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POLYNOMIAL INEQUALITIES IN QUASIDISKS ON WEIGHTED BERGMAN SPACE*

ПОЛІНОМІАЛЬНІ НЕРІВНОСТІ У КВАЗІДИСКАХ НА ЗВАЖЕНИХ ПРОСТОРАХ БЕРГМАНА

We continue studying on the Nikol'skii and Bernstein – Walsh type estimations for complex algebraic polynomials in the bounded and unbounded quasidisks on the weighted Bergman space.

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1. Introduction. Let $\mathbb{C}$ be a complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ such that $0 \in G$, $\Omega := \overline{\mathbb{C}} \backslash \overline{G} = \operatorname{ext} L$, $\Delta := \{w \colon |w| > 1\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ such that $\Phi(\infty) = \infty$ and $\lim_{z \to \infty} \dfrac{\Phi(z)}{z} > 0$, $\Psi := \Phi^{-1}$. For $R > 1$, we take $L_R := \{z \colon |\Phi(z)| = R\}$, $G_R := \operatorname{int} L_R$ and $\Omega_R := \operatorname{ext} L_R$. Let $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$.

In this work we consider the following weight function $h(z)$: Let $\{z_j\}_{j=1}^m$ be the fixed system of distinct points on the curve L. For some fixed R_0, $1 < R_0 < \infty$, consider generalized Jacobi weight function $h(z)$ which is defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \quad z \in \overline{G}_{R_0}, \tag{1.1}$$

where $\gamma_j > -2$, for all $j = 1, 2, \ldots, m$, and h_0 is uniformly separated from zero on L, i.e., there exists a constant $c_0 = c_0(G) > 0$ such that $h_0(z) \geq c_0 > 0$ for all $z \in G_{R_0}$.

Let $0 < p \leq \infty$. For the Jordan region G, we introduce

$$\|P_n\|_p := \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \, |P_n(z)|^p \, d\sigma_z \right)^{1/p}, \quad 0 < p < \infty, \tag{1.2}$$

$$\|P_n\|_\infty := \|P_n\|_{A_\infty(1,G)} := \max_{z \in \overline{G}} |P_n(z)|, \quad p = \infty. \tag{1.3}$$

In this work, firstly we study the following Nikol'skii-type inequality:

$$\|P_n\|_\infty \leq c_1 \lambda_n(G, h, p) \, \|P_n\|_p, \tag{1.4}$$

where $c_1 = c_1(G, h, p) > 0$ is a constant independent of n and P_n, and $\lambda_n(G, h, p) \to \infty$, $n \to \infty$, depending on the geometrical properties of the region G and the weight function h. The estimates

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of (1.4)-type for some $0 < p < \infty$ and $h(z)$ were investigated in [21, p. 122 – 133], [14], [20] (Sect. 5.3), [25], [2 – 8] (see also references therein) and others.

Secondly, we find pointwise estimations in unbounded region Ω:

$$|P_n(z)| \leq c_2 \frac{\eta_n(G,h,p)}{d(z,L)} \|P_n\|_p \, |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{1.5}$$

where $c_2 = c_2(G,h,p) > 0$ is a constant independent of n and P_n, and $\eta_n(G,h,p) \to \infty$, $n \to \infty$, again depending on the region G and h. We note that, according to the maximum principle the inequality (1.4) holds also for $\left| \dfrac{P_n(z)}{\Phi^{n+1}(z)} \right|$ for any points $z \in \Omega$. But we will try to have a more accurate estimate for $\left| \dfrac{P_n(z)}{\Phi^{n+1}(z)} \right|$ of the (1.5)-type.

Analogous results of (1.5)-type for some norms and for different unbounded regions were obtained by N. A. Lebedev, P. M. Tamrazov, V. K. Dzjadyk (see, for example, [17, p. 418 – 428]), V. V. Andrievskii [13], N. Stylianopoulos [26], F. G. Abdullayev et al. [7, 8] and others.

Finally, combining obtained estimates for $|P_n(z)|$ on bounded and unbounded regions, we get the evaluation for $|P_n(z)|$ in whole complex plane.

2. Definitions and main results. Throughout this paper, c, c_0, c_1, $c_2, \ldots$ are positive and ε_0, ε_1, $\varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k,m}$ means $i = k, k+1, \ldots, m$. Let the function φ maps G conformally and univalently onto $B := \{w \colon |w| < 1\}$ which is normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$; let $\psi := \varphi^{-1}$.

Definition 2.1 [22, p. 286 – 294]. *A bounded Jordan region G is called a K-quasidisk, $0 \leq \leq k < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$, where $K = \dfrac{1+k}{1-k}$. In that case the curve $L := \partial G$ is called a K-quasicircle.*

Let G be a region bounded by two arcs of circle, symmetric with respect to the OX-axis and OY-axis, each of the arcs crosses the OX-axis at $\pm\varepsilon_0$, where $\varepsilon_0 > 0$ and the angle between the arcs is $\pi(1-k)$ where $0 \leq k < 1$. This region is a K-quasidisk.

A region G (curve L) is called a *quasidisk* (*quasicircle*), if it is a K-quasidisk (K-quasicircle) for some $0 \leq k < 1$. A Jordan curve L is called a quasicircle or quasiconformal curve, if it is the image of the unit circle under a quasiconformal mapping of $\mathbb{C}$ (see [18, p. 105; 22, p. 286]). On the other hand, it is given some geometric criteria of quasiconformality of the curves (see also [10, p. 81; 23, p. 107; 19, p. 341]). Now we give one of them.

Let z_1, z_2 be an arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2. Lesley [19, p. 341] defined the curve L as "*c-quasiconformal*", if for all $z_1, z_2 \in L$ and $z \in L(z_1, z_2)$ there exists a constant c, independent from points z_1, z_2 and z, such that

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \leq c. \tag{2.1}$$

A simple example of c-quasiconformal curves can be given a polygon whose smallest interior or exterior opening angle $2\arcsin(1/c)$. It is well known that quasicircles can be nonrectifiable (see, for example, [15; 18, p. 104]).

In this work, we study similar problems to (1.4) and (1.5) for quasidisks. Now, we start to formulate the new results.

Theorem 2.1. *Let* $p>0$, G *be a* K*-quasidisk for some* $0\leq k<1$ *and* $h(z)$ *be defined by* (1.1). *Then, for any* $P_n\in\wp_n$, $n\in\mathbb{N}$,

$$|P_n(z_j)|\leq c_1 n^{(2+\gamma_j)(1+k)/p}\left\|P_n\right\|_p, \quad j=\overline{1,m}, \tag{2.2}$$

and, consequently,

$$\left\|P_n\right\|_\infty\leq c_2 n^{(2+\gamma)(1+k)/p}\left\|P_n\right\|_p, \tag{2.3}$$

where $\gamma:=\max\left\{0;\gamma_j,\, j=\overline{1,m}\right\}$.

For any $\rho>1$ we divide Ω_ρ as $\Omega_\rho:=\bigcup\limits_{j=1}^{m}\Omega_\rho^j$, where Ω_ρ^j defined below in (4.6). For given $\vartheta>0$, we define

$$\nu_n:=\begin{cases}1, & \text{if}\quad \gamma_j<\vartheta \quad \text{for all}\quad j=\overline{1,m},\\ \ln n, & \text{if}\quad \text{exists}\quad j_0,\quad \gamma_{j_0}=\vartheta\quad \text{and}\quad \gamma_j<\vartheta\quad \text{for}\quad j\neq j_0,\\ n^{\gamma_{j_0}/\vartheta-1}, & \text{if}\quad \text{exists}\quad j_0,\quad \gamma_{j_0}>\vartheta\quad \text{and}\quad \gamma_j\leq\vartheta\quad \text{for}\quad j\neq j_0.\end{cases} \tag{2.4}$$

Theorem 2.2. *Let* $p>0$, L *be a* K*-quasicircle for some* $0\leq k<1$ *and* $h(z)$ *be defined by* (1.1). *Then there exists a constant* $c_3=c_3(G,p,\gamma_j)>0$ *such that for any* $P_n\in\wp_n$, $n\in\mathbb{N}$, *and for* $j=\overline{1,m}$ *the inequality*

$$|P_n(z)|\leq c_3\left(\frac{\sqrt{n\mu_n}}{d(z,L_R)}\right)^{2/p}\left\|P_n\right\|_p|\Phi(z)|^{n+1}, \quad z\in\Omega_R^j, \tag{2.5}$$

holds, where $R=1+\dfrac{\varepsilon_1}{n}$ *and* $\mu_n=\nu_n$ *defined in* (2.4) *with* $\vartheta=\dfrac{1}{1+k}$.

According to the Bernstein lemma [30], the estimation (2.3) is again true for the $z\in\overline{G}_{R_1}$ with a different constant. Therefore, combining estimation (2.3) (for the $z\in\overline{G}_{R_1}$) with (2.5), we obtain an estimation for the growth of $|P_n(z)|$ in the whole complex plane:

***Corollary* 2.1.** *Under the assumptions of Theorems* 2.1 *and* 2.2, *the following is true*:

$$|P_n(z)|\leq c_4\left\|P_n\right\|_p\begin{cases} n^{(2+\gamma)(1+k)/p}, & z\in\overline{G}_R,\\ \left(\dfrac{\sqrt{n\mu_n}}{d(z,L_R)}\right)^{2/p}|\Phi(z)|^{n+1}, & z\in\Omega_R^j,\end{cases} \tag{2.6}$$

where $c_4=c_4(G,p,h)>0$.

***Corollary* 2.2.** *For any compact subset* $F\Subset\Omega$ *and* $P_n\in\wp_n$, $n\in\mathbb{N}$, *we have*

$$|P_n(z)|\leq c_5\left(\frac{\sqrt{n\mu_n}}{d(z,L)}\right)^{2/p}\left\|P_n\right\|_p|\Phi(z)|^{n+1}, \quad z\in F, \tag{2.7}$$

where $c_5=c_5(G,F)>0$.

From the conditions of the theorems, we see that they hold for K-quasidisks with $0 \le k < 1$. But calculating the coefficient of quasiconformality for some curves is not an easy task. Therefore, we define more general class of curves with another characteristics. One of them is the following:

Definition 2.2. *We say that* $L = \partial G \in Q_\alpha$, $0 < \alpha \le 1$, *if* L *is a quasicircle and* $\Phi \in \operatorname{Lip}\alpha$ *on* $\overline{\Omega}$.

We note that the classes Q_α are sufficiently large. A detailed information on it and the related topics are contained in [**?**, **?**, 19] (see also the references cited therein). We consider only some cases:

***Remark* 2.1.** (a) If L is a Dini-smooth curve [**?**, p. 48], then $L \in Q_1$.

(b) If L is a piecewise Dini-smooth curve and largest exterior angle on L has opening $\alpha\pi$, $0 < \alpha \le 1$ [**?**, p. 52], then $L \in Q_\alpha$.

(c) If L is a smooth curve having continuous tangent, then $L \in Q_\alpha$ for all $0 < \alpha < 1$.

(d) If G is "L-shaped" region, then $\Phi \in \operatorname{Lip}\frac{2}{3}$, $\Psi \in \operatorname{Lip}\frac{1}{2}$.

(e) If L is quasismooth (in the sense of Lavrentiev), that is, for every pair $z_1, z_2 \in L$, there exists a constant $c > 1$ such that $s(z_1, z_2) \le c|z_1 - z_2|$, then $\Phi \in \operatorname{Lip}\alpha$ for $\alpha = \dfrac{\pi}{2(\pi - \arcsin(1/c))}$ and $\Psi \in \operatorname{Lip}\beta$ for $\beta = \dfrac{2}{(1+c)^2}$, where $s(z_1, z_2)$ represents the smallest of the lengths of the arcs joining z_1 to z_2 on L [**?**, 29].

(f) If L is a c-quasiconformal, then $\Phi \in \operatorname{Lip}\alpha$ for $\alpha = \dfrac{\pi}{2(\pi - \arcsin(1/c))}$ and $\Psi \in \operatorname{Lip}\beta$ for $\beta = \dfrac{2(\arcsin(1/c))^2}{\pi(\pi - \arcsin(1/c))}$.

For $\partial G \in Q_\alpha$ we have the following results:

Theorem 2.3. *Let* $p > 0$, $L \in Q_\alpha$ *for some* $\frac{1}{2} \le \alpha \le 1$ *and* $h(z)$ *be defined by* (1.1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$,

$$|P_n(z_j)| \le c_6 n^{(2+\gamma_j)/p\alpha} \|P_n\|_p, \quad j = \overline{1, m}, \tag{2.8}$$

and, *consequently*,

$$\|P_n\|_\infty \le c_7 n^{(2+\gamma)/p\alpha} \|P_n\|_p, \tag{2.9}$$

where $\gamma := \max\left\{0; \gamma_j,\ j = \overline{1, m}\right\}$.

Theorem 2.4. *Let* $p > 0$, $L \in Q_\alpha$ *for some* $\frac{1}{2} \le \alpha \le 1$ *and* $h(z)$ *be defined by* (1.1). *Then, there exists* $c_8 = c_8(G, p, \gamma_j) > 0$ *such that for any* $P_n \in \wp_n$, $n \in \mathbb{N}$,

$$|P_n(z)| \le c_8 \left(\frac{\sqrt{n\delta_n}}{d(z, L_R)}\right)^{2/p} \|P_n\|_p |\Phi(z)|^{n+1}, \qquad z \in \Omega_R^j, \quad j = \overline{1, m}, \tag{2.10}$$

where $R = 1 + \frac{\varepsilon_1}{n}$ *and* $\delta_n = \nu_n$ *defined in* (2.4) *with* $\vartheta = \alpha$.

***Corollary* 2.3.** *Under the conditions Theorems* 2.3 *and* 2.4, *we have*

$$|P_n(z)| \le c_9 \|P_n\|_p \begin{cases} n^{(2+\gamma)/\alpha p}, & z \in \overline{G}_R, \\ \left(\dfrac{\sqrt{n\delta_n}}{d(z, L_R)}\right)^{2/p} |\Phi(z)|^{n+1}, & z \in \Omega_R^j, \end{cases} \tag{2.11}$$

where $c_9 = c_9(G, p, \gamma_j) > 0$.

***Corollary* 2.4.** *Let* $p > 0,$ L *be a* c*-quasiconformal for some* $c > 1$ *and* $h(z)$ *be defined by* (1.1). *Then, for any* $P_n \in \wp_n,$ $n \in \mathbb{N}$, *the estimations* (2.8), (2.9) *and* (2.10) *hold with* $\alpha = \dfrac{\pi}{2(\pi - \arcsin(1/c))}$.

Note that, the estimations, analogous to Theorems 2.1 and 2.3 for $p > 1$ and to Theorems 2.2 and 2.4 for $h(z) \equiv 1$ were investigated in [6] ([Theorems 5.1 and 5.2) and [**?**], respectively.

Sharpness of the inequalities. The sharpness of the estimations (2.2) – (2.7) can be discussed by comparing them with the following result.

***Remark* 2.2.** (a) For any $n \in \mathbb{N}$ there exists a polynomial $Q_n^* \in \wp_n$ such that the inequality

$$\|Q_n^*\|_\infty \geq c_{10} n \, \|Q_n^*\|_{A_2(B)}\,, \qquad (2.12)$$

is true for the unit disk B and the weight function $h(z) \equiv 1$.

(b) For any $n \in \mathbb{N}$ there exists a polynomial $P_n^* \in \wp_n$, a region $G_1^* \subset \mathbb{C}$ and a compact subset $F^* \Subset \Omega \backslash \overline{G}_1^*$ such that

$$|P_n^*(z)| \geq c_{11} \frac{\sqrt{n}}{d(z,L)} \, \|P_n^*\|_{A_2(G_1^*)} \, |\Phi(z)|^{n+1} \qquad \text{for all} \quad z \in F. \qquad (2.13)$$

(c) For any $n \in \mathbb{N}$ there exists a polynomial $T_n^* \in \wp_n$ such that the inequality

$$\|T_n^*\|_\infty \geq c_{12} n^2 \, \|T_n^*\|_{A_2(h^*,B)} \qquad (2.14)$$

is true for the unit disk B and the weight function $h^*(z) = |z - z_1|^2$.

3. Some auxiliary results. Throughout this paper, for $a > 0$ and $b > 0$, we use the expression "$a \preceq b$" (order inequality), if $a \leq cb$. The expression "$a \asymp b$" means that "$a \preceq b$" and "$b \preceq a$".

Let L be a K-quasiconformal curve, then there exists a quasiconformal reflection $y(\cdot)$ across L such that $y(G) = \Omega, y(\Omega) = G$ and $y(\cdot)$ is fixed the points of L. There exists a quasiconformal reflection $y(\cdot)$ satisfying the following conditions [10; 12, p. 26]:

$$|y(\zeta) - z| \asymp |\zeta - z| \qquad \text{for all} \quad z \in L \qquad \text{and} \qquad \varepsilon < |\zeta| < \frac{1}{\varepsilon},$$

$$\left|y_{\overline{\zeta}}\right| \asymp |y_\zeta| \asymp 1, \qquad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \qquad (3.1)$$

$$\left|y_{\overline{\zeta}}\right| \asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon; \qquad \left|y_{\overline{\zeta}}\right| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon},$$

and for the Jacobian $J_y = |y_z|^2 - |y_{\overline{z}}|^2$ of $y(\cdot)$, the relation $\left|y_{\overline{\zeta}}\right|^2 \leq \dfrac{1}{1-k^2} J_y$ holds, where $k = \dfrac{K^2-1}{K^2+1}$. Such quasiconformal reflection $y(\cdot)$ is called a regular quasiconformal reflection across L.

Let L be a quasicircle and $y(\cdot)$ be a regular quasiconformal reflection across L. For any $R > 1$, we put $L^* := y(L_R)$, $G^* := \operatorname{int} L^*$, $\Omega^* := \operatorname{ext} L^*$, and denote by Φ_R the conformal mapping of Ω^* onto Δ with the normalization $\Phi_R(\infty) = \infty$, $\lim_{z\to\infty} \dfrac{\Phi_R(z)}{z} > 0$, and let $\Psi_R := \Phi_R^{-1}$. Moreover, for any $t > 1$, we set $L_t^* := \{z \colon |\Phi_R(z)| = t\}$, $G_t^* := \operatorname{int} L_t^*$, $\Omega_t^* := \operatorname{ext} L_t^*$. According to [11], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$, we have

$$d(z,L) \asymp d(t,L_R) \asymp d(z,L_R^*), \tag{3.2}$$

$$|\Phi_R(z)| \leq |\Phi_R(t)| \leq 1 + c(R-1).$$

Lemma 3.1 [1]**.** *Let L be a quasicircle, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z\colon\ |z - z_1| \leq d(z_1, L_{r_0})\}$, where $L_{r_0} := \{\zeta\colon\ |\varphi(\zeta)| = r_0,\ 0 < r_0 < 1\}$ and $r_0 := r_0(G)$ is a constant, depending on G, $w_j = \Phi(z_j)$, $j = 1,2,3$. Then*

(a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

(b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon < 1$ and $c > 1$.

Lemma 3.2. *Let G be a K-quasidisk for some $0 \leq k < 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+k}$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact is derived to appropriate the results for the estimation $|\Psi'(\tau)|$ (see [22, p. 287] (Lemma 9.9) and [12] (Theorem 2.8)).

Lemma 3.3 ([?], Lemma 2.3)**.** *Let L be a quasicircle. For arbitrary $R > 1$, there exist numbers $\rho_1,\ \rho_2, \rho_3$ and ρ_4 such that $\rho_1 < \rho_2$ and $\rho_3 < \rho_4$ and the following conditions are satisfied:*

(1) $\overline{G}^*_{\rho_1} \subseteq \overline{G} \subseteq \overline{G}^*_{\rho_2}$ *and* $\overline{G}^*_{\rho_3} \subseteq \overline{G}_R \subseteq \overline{G}^*_{\rho_4}$,

(2) $\rho_1 - 1 \asymp \rho_2 - 1 \asymp \rho_3 - 1 \asymp \rho_4 - 1 \asymp R - 1$.

Lemma 3.4 ([4], Lemma 3.3)**.** *Let L be a quasicircle; $h(z)$ is defined as in* (1.1). *Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{A_p(h,G_R)} \preceq [1 + c(R-1)]^{n+1/p} \|P_n\|_p\,, \quad p > 0.$$

4. Proofs of the main results. ***4.1. Proofs of Theorems* 2.1 *and* 2.3.** We begin with the proof of the Theorem 2.3. Let $L \in Q_\alpha$ for some $\frac{1}{2} \leq \alpha \leq 1$, $R = 1 + \frac{1}{n}$ and $R_1 := 1 + \varepsilon_1(R-1)$. Denote by $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, the zeros of $P_n(z)$ lying on Ω (if such zeros exist) and let

$$B_m(z) := \prod_{j=1}^{m} \tilde{B}_j(z) = \prod_{j=1}^{m} \frac{\Phi(z) - \Phi(\zeta_j)}{1 - \overline{\Phi(\zeta_j)}\Phi(z)} \tag{4.1}$$

denote a Blaschke function with respect of the zeros of $P_n(z)$. For any $p > 0$ and $z \in \Omega$, let us set

$$H_{n,p}(z) := \left(\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right)^{p/2}. \tag{4.2}$$

The function $H_{n,p}(z)$, $H_{n,p}(\infty) = 0$, is analytic in Ω, continuous on $\overline{\Omega}$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $H_{n,p}(z)$ and for this branch we maintain the same designation. Cauchy integral representation for the region Ω is given as:

$$H_{n,p}(z)=-\frac{1}{2\pi i}\int\limits_{L_{R_1}}H_{n,p}(\zeta)\frac{d\zeta}{\zeta-z},\quad z\in\Omega_{R_1}.$$

Since $|B_m(\zeta)|=1$ for $\zeta\in L$, then for arbitrary ε, $0<\varepsilon<\varepsilon_1$, there exists a circle $|w|=1+\dfrac{\varepsilon}{n}$, such that for any $j=\overline{1,m}$ the following is satisfied:

$$\left|\tilde{B}_j(\zeta)\right|>1-\varepsilon,\quad \zeta\in L_{1+\varepsilon/n}.$$

Then $|B_m(\zeta)|>(1-\varepsilon)^m\succeq 1$ for $\zeta\in L_{R_1}$ and $|B_m(z)|\le 1$ for $z\in\Omega_{R_1}$. On the other hand, $|\Phi(\zeta)|=R_1>1$ for $\zeta\in L_{R_1}$. Therefore for any $z\in\Omega_{R_1}$ from the inequality

$$|H_{n,p}(z)|\le\frac{1}{2\pi}\int\limits_{L_{R_1}}\left|H_{n,p}(\zeta)\right|\frac{|d\zeta|}{|\zeta-z|}$$

we have

$$|P_n(z)|^{p/2}\le\frac{\left|B_m(z)\ \Phi^{n+1}(z)\right|^{p/2}}{2\pi}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{B_m(\zeta)\ \Phi^{n+1}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta-z|}\preceq$$

$$\preceq\left|\Phi^{n+1}(z)\right|^{p/2}\int\limits_{L_{R_1}}\left|P_n(\zeta)\right|^{p/2}\frac{|d\zeta|}{|\zeta-z|}. \tag{4.3}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w=\Phi(z)$ and applying the Hölder inequality, we obtain

$$\left(\int\limits_{L_{R_1}}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z|}\right)^2=\left(\int\limits_{L_{R_1}}h^{1/2}(\zeta)\,|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{h^{1/2}(\zeta)|\zeta-z|}\right)^2\le$$

$$\le\int\limits_{|t|=R_1}h(\Psi(t))\,|P_n(\Psi(t))|^p\left|\Psi'(t)\right|^2|dt|\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2}=$$

$$=\int\limits_{|t|=R_1}|f_{n,p}(t)|^p\,|dt|\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))\,|\Psi(t)-\Psi(w)|^2}=:\mathcal{A}_n\mathcal{B}_n(w), \tag{4.4}$$

where $f_{n,p}(t):=h^{1/p}(\Psi(t))P_n(\Psi(t))(\Psi'(t))^{2/p}$, $|t|=R_1$.

For the estimate integral $\mathcal{A}_n$, we separate the circle $|t|=R_1$ to n equal parts δ_n with $\operatorname{mes}\delta_n=$ $=\dfrac{2\pi R_1}{n}$ and by applying the mean value theorem, we get

$$\mathcal{A}_n=\int\limits_{|t|=R_1}|f_{n,p}(t)|^p\,|dt|=\sum_{k=1}^{n}\int\limits_{\delta_k}|f_{n,p}(t)|^p\,|dt|=\sum_{k=1}^{n}\left|f_{n,p}\left(t_k'\right)\right|^p\operatorname{mes}\delta_k,\quad t_k'\in\delta_k.$$

On the other hand, by applying mean value estimation, we have

$$\left|f_{n,p}\left(t'_k\right)\right|^p \le \frac{1}{\pi\left(\left|t'_k\right|-1\right)^2} \iint\limits_{\left|\xi-t'_k\right|<\left|t'_k\right|-1} \left|f_{n,p}\left(\xi\right)\right|^p d\sigma_\xi,$$

and therefore

$$\mathcal{A}_n \preceq \sum_{k=1}^{n} \frac{\operatorname{mes}\delta_k}{\pi\left(\left|t'_k\right|-1\right)^2} \iint\limits_{\left|\xi-t'_k\right|<\left|t'_k\right|-1} \left|f_{n,p}(\xi)\right|^p d\sigma_\xi, \quad t'_k \in \delta_k.$$

By taking into account the fact that at most two of the discs with center t'_k are intersecting, we obtain

$$\mathcal{A}_n \preceq \frac{\operatorname{mes}\delta_1}{\left(\left|t'_1\right|-1\right)^2} \iint\limits_{1<|\xi|<R} \left|f_{n,p}(\xi)\right|^p d\sigma_\xi \preceq n \iint\limits_{1<|\xi|<R} \left|f_{n,p}(\xi)\right|^p d\sigma_\xi.$$

According to Lemma 3.4, we get

$$\mathcal{A}_n \preceq n \iint\limits_{G_R\setminus G} h(\zeta)\left|P_n(\zeta)\right|^p d\sigma_\zeta \preceq n\left\|P_n\right\|_p^p. \tag{4.5}$$

To estimate the integral $\mathcal{B}_n(w)$, we introduce for $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, $j = \overline{1,m}$, and for any $\rho > 1$

$$\Delta_1(\rho) := \left\{t = re^{i\theta} : r > \rho,\ \frac{\varphi_m+\varphi_1}{2} \le \theta < \frac{\varphi_1+\varphi_2}{2}\right\},$$

$$\Delta_j(\rho) := \left\{t = re^{i\theta} : r > \rho, \frac{\varphi_{j-1}+\varphi_j}{2} \le \theta < \frac{\varphi_j+\varphi_{j+1}}{2}\right\}, \quad j = \overline{2,m-1},$$

$$\Delta_m(\rho) := \left\{t = re^{i\theta} : r > \rho, \frac{\varphi_{m-1}+\varphi_m}{2} \le \theta < \frac{\varphi_m+\varphi_1}{2}\right\}.$$

Let $\Omega^j := \Psi(\Delta_j)$, $\Omega^j_\rho := \Psi(\Delta_j(\rho))$, $\Delta_j := \Delta_j(1)$ and

$$L^j := L \cap \overline{\Omega}^j, \qquad L^j_\rho := L_\rho \cap \overline{\Omega}^j_\rho. \tag{4.6}$$

Then we get

$$\mathcal{B}_n(w) = \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t))\left|\Psi(t)-\Psi(w)\right|^2} \preceq$$

$$\preceq \sum_{j=1}^{m} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\prod_{j=1}^{m}\left|\Psi(t)-\Psi(w_j)\right|^{\gamma_j}\left|\Psi(t)-\Psi(w)\right|^2} \asymp$$

$$\asymp \sum_{j=1}^{m} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\left|\Psi(t)-\Psi(w_j)\right|^{\gamma_j}\left|\Psi(t)-\Psi(w)\right|^2} := \sum_{j=1}^{m} B_{n,j}(w), \tag{4.7}$$

since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct. It remains to estimate the integrals $B_{n,j}(w)$ for each $j=\overline{1,m}$.

Firstly, we assume that $z \in L_R$. For simplicity of our next calculations, we assume that $m=1$. We put $\Phi(L^1_{R_1}) = \bigcup\limits_{i=1}^{3} K_i(R_1)$, where

$$K_1(R_1) := \left\{t \in \Phi(L^1_{R_1})\colon\ |t-w_1| < \frac{c_1}{n}\right\},$$

$$K_2(R_1) := \left\{t \in \Phi(L^1_{R_1})\colon \frac{c_1}{n} \le |t-w_1| < c_2\right\},$$

$$K_3(R_1) := \left\{t \in \Phi(L^1_{R_1})\colon c_2 \le |t-w_1| < c_3 < \operatorname{diam} \overline{G}\right\}$$

and $\Phi(L^1_R) = \bigcup\limits_{i=1}^{3} K_i(R)$, where

$$K_1(R) := \left\{\tau \in \Phi(L^1_R)\colon\ |\tau-w_1| < \frac{2c_1}{n}\right\},$$

$$K_2(R) := \left\{\tau \in \Phi(L^1_R)\colon \frac{2c_1}{n} \le |\tau-w_1| < c_2\right\},$$

$$K_3(R) := \left\{\tau \in \Phi(L^1_R)\colon c_2 \le |\tau-w_1| < c_3 < \operatorname{diam} \overline{G}\right\}.$$

Let $w \in \Phi(L^1_R)$ an arbitrary fixed point. We will estimate the following integral for each cases with $w \in K_i(R)$ and $t \in K_i(R_1)$, $i=1,2,3$:

$$B_{n,1}(w) := \int\limits_{\Phi(L^1_{R_1})} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} =$$

$$= \sum_{i=1}^{3} \int\limits_{K_i(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2} =:$$

$$=: \sum_{i=1}^{3} B^i_{n,1}(w). \tag{4.8}$$

Case 1. Let $w \in K_1(R)$. We put $K^1_j(R_1) := \left\{t \in \Phi(L^1_{R_1})\colon\ |t-w_1| < |t-w|\right\}$, $K^2_j(R_1) :=$ $:= K_j(R) \backslash K^1_j(R_1)$, $j=1,2$. Since $\Phi \in \operatorname{Lip}\alpha$ on $\overline{\Omega}$, according to Lemma 3.1 we have

$$B^1_{n,1}(w) := \int\limits_{K^1_1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}} + \int\limits_{K^2_1(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^{2+\gamma_1}} \preceq$$

$$\preceq \int\limits_{K^1_1(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)/\alpha}} + \int\limits_{K^2_1(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)/\alpha}} \preceq n^{(2+\gamma_1)/\alpha-1}, \quad \gamma_1>0,$$

$$B^1_{n,1}(w) = \int\limits_{K_1(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{-\gamma_1}|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{K_1(R_1)} \frac{|dt|}{|t-w|^{2/\alpha}} \preceq n^{2/\alpha-1}, \quad \gamma_1 \le 0,$$

$$B^2_{n,1}(w) := \sum_{i=1}^{2} \int\limits_{K^i_2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}|\Psi(t)-\Psi(w)|^2} \preceq$$

$$\preceq \int\limits_{K^1_2(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)/\alpha}} + \int\limits_{K^2_2(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)/\alpha}} \preceq n^{(2+\gamma_1)/\alpha-1}, \quad \gamma_1 > 0,$$

$$B^2_{n,1}(w) = \int\limits_{K_2(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{-\gamma_1}|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{K_2(R_1)} \frac{|dt|}{|t-w|^{2/\alpha}} \preceq n^{2/\alpha-1}, \quad \gamma_1 \le 0.$$

Since $|t-w_1| \ge c_2$ and $|t-w| \ge ||t-w_1|-|w-w_1|| \ge c_2 - \dfrac{2c_1}{n} \succeq 1$, for $t \in K_3(R_1)$ and $w \in K_1(R)$, then we obtain

$$B^3_{n,1}(w) \preceq \int\limits_{K_3(R_1)} |dt| \preceq |K_3(R_1)| \preceq 1, \quad \gamma_1 > 0,$$

$$B^3_{n,1}(w) \preceq \int\limits_{K_3(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{K_3(R_1)} |dt| \preceq 1, \quad \gamma_1 \le 0.$$

Case 2. Let $w \in K_2(R)$. Then

$$B^1_{n,1}(w) \preceq \int\limits_{K_1(R_1)} \frac{|dt|}{|t-w_1|^{\gamma_1/\alpha}|t-w|^{2/\alpha}} \preceq$$

$$\preceq \frac{1}{(R_1-1)^{\gamma_1/\alpha}\left(\dfrac{c_2}{n}\right)^{2/\alpha}} \int\limits_{K_1(R_1)} |dt| \preceq n^{(2+\gamma_1)/\alpha-1}, \quad \gamma_1 > 0,$$

$$B^1_{n,1}(w) \preceq \int\limits_{K_1(R_1)} \frac{|dt|}{|t-w|^{2/\alpha}} \preceq n^{2/\alpha-1}, \quad \gamma_1 \le 0,$$

$$B^2_{n,1}(w) := \sum_{i=1}^{2} \int\limits_{K^i_2(R_1)} \frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}|\Psi(t)-\Psi(w)|^2} \preceq$$

$$\preceq \int\limits_{K^1_2(R_1)} \frac{|dt|}{|t-w_1|^{(2+\gamma_1)/\alpha}} + \int\limits_{K^2_2(R_1)} \frac{|dt|}{|t-w|^{(2+\gamma_1)/\alpha}} \preceq n^{(2+\gamma_1)/\alpha-1}, \quad \gamma_1 > 0,$$

$$B^2_{n,1}(w) = \int\limits_{K_2(R_1)} \frac{|\Psi(t)-\Psi(w_1)|^{-\gamma_1}|dt|}{|\Psi(t)-\Psi(w)|^2} \preceq \int\limits_{K_2(R_1)} \frac{|dt|}{|t-w|^{2/\alpha}} \preceq n^{2/\alpha-1}, \quad \gamma_1 \le 0.$$

Since $|t-w_1|\geq c_2$ and $|t-w|\geq 1$ for $t\in K_3(R_1)$ and $w\in K_2(R)$, then

$$B^3_{n,1}(w):=\int\limits_{K_3(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}\,|\Psi(t)-\Psi(w)|^2}\preceq$$

$$\preceq\left(\frac{1}{c_2}\right)^{\gamma_1/\alpha}\int\limits_{K_3(R_1)}\frac{|dt|}{|t-w|^{2/\alpha}}\preceq n^{2/\alpha-1},\quad \gamma_1>0,$$

$$B^3_{n,1}(w)=\int\limits_{K_3(R_1)}\frac{|\Psi(t)-\Psi(w_1)|^{-\gamma_1}\,|dt|}{|\Psi(t)-\Psi(w)|^2}\preceq$$

$$\preceq\int\limits_{K_3(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w)|^2}\preceq\int\limits_{K_3(R_1)}\frac{|dt|}{|t-w|^{2/\alpha}}\preceq n^{2/\alpha-1},\quad \gamma_1\leq 0.$$

Case 3. Let $w\in K_3(R)$. Then

$$B^1_{n,1}(w)\preceq\int\limits_{K_1(R_1)}\frac{|dt|}{|t-w_1|^{\gamma_1/\alpha}}\preceq n^{\gamma_1/\alpha-1},\quad \gamma_1>0,$$

$$B^1_{n,1}(w)\preceq\int\limits_{K_1(R_1)}\frac{|dt|}{|t-w|^{2/\alpha}}\preceq\left(c_2-\frac{2c_1}{n}\right)^{-2/\alpha}\int\limits_{K_1(R_1)}|dt|\preceq 1,\quad \gamma_1\leq 0,$$

$$B^2_{n,1}(w)\preceq\int\limits_{K_2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{2+\gamma_1}}\preceq$$

$$\preceq\int\limits_{K_2(R_1)}\frac{|dt|}{|t-w_1|^{(2+\gamma_1)/\alpha}}\preceq n^{(2+\gamma_1)/\alpha-1},\quad \gamma_1>0,$$

$$B^2_{n,1}(w)\preceq\int\limits_{K_2(R_1)}|dt|\preceq 1,\quad \gamma_1\leq 0,$$

$$B^3_{n,1}(w)\preceq\int\limits_{K_3(R_1)}\frac{|dt|}{|t-w|^{2/\alpha}}\preceq n^{2/\alpha-1},\quad \gamma_1>0,$$

$$B^3_{n,1}(w)\preceq\int\limits_{K_3(R_1)}\frac{|dt|}{|t-w|^{2/\alpha}}\preceq n^{2/\alpha-1},\quad \gamma_1\leq 0.$$

By combining the estimates obtained in the Cases 1–3 with (4.3)–(4.5), (4.7) and (4.8), for any $p>0$ and for all $z\in L_R$, we obtain

$$|P_n(z)|\preceq\Gamma_n\,\|P_n\|_p\,, \tag{4.9}$$

where

$$\Gamma_n := \begin{cases} \sum_{j=1}^{m} n^{\frac{\gamma_j+2}{p\alpha}}, & \text{if at least for one } \quad j, \quad \gamma_j > 0, \quad j = \overline{1,m}, \\ n^{2/p\alpha}, & \text{if } \quad \gamma_j \le 0, \quad \text{for all } \quad j = \overline{1,m}. \end{cases}$$

The estimation (4.9) satisfied for all $z \in L_R$. We show that it occurs on L. For $R > 1$, let $w = \varphi_R(z)$ denotes the univalent conformal mapping of G_R onto B normalized by $\varphi_R(0) = 0$, $\varphi'_R(0) > 0$, and let $\{\xi_j\}$, $1 \le j \le m \le n$, zeros of $P_n(z)$, lying on G_R (if such zeros exist). Let

$$B_{m,R}(z) := \prod_{j=1}^{m} \tilde{B}_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\xi_j)}{1 - \overline{\varphi_R(\xi_j)}\varphi_R(z)} \tag{4.10}$$

denotes a Blaschke function with respect to zeros $\{\xi_j\}$, $1 \le j \le m \le n$, of $P_n(z)$. Clearly,

$$|B_{m,R}(z)| \equiv 1, \quad z \in L_R; \qquad |B_{m,R}(z)| < 1, \quad z \in G_R.$$

For any $z \in G_R$, let us set

$$H_n(z) := \frac{P_n(z)}{B_{m,R}(z)}.$$

The function $H_n(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. Then, applying maximal modulus principle to $H_n(z)$, we have

$$\left|\frac{P_n(z)}{B_{m,R}(z)}\right| \le \max_{\zeta \in \overline{G_R}} \left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right| \le \max_{\zeta \in L_R} |P_n(\zeta)| \preceq \Gamma_n \left\|P_n\right\|_p \quad \forall z \in L,$$

and, therefore, we find

$$\max_{z \in L} |P_n(z)| \preceq n^{(\gamma+2)/\alpha p} \left\|P_n\right\|_p \quad \forall p > 0,$$

where $\gamma := \max\left\{0; \gamma_j, \, j = \overline{1,m}\right\}$, and the proof (2.9) is completed.

Now, we will begin to proof (2.8). For the arbitrary fixed $R = 1 + \frac{1}{n}$, let us set $L^* := y(L_R)$. According to Lemma 3.3, the number $\rho_1 := 1 + c_1(R-1)$ can be chosen as $\overline{G}^*_{\rho_1} \subseteq G$. Let $\rho := 1 + \frac{\rho_1 - 1}{2}$. Let $\{\zeta_j\}$, $1 \le j \le m \le n$, zeros of $P_n(z)$ lying in Ω^* and let

$$B^*_m(z) := \prod_{j=1}^{m} B^*_{j,R}(z) = \prod_{j=1}^{m} \frac{\Phi_R(z) - \Phi_R(\zeta_j)}{1 - \overline{\Phi_R(\zeta_j)}\Phi_R(z)}, \quad z \in \Omega^*,$$

denotes a Blaschke function with respect of the zeros $\{\zeta_j\}$ of $P_n(z)$ in Ω^*. For any $p > 0$ and $z \in \Omega^*$, we define

$$S^*_{n,p}(z) := \left(\frac{P_n(z)}{B^*_m(z)\Phi_R^{n+1}(z)}\right)^{p/2}.$$

The function $S^*_{n,p}(z)$ is analytic in Ω^*, continuous on $\overline{\Omega^*}$, $S^*_{n,p}(\infty)=0$ and does not have zeros in Ω^*. We take an arbitrary continuous branch of the $S^*_{n,p}(z)$ and for this branch we maintain the same designation. Then, by the Cauchy integral formula for the region $\Omega^*_{R_1}$, we have

$$S^*_{n,p}(z)=-\frac{1}{2\pi i}\int\limits_{L^*_\rho} S^*_{n,p}(\zeta)\frac{d\zeta}{\zeta-z}, \quad z\in\Omega^*_{R_1}. \tag{4.11}$$

Since $|B^*_m(\zeta)|=1$, for $\zeta\in L$, then, for arbitrary ε, $0<\varepsilon<\varepsilon_1$, there exists a circle $|w|=1+\dfrac{\varepsilon}{n}$, such that for any $j=\overline{1,m}$ the following is satisfied:

$$\left|B^*_{j,R}(\zeta)\right|>1-\varepsilon, \quad \zeta\in L_{1+\varepsilon/n}.$$

Then $|B^*_m(\zeta)|>(1-\varepsilon)^m\succeq 1$ for $\zeta\in L^*_\rho$ and $|B^*_m(z)|\le 1$ for $z\in\Omega^*_\rho$. On the other hand, $|\Phi_R(\zeta)|=R>1$ for $\zeta\in L^*_\rho$. Therefore, for any $z=z_j$, $j=\overline{1,m}$, from (4.11) we get

$$|P_n(z_j)|^{p/2}\le\frac{\left|B^*_m(z)\Phi^{n+1}_R(z_j)\right|^{p/2}}{2\pi}\int\limits_{L^*_\rho}\left|\frac{P_n(\zeta)}{B^*_m(\zeta)\ \Phi^{n+1}_R(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta-z_j|}\preceq$$

$$\preceq\left|\Phi^{n+1}_R(z_j)\right|^{p/2}\int\limits_{L^*_\rho}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z_j|}. \tag{4.12}$$

According to Lemma 3.3, there exists a number $\rho_2>\rho_1$ such that $\overline{G}\subseteq\overline{G}^*_{\rho_2}$ and $\rho_2-1\asymp R-1$. Using (3.2) we have $|\Phi_R(z_j)|\le\rho_2\le 1+c(R-1)$, and so, $\left|\Phi^{n+1}_R(z_j)\right|\preceq 1$. Therefore, from (4.12) we have

$$|P_n(z_j)|^{p/2}\preceq\int\limits_{L^*_\rho}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z_j|}. \tag{4.13}$$

Let

$$b_m(z):=\prod_{j=1}^m b^*_j(z)=\prod_{j=1}^m\frac{\Phi_R(z)-\Phi_R(z_j)}{1-\overline{\Phi_R(z_j)}\Phi_R(z)}$$

denote a Blaschke function for the weight function $h(z)$ with respect to they singular points $\{z_j\}\in L$, $j=\overline{1,m}$. Multiplying the numerator and the denominator of the last integrand by $\prod_{j=1}^m\left|\dfrac{\zeta-z_j}{b^*_j(\zeta)}\right|^{\gamma_j/2}$, replacing the variable $w=\Phi_R(z)$ and applying the Hölder inequality, from (4.13) we obtain

$$|P_n(z_j)|^{p/2}\preceq\left(\int\limits_{|t|=\rho}\prod_{j=1}^m\left|\frac{\Psi_R(t)-\Psi_R(w_j)}{b^*_j(\Psi_R(t))}\right|^{\gamma_j}|P_n(\Psi_R(t))|^p\left|\Psi'_R(t)\right|^2|dt|\right)^{1/2}\times$$

$$\times\left(\int\limits_{|t|=\rho}\prod_{j=1}^m\left|\frac{\Psi_R(t)-\Psi_R(w_j)}{b^*_j(\Psi_R(t))}\right|^{-\gamma_j}\frac{|dt|}{|\Psi_R(t)-\Psi_R(w_j)|^2}\right)^{1/2}. \tag{4.14}$$

Since $\left|b_j^*(\zeta)\right| = 1,\ j = \overline{1,m}$, for $\zeta \in L^*$, then, for arbitrary $\varepsilon > 0$ there exists a circle $|w| = 1 + \dfrac{\varepsilon}{n}$, such that for any $j = 1, 2, \ldots, m$ the following is satisfied: $\left|b_j^*(\zeta)\right| > 1 - \varepsilon$. Then $\left|b_j^*(\zeta)\right| > (1-\varepsilon) \succeq 1$ for $\zeta \in L_\rho^*$ and $\left|b_j^*(z)\right| \le 1$ for $z \in \Omega_\rho^*$. According this reason, from (4.14) we get

$$|P_n(z_j)|^{p/2} \preceq \left(\int\limits_{L_\rho^*} |g_{n,p}(t)|^p \, |dt|\right)^{1/2} \left(\int\limits_{L_\rho^*} \prod_{j=1}^m \frac{|dt|}{|\Psi_R(t) - \Psi_R(w_j)|^{2+\gamma_j}}\right)^{1/2} := J_{n,1} J_{n,2}, \quad (4.15)$$

where

$$g_{n,p}(t) := \prod_{j=1}^m \left[\Psi_R(t) - \Psi_R(w_j)\right]^{\gamma_j/p} P_n(\Psi_R(t)) \left[\Psi_R'(t)\right]^{2/p}, \quad |t| = \rho.$$

The integral $J_{n,1}$ we will estimate analogously to estimation of the integral $\mathcal{A}_n$ from (4.4). For this, we separate the circle $|t| = \rho$ to n equal parts η_n with $\operatorname{mes}\eta_n = \dfrac{2\pi\rho}{n}$ and by applying the mean value theorem to the integral $J_{n,1}$, we have

$$(J_{n,1})^2 = \int\limits_{|t|=\rho} |g_{n,p}(t)|^p \, |dt| = \sum_{k=1}^n \int\limits_{\eta_k} |g_{n,p}(t)|^p \, |dt| = \sum_{k=1}^n \left|g_{n,p}(t_k')\right|^p \operatorname{mes}\eta_k, \quad t_k' \in \eta_k.$$

On the other hand, by applying mean value estimation

$$\left|g_{n,p}(t_k')\right|^p \le \frac{1}{\pi\left(\left|t_k'\right| - 1\right)^2} \iint\limits_{\left|\xi - t_k'\right| < \left|t_k'\right| - 1} |g_{n,p}(\xi)|^p \, d\sigma_\xi,$$

we obtain

$$(J_{n,1})^2 \preceq \sum_{k=1}^n \frac{\operatorname{mes}\eta_k}{\pi\left(\left|t_k'\right| - 1\right)^2} \iint\limits_{\left|\xi - t_k'\right| < \left|t_k'\right| - 1} |g_{n,p}(\xi)|^p \, d\sigma_\xi, \quad t_k' \in \eta_k.$$

By taking into account, at most two of the discs with center t_k' are intersecting, we have

$$(J_{n,1})^2 \preceq \frac{\operatorname{mes}\eta_1}{(|t_1'| - 1)^2} \iint\limits_{1<|\xi|<\rho_1} |g_{n,p}(\xi)|^p \, d\sigma_\xi \preceq n \iint\limits_{1<|\xi|<\rho_1} |g_{n,p}(\xi)|^p \, d\sigma_\xi.$$

By replacing the variable $w = \Phi_R(z)$ and according to inclusion $\overline{G}_{\rho_1}^* \subseteq G$, for $J_{n,1}$ we get

$$(J_{n,1})^2 \preceq n \iint\limits_{G_{\rho_1}^* \setminus G^*} h(\zeta) \, |P_n(\zeta)|^p \, d\sigma_\zeta \preceq n \, \|P_n\|_p^p. \quad (4.16)$$

Let's estimate

$$(J_{n,2})^2 = \int\limits_{|t|=\rho} \prod_{j=1}^m \frac{|dt|}{|\Psi_R(t) - \Psi_R(w_j)|^{2+\gamma_j}}.$$

Since the points $\{z_j\}_{j=1}^{m} \in L$ are distinct, we obtain

$$(J_{n,2})^2 \asymp \int\limits_{|t|=\rho} \frac{|dt|}{|\Psi_R(t) - \Psi_R(w_j)|^{2+\gamma_j}}.$$

Now, since $\Phi \in \mathrm{Lip}\,\alpha$, we have

$$(J_{n,2})^2 \preceq \int\limits_{|t|=\rho} \frac{|dt|}{|t - w_j|^{(2+\gamma_j)/\alpha}} \preceq n^{(2+\gamma_j)/\alpha - 1}. \tag{4.17}$$

Combining estimations (4.13) – (4.17), we get

$$|P_n(z_j)| \preceq n^{(2+\gamma_j)/\alpha p} \|P_n\|_p,$$

and the proof (2.8) is completed.

The proof of the Theorem 2.1 (estimations (2.2) and (2.3)) obtained analogously to the proof of Theorem 2.3. In this case, we change estimation under the condition $\Phi \in \mathrm{Lip}\,\alpha$ with estimation from the Lemma 3.2.

***4.2. Proofs of Theorems* 2.2 *and* 2.4.** Let us begin to proof of Theorem 2.4. Let $z \in \Omega_R$ be an arbitrary fixed point. Then $z \in \Omega_R^j$ for some $j = \overline{1, m}$. From (4.3) we have

$$|P_n(z)|^{p/2} \preceq \frac{|\Phi^{n+1}(z)|^{p/2}}{d(z, L_R)} \int\limits_{L_R} |P_n(\zeta)|^{p/2} |d\zeta|. \tag{4.18}$$

Analogously to the estimations (4.4) – (4.9), for each $j = \overline{1, m}$ we obtain

$$\left(\int\limits_{L_R} |P_n(\zeta)|^{p/2} |d\zeta| \right)^2 \preceq n \|P_n\|_p^p \int\limits_{|t|=R} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j}}.$$

Further

$$\int\limits_{|t|=R} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j}} \preceq \int\limits_{|t|=R} \frac{|dt|}{|t - w_j|^{\gamma_j/\alpha}} \preceq \delta_n. \tag{4.19}$$

Therefore, from (4.18) we get

$$|P_n(z)| \preceq \left(\frac{\sqrt{n\delta_n}}{d(z, L_R)} \right)^{2/p} |\Phi(z)|^{n+1} \|P_n\|_p, \quad z \in \Omega_R^j,$$

and we obtain the proof of (2.5).

The proof of the Theorem 2.2 will be obtained from (4.19), according to Lemma 3.2

$$\int\limits_{|t|=R} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j}} \preceq \int\limits_{|t|=R} \frac{|dt|}{|t - w_j|^{\gamma_j(1+\kappa)}} \preceq \mu_n.$$

***4.3. Proof of Remark* 2.2.** The estimations (2.12) and (2.13) are shown in [**?**] (Theorem 17). Let us prove (2.14). Denote by $\{K_n(z)\}$, $\deg K_n = n,\ n = 0, 1, 2, \ldots,$ the system of orthonormal polynomials with the weight function $h(z)$ for the region G, i.e., $K_n(z) := \alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots$ $\ldots + \alpha_0,\ \alpha_n > 0$ and

$$\iint\limits_G h(z) K_n(z) \overline{K_m(z)}\, d\sigma_z = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker's symbol.

Let $\wp_n^p := \wp_{n,h}^p = \wp_{n,h,G}^p$ be the space $\wp_n$ with the norm (1.2) for $p \geq 1$. Similarly, $\wp_n^\infty := \wp_{n,\overline{G}}^\infty$ be the space $\wp_n$ with the norm (1.3).

Consider a sequence of lineer operators $I_{n,h} \colon \wp_{n,h}^2 \to \wp_{n,\overline{G}}^\infty$, $I_{n,h}(P_n) = P_n$, with the norms

$$\|I_{n,h}\| := \sup\left\{ \|P_n\|_\infty : P_n \in \wp_n, \|P_n\|_{A_2(h,G)} \leq 1 \right\}.$$

In [3] (Theorem 1), it was proved the following theorem.

Theorem A. *Suppose that there exists* $\xi \in L$ *such that* $\|K_n\|_\infty \asymp |K_n(\xi)|$ *and, for a certain number* $\beta \geq 0$, $\|K_n\|_\infty \asymp n^\beta$. *Then* $\|I_{n,h}\| \asymp n^{\beta+1/2}$.

Let $G = B$ and $h(z) = |z-1|^2$. Then [27, p. 76]

$$K_n(z) = \frac{2}{\sqrt{\pi(n+1)(n+2)(n+3)}} \sum_{j=0}^{n} (j+1) z^j (1 + z + \ldots + z^{n-j}).$$

Therefore,

$$\|K_n\|_{A_\infty(1,\overline{B})} = K_n(1) = \frac{2}{\sqrt{\pi(n+1)(n+2)(n+3)}} \sum_{j=0}^{n} (j+1)(n-j+1) =$$

$$= \frac{1}{3\sqrt{\pi}} \sqrt{(n+1)(n+2)(n+3)}.$$

On the other hand, according to [3] (Lemma 1), we obtain

$$\|I_{n,h}\| = \left(\sum_{j=0}^{n} |K_j(1)|^2 \right)^{1/2} = \frac{1}{3\sqrt{\pi}} \left(\sum_{j=0}^{n} (j+1)(j+2)(j+3) \right)^{1/2} =$$

$$= \frac{1}{6\sqrt{\pi}} \sqrt{(n+1)(n+2)(n+3)(n+4)}.$$

Therefore, we can choose $T_n^* \in \wp_n$ such that $\|I_{n,h}\| = \|T_n^*\|_{A_\infty(1,\overline{B})}$.

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