theorem A it can be assumed, without loss of generality, that $\lambda_n \leq n$.

2. Formulation of the basic problem. S. Banach [3] proved that for any function $f(x) \in L_2(0,1)$ there exists ONS $\{\varphi_n(x)\}$ such that the Fourier series of the function $f(x)$ of this system diverges a.e. $[0,1]$.

Besides, A. Olevski [4] proves that if $f(x) \in L_2(0,1)$ is an arbitrary function and $(a_n) \in \ell_2$ is any sequence of numbers, then there exists ONS $\{\varphi_n(x)\}$ such that $a_n = c\int_0^1 f(x)\varphi_n(x)\,dx$, $n = 1, 2, \ldots$, and c is some number.

Let N_k be an increasing sequence of natural numbers

$$a_n = \frac{1}{((N_{k+1} - N_k)\lambda_n)^{1/2}}, \quad N_k \leq n < N_{k+1},$$

where

$$\sum_{n=1}^{\infty} (\lambda_n)^{-1} < \infty.$$

Let $f_0(x) = 1$, $x \in [0,1]$. By virtue of A. Olevski's theorem there exists ONS $\{\varphi_n(x)\}$ such that

[1] The upper estimate is true and more exact, but in future it will be sufficient to use this estimate.

$$\widehat{\varphi}_n(f_0) = \int\limits_0^1 f_0(x)\varphi_n(x)\,dx = c\cdot a_n, \quad n = 1, 2, \ldots .$$

Thus

$$\sum_{n=N_1}^{\infty} \widehat{\varphi}_n^2(f_0) = O(1)\sum_{k=1}^{\infty}\sum_{n=N_k}^{N_{k+1}-1}\frac{1}{(N_{k+1}-N_k)\lambda_n} = O(1)\sum_{k=1}^{\infty}\lambda_{n_k}^{-1} < +\infty.$$

On the other hand,

$$\sum_{n=N_1}^{\infty} \widehat{\varphi}_n^2(f_0)\lambda_n = c^2\sum_{k=1}^{n}\sum_{n=N_k}^{N_{k+1}-1}\frac{1}{(N_{k+1}-N_k)\lambda_n}\lambda_n = +\infty.$$

Consequently, in our case condition (3) is not satisfied for the Fourier coefficients of the functions $f_0(x) = 1$. Thus, "good" differential properties of functions do not provide the convergence of Fourier series of these functions with respect to general ONS.

From the above example we can conclude that if we want the Fourier coefficients of smooth functions to satisfy condition (3), it is necessary that the functions $\varphi_n(x)$ from ONS $\{\varphi_n(x)\}$ satisfy some conditions.

In the present paper we give certain conditions which are imposed on functions of ONS $\{\varphi_n(x)\}$ under which the Fourier coefficients of the function of class $\mathrm{Lip}\,1$ satisfy condition (3).

***Remark* 2.** As it was shown on an example, there exists ONS $\{\varphi_n(x)\}$ for which the Fourier coefficients $f_0(x) = 1$ do not satisfy condition (3).

Thus if we want the Fourier coefficients of the function from class $\mathrm{Lip}\,1$ to satisfy condition (3) with respect to the system $\{\varphi_n(x)\}$ we should require that

$$\sum_{n=1}^{\infty}\widehat{\varphi}_n^2(f_0)\lambda_n = \sum_{n=1}^{\infty}\left(\int\limits_0^1 \varphi_n(x)\,dx\right)^2\lambda_n < +\infty,$$

for otherwise the formulation of the problem with respect to the functional class $\mathrm{Lip}\,1$ has no meaning since $1 \in \mathrm{Lip}\,1$.

3. Main results.

Theorem 1. *Let* $\{\varphi_n(x)\}$ *be ONS on* $[0,1]$ *and a sequence of nondecreasing numbers* (λ_n) *satisfies the following conditions*:

(a) $\lim_{n\to\infty}\lambda_n = +\infty$; $L_n(x) = O(\lambda_n)$ (*see* (2));

(5)

(b) $\sum_{n=1}^{\infty}\left(\int_0^1 \varphi_n(x)\,dx\right)^2\lambda_n < +\infty.$

If for any sequence $(a_n) \in \ell_2$ *the following condition*:

$$S_n(a) = O(1)\left(\sum_{k=1}^{n} a_k^2\lambda_k\right)^{1/2} \tag{6}$$

holds, then the Fourier series of any function $f(x) \in \mathrm{Lip}\,1$

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n(f)\varphi_n(x) \tag{7}$$

converges a.e. on $[0, 1]$.

Proof. The equality (see [5])

$$\int_0^1 f(x)\Phi(x)\,dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right) \int_0^{i/n} \Phi(x)\,dx +$$

$$+ \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{i/n} \left(f(x) - f\left(\frac{i}{n}\right)\right) \Phi(x)\,dx + f(1)\int_0^1 \Phi(x)\,dx \tag{8}$$

holds, where $f(x)$ and $\Phi(x)$ are functions from $L_2(0,1)$ and $f(x)$ is finite at every point $[0,1]$.

We have the equality

$$\sum_{k=1}^{n} \widehat{\varphi}_k^2(f)\lambda_k = \int_0^1 f(x)\sum_{k=1}^{n} \widehat{\varphi}_k(f)\lambda_k\varphi_k(x)\,dx \equiv$$

$$\equiv \int_0^1 f(x)\Phi_n(\widehat{\varphi}, x)\,dx, \tag{9}$$

where $\Phi_n(\widehat{\varphi}, x) = \Phi_n(a, x)$ for $(a_n) = (\widehat{\varphi}_n)$.

In equality (8) assume that $\Phi_n(a, x) = \Phi_n(\widehat{\varphi}, x)$, we get

$$\int_0^1 f(x)\Phi_n(\widehat{\varphi}, x)\,dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right) \int_0^{i/n} \Phi_n(\widehat{\varphi}, x)\,dx +$$

$$+ \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{i/n} \left(f(x) - f\left(\frac{i}{n}\right)\right) \Phi_n(\widehat{\varphi}, x)\,dx + f(1)\int_0^1 \Phi_n(\widehat{\varphi}, x)\,dx =$$

$$= M_1 + M_2 + M_3. \tag{10}$$

Let now $f(x) \in \mathrm{Lip}\,1$. Then by (6) and Remark 1 we have

$$|M_1| = O(1)\,\frac{1}{n}\sum_{i=1}^{n-1} \left| \int_0^{i/n} \Phi_n(\widehat{\varphi}, x)\,dx \right| = O(1)S_n(\widehat{\varphi}), \tag{11}$$

$$|M_2| = O(1)\,\frac{1}{n}\sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{i/n} |\Phi_n(\widehat{\varphi}, x)|\,dx = O(1)\,\frac{1}{n}\int_0^1 |\Phi_n(\widehat{\varphi}, x)|\,dx =$$

$$= O(1)\,\frac{1}{n}\left(\int\limits_0^1 \Phi_n^2(\widehat{\varphi},x)\,dx\right)^{1/2} = O(1)\,\frac{1}{n}\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k^2\right)^{1/2} =$$

$$= O(1)\,\frac{\sqrt{\lambda_n}}{n}\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k\right)^{1/2} = O(1)\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k\right)^{1/2}, \tag{12}$$

$$|M_3| = O(1)\left|\sum_{k=1}^{n}\widehat{\varphi}_k(f)\lambda_k\int\limits_0^1\varphi_n(x)\,dx\right| =$$

$$= O(1)\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k\right)^{1/2}\left(\sum_{k=1}^{n}\left(\int\limits_0^1\varphi_k(x)\,dx\right)^2\lambda_k\right)^{1/2} =$$

$$= O(1)\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k\right)^{1/2}. \tag{13}$$

(Here condition b) of Theorem 1 is taken into account.)

Finally, in (10) taking into account (9), (11), (12) and (13), we get

$$\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k = O(1)\left(\sum_{k=1}^{n}\widehat{\varphi}_k^2(f)\lambda_k\right)^{1/2},$$

i.e.,

$$\sum_{k=1}^{\infty}\widehat{\varphi}_k^2(f)\lambda_k < +\infty.$$

Taking into account now the last inequality from the statement of Theorem A, it follows that series (7) converges a.e. on $[0,1]$.

Theorem 1 is proved.

It should be noted that condition (6) is an important factor for the convergence of the Fourier series of functions from class $\operatorname{Lip} 1$. We will show later what we will have if the condition is not satisfied.

Theorem 2. *Let* $\{\varphi_n(x)\}$ *be ONS on* $[0,1]$ *and* $\lambda_n \uparrow \infty$. *If for some sequence* $(b_n) \in \ell_2$ *condition* (6) *does not hold, i.e.,*

$$\lim_{n\to\infty}\frac{S_n(b)}{\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2}} = +\infty,$$

then there exists the function $f_0(x) \in \operatorname{Lip} 1$ *such that*

$$\sum_{n=1}^{\infty}\widehat{\varphi}_n^2(f_0)\lambda_n = +\infty.$$

Proof. Let us assume from the beginning that

$$\sum_{k=1}^{\infty}\left(\int_0^1 \varphi_k(x)\,dx\right)^2 \lambda_k < +\infty. \tag{14}$$

Otherwise we will get that the Fourier coefficients of the function $f_0(x) = 1$ do not satisfy condition (3) and thus Theorem 2 is valid.

If series (14) is convergent, then we will consider the sequence of functions

$$f_n(x) = \int_0^x \operatorname{sign} \int_0^t \Phi_n(b,u)\,du\,dt, \quad n = 1, 2, \ldots. \tag{15}$$

Assume in equality (10) $f(x) = f_n(x)$ and $\Phi_n(a,x) = \Phi_n(b,x)$ we will have

$$\int_0^1 f_n(x)\Phi_n(b,x)\,dx = \sum_{i=1}^{n-1}\left(f_n\left(\frac{i}{n}\right) - f_n\left(\frac{i+1}{n}\right)\right)\int_0^{i/n}\Phi_n(b,x)\,dx +$$

$$+\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{i/n}\left(f_n(x) - f_n\left(\frac{i}{n}\right)\right)\Phi_n(b,x)\,dx + f_n(1)\int_0^1 \Phi_n(b,x)\,dx =$$

$$= H_1 + H_2 + H_3. \tag{16}$$

Since for $x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$

$$\left|f_n(x) - f_n\left(\frac{i}{n}\right)\right| \le \frac{1}{n},$$

using Remark 1 we obtain

$$|H_2| \le \frac{1}{n}\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{i/n}|\Phi_n(b,x)|\,dx = \frac{1}{n}\int_0^1 |\Phi_n(b,x)|\,dx \le \frac{1}{n}\left(\sum_{k=1}^{n} b_k^2\lambda_k^2\right)^{1/2} =$$

$$= O(1)\,\frac{\sqrt{\lambda_n}}{n}\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2} = O(1)\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2}. \tag{17}$$

Taking into account (14) we get

$$|H_3| \le |f_n(1)|\left|\int_0^1 \Phi_n(b,x)\,dx\right| = |f_n(1)|\left|\sum_{k=1}^{n} b_k\lambda_k\int_0^1 \varphi_k(x)\,dx\right| =$$

$$= O(1)\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2}\left(\sum_{k=1}^{n}\left(\int_0^1 \varphi_k(x)\,dx\right)^2\lambda_k\right)^{1/2} =$$

$$= O(1)\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2}. \tag{18}$$

For estimating M_1 we will need the following lemma.

Lemma 1. *Let I_n be a set of all $i \in \{1,2,\ldots,n\}$ for each of which there exists the point $x_{in} \in \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$ for which the condition*

$$\operatorname{sign}\int_0^{x_{in}} \Phi_n(b,t)\,dt \neq \operatorname{sign}\int_0^{i/n} \Phi_n(b,t)\,dt$$

holds. Then (*see Remark* 2)

$$\sum_{i\in I_n}\left|\int_0^{i/n}\Phi_n(b,x)\,dx\right| = O(1)\sqrt{\lambda_n}\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2}.$$

Proof. In view of condition of the lemma there exists the point $y_{in} \in \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$ such that $\int_0^{y_{in}} \Phi_n(b,x)\,dx = 0$. Then

$$\sum_{i\in I_n}\left|\int_0^{i/n}\Phi_n(b,x)\,dx\right| = \sum_{i\in I_n}\left|\int_0^{y_{in}}\Phi_n(b,x)\,dx\right| + \sum_{i\in I_n}\left|\int_{y_{in}}^{i/n}\Phi_n(b,x)\,dx\right| \le$$

$$\le \sum_{i\in I_n}\int_{\frac{i-1}{n}}^{i/n}|\Phi_n(b,x)|\,dx \le \int_0^1|\Phi_n(b,x)|\,dx \le \left(\sum_{k=1}^{n} b_k^2\lambda_k^2\right)^{1/2} =$$

$$= O(1)\sqrt{\lambda_n}\left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2}. \tag{19}$$

The lemma is proved.

Assume now that $E_n = \{1,2,\ldots,n-1\}\setminus I_n$. Then according to the property of E_n we get

$$\sum_{i=1}^{n-1}\left(f_n\left(\frac{i}{n}\right) - f_n\left(\frac{i+1}{n}\right)\right)\int_0^{i/n}\Phi_n(b,x)\,dx =$$

$$= -\sum_{i=1}^{n-1}\int_{i/n}^{(i+1)/n}\operatorname{sign}\int_0^{x}\Phi_n(b,t)\,dt\,dx\int_0^{i/n}\Phi_n(b,x)\,dx =$$

$$= -\frac{1}{n}\sum_{i\in E_n}\left|\int_0^{i/n}\Phi_n(b,x)\,dx\right| - \sum_{i\in I_n}\int_{i/n}^{(i+1)/n}\operatorname{sign}\int_0^{x}\Phi_n(b,t)\,dt\,dx\int_0^{i/n}\Phi_n(b,x)\,dx.$$

Hence, in view of (19)

$$\left|\sum_{i=1}^{n-1}\left(f_n\left(\frac{i}{n}\right)-f_n\left(\frac{i+1}{n}\right)\right)\int\limits_0^{i/n}\Phi_n(b,x)\,dx\right|\geq$$

$$\geq\frac{1}{n}\sum_{i\in E_n}\left|\int\limits_0^{i/n}\Phi_n(b,x)\,dx\right|-\frac{1}{n}\sum_{i\in I_n}|\Phi_n(b,x)\,dx|=$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left|\int\limits_0^{i/n}\Phi_n(b,x)\,dx\right|-\frac{2}{n}\sum_{i\in I_n}\left|\int\limits_0^{i/n}\Phi_n(b,x)\,dx\right|\geq$$

$$\geq S_n(b)-O(1)\frac{1}{n}\sqrt{\lambda_n}\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}\geq S_n(b)-O(1)\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}. \tag{20}$$

At last, applying (17), (18) and (20) from (16) we have

$$\left|\int\limits_0^1 f_n(x)\Phi_n(b,x)\,dx\right|\geq S_n(b)-O(1)\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}.$$

From here

$$\frac{1}{\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}}\left|\int\limits_0^1 f_n(x)\Phi_n(b,x)\,dx\right|\geq\frac{S_n(b)}{\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}}-O(1). \tag{21}$$

According to the conditions of Theorem 2 from (21) we get

$$\overline{\lim_{n\to\infty}}\frac{1}{\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}}\left|\int\limits_0^1 f_n(x)\Phi_n(b,x)\,dx\right|=\overline{\lim_{n\to\infty}}\frac{S_n(b)}{\left(\sum_{k=1}^{n}b_k^2\lambda_k\right)^{1/2}}=+\infty. \tag{22}$$

Evidently,

$$\|f_n\|_{\mathrm{Lip}\,1}=\|f_n\|_C+\sup_{x,y}\frac{|f_n(x)-f_n(y)|}{|x-y|}=2.$$

Thus, in virtue of Banach – Steinhaus' theorem (see (22)) there exists the function $f_0(x)\in\mathrm{Lip}\,1$ such that

$$\overline{\lim_{n\to\infty}}\frac{\left|\int_0^1 f_0(x)\Phi_n(b,x)\,dx\right|}{\left(\sum_{n=1}^{n}b_k^2\lambda_k\right)^{1/2}}=+\infty. \tag{23}$$

Since

$$\left|\int\limits_0^1 f_0(x)\Phi_n(b,x)\,dx\right|=\left|\sum_{k=1}^{n}b_k\lambda_k\int\limits_0^1 f_0(x)\varphi_n(x)\,dx\right|=$$

$$= \left|\sum_{k=1}^{n} b_k\lambda_k\widehat{\varphi}_k(f_0)\right| \leq \left(\sum_{k=1}^{n} b_k^2\lambda_k\right)^{1/2} \left(\sum_{k=1}^{n} \widehat{\varphi}_k^2(f_0)\lambda_k\right)^{1/2},$$

from (23) we get

$$\sum_{k=1}^{\infty} \widehat{\varphi}_k^2(f_0)\lambda_k = +\infty.$$

Theorem 2 is proved.

As is was stated above, the Fourier coefficients of functions from $\mathrm{Lip}\,1$ do not, in general, satisfy condition (3).

Theorem 3. *From any ONS $\{\varphi_n(x)\}$ one can isolate a subsystem $\{\varphi_{n_k}(x)\} = \{\psi_k(x)\}$ with respect to which for any sequence $(a_n) \in \ell_2$ the condition*

$$S_n(a) = \frac{1}{n}\sum_{i=1}^{n-1}\left|\int_0^{i/n} P_n(a,x)\,dx\right| = O(1)\left(\sum_{k=1}^{n} a_k^2\lambda_k\right)^{1/2}$$

holds, where $P_n(a,x) = \sum_{k=1}^{n} a_k\lambda_k\psi_k(x)$.

Proof. According to Parceval's theorem

$$\sum_{m=1}^{\infty}\left(\int_0^{i/n} \varphi_m(x)\,dx\right)^2 \leq 1, \quad i = 1,\ldots,n,$$

by $k(n)$ we denote the natural number for which

$$\sum_{m=k(n)}^{\infty}\left(\int_0^{i/n} \varphi_m(x)\,dx\right)^2 \leq 2^{-2n}.$$

Thus, when $m \geq k(n)$

$$\left|\int_0^{i/n} \varphi_m(x)\,dx\right| \leq 2^{-n}, \quad i = 1,2,\ldots,n. \tag{24}$$

Assume that $k(n+1) \geq k(n) + 2^n$ $(k(n+1) > k(n))$.

Let

$$\psi_{2^n+1}(x) = \varphi_{k(n)+1}(x), \qquad \psi_{2^n+s}(x) = \varphi_{k(n)+s}(x),$$

where $1 \leq s \leq 2^n$, $n = 1,2,\ldots$. In this way we obtain the sequence of functions $(\psi_m(x))$ and for $m = 2^n + l$ $(1 \leq l \leq 2^n)$ (see (24))

$$\left|\int_0^{i/n} \psi_m(x)\,dx\right| = \left|\int_0^{i/n} \varphi_{k(n)+l}(x)\,dx\right| < 2^{-n}, \quad i = 1,2,\ldots,n. \tag{25}$$

The number $j(i),\ i=1,\ldots,n,\ j=1,\ldots,2^m$, is chosen so that $\left|\dfrac{i}{n}-\dfrac{j(i)}{2^m}\right|\le 2^{-m},\ n=2^m+l$. Then by using (25), we get

$$\left|\int\limits_0^{i/n}\sum_{s=2^m}^{2^{m+1}-1}a_s\lambda_s\psi_s(x)\,dx\right|\le$$

$$\le\left|\int\limits_{i/n}^{j(i)/2^m}\sum_{s=2^m}^{2^{m+1}-1}a_s\lambda_s\psi_s(x)\,dx\right|+\left|\int\limits_0^{j(i)/2^m}\sum_{s=2^m}^{2^{m+1}-1}a_s\lambda_s\psi_s(x)\,dx\right|\le$$

$$\le 2^{-m/2}\left(\sum_{s=2^m}^{2^{m+1}-1}a_s^2\lambda_s^2\right)^{1/2}+\sum_{s=2^m}^{2^{m+1}-1}|a_s\lambda_s|\cdot 2^{-m}. \tag{26}$$

Assume now that $n=2^d+n_1,\ n_1<2^d$, then according to (26)

$$\left|\int\limits_0^{i/n}\sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1}a_s\lambda_s\psi_s(x)\,dx\right|\le$$

$$\le\sum_{m=0}^{d-1}2^{-m/2}\left(\sum_{s=2^m}^{2^{m+1}-1}a_s^2\lambda_s^2\right)^{1/2}+\sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1}|a_s|\lambda_s\cdot 2^{-m}\le$$

$$\le\sum_{m=0}^{d-1}2^{-m/2}\sqrt{\lambda_{2^{m+1}}}\left(\sum_{s=2^m}^{2^{m+1}-1}a_s^2\lambda_s^2\right)^{1/2}+\sum_{m=0}^{d-1}2^{-m}\cdot\sqrt{\lambda_{2^{m+1}}}\cdot 2^{m/2}\left(\sum_{s=2^m}^{2^{m+1}-1}a_s^2\lambda_s\right)^{1/2}\le$$

$$\le\left(\sum_{m=0}^{d-1}2^{-m}\lambda_{2^{m+1}}\right)^{1/2}\left(\sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1}a_s^2\lambda_s\right)^{1/2}+$$

$$+\left(\sum_{m=0}^{d-1}2^{-m}\lambda_{2^{m+1}}\right)^{1/2}\left(\sum_{m=0}^{d-1}\sum_{s=2^m}^{2^{m+1}-1}a_s^2\lambda_s\right)^{1/2}=$$

$$=O(1)\left(\sum_{k=1}^{2^d-1}a_k^2\lambda_k\right)^{1/2}. \tag{27}$$

Further $(n<2^{d+1})$,

$$\left|\int\limits_0^{i/n}\sum_{s=2^m}^{n}a_s\lambda_s\psi_s(x)\,dx\right|\le$$

$$\leq \int\limits_{i/n}^{j(i)/2^d} \left|\sum_{s=2^d}^{n} a_s\lambda_s\psi_s(x)\,dx\right| + \left|\sum_{s=2^d}^{n} a_s\lambda_s \int\limits_0^{j(i)/2^d} \psi_s(x)\,dx\right| \leq$$

$$\leq 2^{-d/2}\left(\sum_{s=2^d}^{n} a_s^2\lambda_s^2\right)^{1/2} + \sum_{s=2^d}^{n} |a_s|\lambda_s 2^{-d} \leq$$

$$\leq 2^{-d/2}\sqrt{\lambda_n}\left(\sum_{s=2^d}^{n} a_s^2\lambda_s\right)^{1/2} + 2^{-d/2}\sqrt{\lambda_n}\left(\sum_{s=2^d}^{n} a_s^2\lambda_s\right)^{1/2} = O(1)\left(\sum_{s=2^d}^{n} a_s^2\lambda_s\right)^{1/2}. \tag{28}$$

Finally, from (27) and (28) for any sequence $(a_n) \in \ell_2$ we have

$$S_n(a) = \frac{1}{n}\sum_{i=1}^{n-1}\left|\int\limits_0^{i/n}\sum_{m=1}^{n} a_m\lambda_m\psi_m(x)\,dx\right| = O(1)\left(\sum_{m=1}^{n} a_m^2\lambda_m\right)^{1/2}. \tag{29}$$

From (29) and Theorem 1 it follows that for any function $f(x) \in \mathrm{Lip}\,1$ condition (3) holds in the case of the subsequence $\{\psi_n(x)\}$.

Theorem 3 is proved.

Problems of efficiency. Condition (6) is said to be efficient if it is easily verified for classical ONS (trigonometric system, Walsh [6] and Haar systems).

Theorem 4. *Let $\{\varphi_n(x)\}$ be ONS such that*

$$\int\limits_0^x \varphi_n(x)\,dx = O\left(\frac{1}{n}\right).$$

Then condition (6) *is fulfilled.*

Proof. For any $i = 1, 2, \ldots, n$ we have

$$\left|\int\limits_0^{i/n}\sum_{k=1}^{n} a_k\lambda_k\varphi_k(x)\,dx\right| = \left|\sum_{k=1}^{n} a_k\lambda_k\int\limits_0^{i/n}\varphi_k(x)\,dx\right| =$$

$$= O(1)\sum_{k=1}^{n}\frac{|a_k|}{k}\lambda_k = O(1)\left(\sum_{k=1}^{n}\frac{\lambda_k}{k^2}\right)^{1/2}\left(\sum_{k=1}^{n} a_k^2\lambda_k\right)^{1/2} = O(1)\left(\sum_{k=1}^{n} a_k^2\lambda_k\right)^{1/2}.$$

From here and from Theorem 1 it follows that for the trigonometric and Walsh systems condition (6) is satisfied.

Theorem 5. *If (χ_n) is a Haar system* (*see* [2, *p.* 54]), *then condition* (6) *is fulfilled.*

Proof. Using the definition of the Haar function, we get $(n = 2^p + l,\ l < 2^p)$

$$\left|\int\limits_0^{i/n}\sum_{k=2^s}^{2^{s+1}-1} a_k\lambda_k\chi_k(x)\,dx\right| \leq 2^{-s/2}|a_{k(i)}|\lambda_{k(i)}, \quad \text{where} \quad 2^s \leq k(i) < 2^{s+1}.$$

Therefore,

$$\left|\int_0^{i/n}\sum_{k=1}^{2^p-1}a_k\lambda_k\chi_k(x)\,dx\right| = \left|\sum_{k=0}^{p-1}\int_0^{i/n}\sum_{m=2^k}^{2^{k+1}-1}a_m\lambda_m\chi_m(x)\,dx\right| \le$$

$$\le \sum_{k=0}^{p-1}2^{-k/2}|a_{k(i)}|\lambda_{k(i)} \le \sum_{k=0}^{p-1}|a_{k(i)}|\sqrt{\lambda_{k(i)}}\sqrt{\lambda_{k(i)}}2^{-k/2} \le$$

$$\le \left(\sum_{k=0}^{p-1}a_{k(i)}^2\lambda_{k(i)}\right)^{1/2}\left(\sum_{k=0}^{p-1}2^{-k}\lambda_{k(i)}\right)^{1/2} \le$$

$$\le \left(\sum_{k=0}^{p-1}\sum_{m=2^k}^{2^{k+1}-1}a_m^2\lambda_m\right)^{1/2}\left(\sum_{k=0}^{p-1}2^{-k}\lambda_{k(i)}\right)^{1/2} = O(1)\left(\sum_{k=1}^{2^p}a_k^2\lambda_k\right)^{1/2}, \quad i=1,2,\ldots,n.$$

In a similar way it can be proved that

$$\left|\int_0^{i/n}\sum_{k=2^p}^{n}a_k\lambda_k\chi_k(x)\,dx\right| = O(1)\left(\sum_{k=2^p}^{n}a_k^2\lambda_k\right)^{1/2}.$$

At last we can conclude that

$$S_n(a) = \frac{1}{n}\left|\int_0^{i/n}\sum_{k=1}^{n}a_k\lambda_k\chi_k(x)\,dx\right| = O(1)\left(\sum_{k=1}^{n}a_k^2\lambda_k\right)^{1/2}.$$

Theorem 5 is proved.

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